Electromagnetic Field in Non-Linear Media

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Abstract

It is well-known that the non-linearity of the constitutive relations of the electromagnetic field must be often taken into account in order to compute the performances of the electromagnetic devices. Some qualitative aspects concerning the electromagnetic field analysis in non-linear media are presented in this work: uniqueness, existence and stability. The polarization method for the treatment of the nonlinearity is analyzed, focusing on the convergence criteria, errors and numerical convergent procedures.

1 Stationary magnetic field

1.1 Uniqueness, Existence and Stability

The Equations of the Magnetic Field

Let Ω be a domain and $\partial\Omega$ its boundary. In Ω the magnetic field verifies the following equations:

(1) $$\nabla \times \mathbf{H} = \mathbf{J}$$

(2) $$\nabla \mathbf{B} = 0$$

(3) $$\mathbf{H} = \hat{\boldsymbol{F}}(\mathbf{B})$$

Non-linear function $\hat{\boldsymbol{F}} : \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega)$ refers to the domains with ferromagnetic bodies and permanent magnets. Usually the relation **B** - **H** is local defined in almost all points of the domain Ω,

(4) $$\mathbf{H}(P) = \mathbf{f}(P, \mathbf{B}(P))$$

(Sometimes, we note $\mathbf{B}(P) = \mathbf{B}$). If the function $\boldsymbol{f}$ is Lipschitzian:

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$$|\boldsymbol{f}(\mathbf{B}') - \boldsymbol{f}(\mathbf{B}'')| < \wedge|\mathbf{B}' - \mathbf{B}''|, \quad (\forall)\ \mathbf{B}', \mathbf{B}''$$

and uniform monotone:

$$(\boldsymbol{f}(\mathbf{B}') - \boldsymbol{f}(\mathbf{B}''))(\mathbf{B}' - \mathbf{B}'') > \lambda(\mathbf{B}' - \mathbf{B}'')^2, \quad (\forall)\ \mathbf{B}', \mathbf{B}'',$$

where $\wedge < \wedge_M$ and $\lambda > \lambda_m > 0$ in Ω, then the function $\hat{\boldsymbol{F}}$ is Lipschitzian and uniform monotone. For example, is an isotropic medium we have

$$\wedge = \sup_{\mathbf{B}', \mathbf{B}''} \frac{|\mathbf{f}(\mathbf{B}') - \mathbf{f}(\mathbf{B}'')|}{|\mathbf{B}' - \mathbf{B}''|} = \nu_{\max} = \frac{1}{\mu_{\min}};$$

$$\lambda = \inf_{\mathbf{B}', \mathbf{B}''} \frac{|\mathbf{f}(\mathbf{B}') - \mathbf{f}(\mathbf{B}'')|}{|\mathbf{B}' - \mathbf{B}''|} = \nu_{\min} = \frac{1}{\mu_{\max}}.$$

The boundary conditions

For the sake of the simplicity we choose in the following only simply connected domain having on the boundary $\partial\Omega$ the conditions: $\mathbf{H}_t = \mathbf{f}$ on the connected surface S', and $B_n = g$ on the rest $S'' = \partial\Omega \setminus S'$.

The $\boldsymbol{L}^2(\Omega)$ - Mathematical Model of the Magnetic Field

The magnetic field is a pair $(\mathbf{B}, \mathbf{H})$ with $\mathbf{B}, \mathbf{H} \in \boldsymbol{L}^2(\Omega)$.

Because of this the above boundary condition must be modified, the surface $\partial\Omega$ having zero measure. Let $\boldsymbol{s}_A = (\mathbf{B}, \mathbf{H})$ be the magnetic field fulfilling the condition:

$$\mathbf{B} \in \boldsymbol{L}^2(\Omega) \cap \boldsymbol{C}^\infty(\Omega), \quad curl\mathbf{B} = \mathbf{0}, \quad div\mathbf{B} = 0, \quad \mathbf{n} \times \mathbf{B}(P) = 0 \quad \text{for } P \in S'$$

$$\mathbf{H} \in \boldsymbol{L}^2(\Omega) \cap \boldsymbol{C}^\infty(\Omega), \quad curl\mathbf{H} = \mathbf{0}, \quad div\mathbf{H} = 0, \quad \mathbf{n} \times \mathbf{H}(P) = 0 \quad \text{for } P \in S''.$$

The field $\mathbf{B}$ is determined by its normal component on S'' and $\mathbf{H}$ by its tangential component on S'. Therefore we can say that the magnetic field $\boldsymbol{s}_A$ represents the boundary conditions. Let $\boldsymbol{S}_A$ be the space of the fields $\boldsymbol{s}_A$.

Because $\mathbf{B}, \mathbf{H} \in \boldsymbol{L}^2(\Omega)$, the internal sources of the magnetic field are distributions. We note: $Q = \{Div\mathbf{Z} | \mathbf{Z} \in \mathbf{L}^2\}$ and $\boldsymbol{C} = \{Curl\mathbf{Z} | \mathbf{Z} \in \boldsymbol{L}^2\}$, where $(< Div\mathbf{Z}, \varphi >) =< \mathbf{Z}, grad\varphi >$, $(\forall)\varphi \in K$, and $(< Curl\mathbf{Z}, \psi >) =< \mathbf{Z}, rot\psi >$, $(\forall)\psi \in \mathbf{K}$, K and $\boldsymbol{K}$ being the spaces of the scalar and vectorial functions ∞-differentiable and with compact support in Ω, respectively.

The existence and uniqueness theorem. *There are given:*

I) the constitutive relation $\hat{\boldsymbol{F}}$ Lipschitzian and uniform monotone;

II) the sources $\rho \in Q$, $\boldsymbol{J} \in \boldsymbol{C}$;

III) the boundary conditions s_A.

It results one and only one magnetic field $(\boldsymbol{H}, \boldsymbol{B})$ so that:

a) its component on the space $\boldsymbol{S_A}$ is s_A;

b) $Div\boldsymbol{B} = \rho$, $\quad Curl\boldsymbol{H} = \boldsymbol{J}$;

c) $\boldsymbol{B} = \hat{\boldsymbol{F}}(\boldsymbol{H})$.

Stability theorem. *There are given:*

I) the constitutive relations: $\hat{\boldsymbol{F}}_1, \hat{\boldsymbol{F}}_2$, where $\hat{\boldsymbol{F}}_1$ is Lipschitzian and uniform monotone and $||\hat{\boldsymbol{F}}_1(\boldsymbol{H}) - \hat{\boldsymbol{F}}_2(\boldsymbol{H})|| \le \varepsilon_F$, $\quad (\forall)\boldsymbol{H} \in \boldsymbol{L}^2(\Omega)$;

II) the boundary conditions: $\boldsymbol{s}_{A1} = (\boldsymbol{H}_{A1}, \boldsymbol{B}_{A1}$, $\boldsymbol{s}_{A2} = (\boldsymbol{H}_{A2}, \boldsymbol{B}_{A2})$, where $||\boldsymbol{H}_{A1} - \boldsymbol{H}_{A2}|| \le \varepsilon_H$, $||\boldsymbol{B}_{A1} - \boldsymbol{B}_{A2}|| \le \varepsilon_B$;

III) the current densities $\boldsymbol{J}_1, \boldsymbol{J}_2 \in \boldsymbol{C}$ *that verify:* $| < \psi, \boldsymbol{J}_1 - \boldsymbol{J}_2 > | \leq \varepsilon_J ||curl \psi||$, $(\forall)\psi \in \boldsymbol{K}$;

IV) the magnetic charges $\rho_1, \rho_2 \in Q$ *that verify:* $| < \varphi, \rho_1 - \rho_2 > | \leq \varepsilon_\rho ||grad \varphi||$, $(\forall)\varphi \in K$;

V) the magnetic fields $(\boldsymbol{H}_1, \boldsymbol{B}_1), (\boldsymbol{H}_2, \boldsymbol{B}_2)$, *which have the component* s_{A1}, s_{A2} *on* $\boldsymbol{S}_A$ *and verify the equations:*

$$\mathbf{B}_1 = \hat{\boldsymbol{F}}(\mathbf{H}_1), \quad Curl \mathbf{H}_1 = \mathbf{J}_1, \quad Div \mathbf{B}_1 = \rho_1$$

$$\mathbf{B}_2 = \hat{\boldsymbol{F}}(\mathbf{H}_2), \quad Curl \mathbf{H}_2 = \mathbf{J}_2, \quad Div \mathbf{B}_2 = \rho_2.$$

Then:

a) $||\mathbf{H}_1 - \mathbf{H}_2|| \leq (1 + \wedge_M/\lambda_m)(\varepsilon_H + \varepsilon_J) + (\varepsilon_F + \varepsilon_\rho + \varepsilon_B)/\lambda_m$;

b) $||\mathbf{B}_1 - \mathbf{B}_2|| \leq \wedge_M ||\mathbf{H}_1 - \mathbf{H}_2|| + \eta_F$,

where $\wedge_M$ *and* λ_m *are Lipschitz and uniform monotony factors of the relation* $\hat{\boldsymbol{F}}_1$.

Remarks. 1) Condition I represents the distance between two constitutive relations. The above theorem gives the stability v. constitutive relation and allows to evaluate the error in the numerical computation, where the real **B - H** relation is replace by a linear one (fig. 2a) or by a piece-wise linear one (fig. 2b).

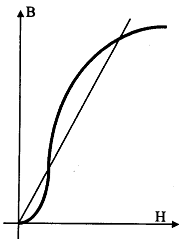

Fig.2a

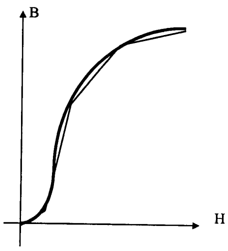

Fig.2b

2) Conditions III and IV represent the distance between two sources. In principal the stability theorem allows to evaluate the error v. the sources. For example, if FEM uses first order finite elements, than the sources have a surface distribution. The distance between this approximate distribution and the real distribution leads to the FEM error.

3) If $\varepsilon_F, \varepsilon_B, \varepsilon_H, \varepsilon_J, \varepsilon_\rho \to 0$, then $(\mathbf{B}_2, \mathbf{H}_2) \to (\mathbf{B}_1, \mathbf{H}_1)$.

Theorem of orthogonality (Tellegen). *If the magnetic field* $(\boldsymbol{B}, \boldsymbol{H})$ *has zero boundary conditions (zero component on* $\boldsymbol{S}_A$*) and zero sources* ($div\boldsymbol{B} = 0$, $curl\boldsymbol{H} = \boldsymbol{0}$)*, then* $< \boldsymbol{H}, \boldsymbol{B} >= 0$.

1.2 The Polarization Method

We replace relation (3) by

$$\mathbf{B} = \mu\mathbf{H} + \mathbf{I}, \tag{5}$$

where the non-linearity is hidden [1] in the polarization

$$\mathbf{I} = \mathbf{B} - \mu\hat{\boldsymbol{F}}(\mathbf{B}) = \hat{\boldsymbol{G}}(\mathbf{B}). \tag{6}$$

We can choose μ so that the function $\hat{\boldsymbol{G}}$ defined by the relation (6) is a contraction:

$$||\hat{\boldsymbol{G}}(\mathbf{B}') - \hat{\boldsymbol{G}}(\mathbf{B}'')||_\nu \leq \theta||\mathbf{B}' - \mathbf{B}''||_\nu, \quad (\forall)\mathbf{B}', \mathbf{B}'',$$

where $\theta < 1$ and $< \mathbf{X}, \mathbf{Y} >_\nu = < \mathbf{X}, \nu\mathbf{Y} >$ and $\nu = 1/\mu$. For example, for $\mu \in (0, 2\lambda_m/\wedge_M^2)$, we have $\theta \leq 1 - 2\mu\lambda_m + \mu^2\wedge_M^2$. The smallest value for contraction factor θ is upperbound by the value $1 - (\lambda_m/\wedge_M)^2$ and it is obtained for $\mu = \lambda_m/\wedge_M^2$. This contraction factor is very close to unit (look, for example, to a ferromagnetic medium having $\wedge = 1/\mu_0$ and $\lambda = 1/1000\mu_0$ for which the smallest upperbound of the contraction factor is $1 - 10^{-6}$). Other procedures for chose the permeability μ are more convenient [2]. In the case of an isotopic media we can choose $\mu < 2\mu_{\min}$, so that the function $\hat{\boldsymbol{G}}$ is a contraction. The contraction factor θ verifies: $\theta = Max(1 - \mu/\mu_M,\ \mu/\mu_m - 1)$. Because $\mu_{\min} \geq \mu_0$ we can replace the nonlinear medium by a linear one having the permeability of the vacuum. In this case $\theta = 1 - \mu_0/\mu_M$. (For above example $\theta \leq 0.999$). The smallest value is $\theta \leq \dfrac{\mu_{\max} - \mu_{\min}}{\mu_{\max} + \mu_{\min}}$, and it is obtained for $\dfrac{1}{\mu_{opt}} = \nu_{opt} = \dfrac{\nu_{\min} + \nu_{\max}}{2}$.

The Iterative Method

The non-linear problem is solved with the following iterative scheme:

a) We give an arbitrary value $\mathbf{I}^{(0)}$.

b) We compute magnetic field $(\mathbf{B}^{(1)}, \mathbf{H}^{(1)})$ which verifies the equations

$$\nabla \times \mathbf{H}^{(1)} = \mathbf{J}$$

$$\nabla\mathbf{B}^{(1)} = 0$$

$$\mathbf{B}^{(1)} = \mu\mathbf{H}^{(1)} + \mathbf{I}^{(0)}.$$

c) We calculate $\mathbf{I}^{(1)}$ with the relation (6):

$$\mathbf{I}^{(1)} = \hat{\boldsymbol{G}}(\mathbf{B}^{(1)}).$$

We repeat steps b) and c) until we have

$$||\Delta\mathbf{I}^{(n)}||_\nu = ||\mathbf{I}^{(n)} - \mathbf{I}^{(n-1)}||_\nu = \sqrt{\int_\Omega \nu(\mathbf{I}^{(n)} - \mathbf{I}^{(n-1)})^2 d\Omega} < \varepsilon,$$

where ε is an imposed error.

For any $\mathbf{I}$ we have one and only one field $\mathbf{B}$ which verifies the equations (1), (2), (5) (the uniqueness theorem). From the theorem of orthogonality it results that the function $\mathbf{I} \to \mathbf{B} = \hat{\boldsymbol{B}}(\mathbf{I})$ is nonexpansive [1], i.e.,

$$||\hat{\boldsymbol{B}}(\mathbf{I}') - \hat{\boldsymbol{B}}(\mathbf{I}'')||_\nu \leq ||\mathbf{I}' - \mathbf{I}''||_\nu.$$

Because the function $\hat{\boldsymbol{G}}$ is contractive, the above iteration scheme is a Picard-Banach fixed point procedure to find the fixed point of the function $\hat{\boldsymbol{W}} = \hat{\boldsymbol{G}} \circ \hat{\boldsymbol{B}}$. Therefore:

The First Theorem of Convergence. *We can choose μ so that the function $\hat{\boldsymbol{G}}$ is a contraction and then the Polarization Method using $\boldsymbol{B}$-correction is convergent.*

Remark. If we use $\mathbf{M} = \mathbf{I}/\mu$, then we have $||\mathbf{M}||_\mu$ instead of $||\mathbf{I}||_\nu$.

The H-correction.

A dual formulation may be used for treatment of the non-linearity. Instead of relations (5) and (6), we have

$$\mathbf{H} = \nu\mathbf{B} - \mathbf{M},$$

where the non-linearity is hidden in the magnetization $\mathbf{M} = \mathbf{H} - \mu\hat{\boldsymbol{F}}^{-1}(\mathbf{H}) = \hat{\boldsymbol{G}}'(\mathbf{H})$. Because $\hat{\boldsymbol{F}}$ is Lipschitzian and uniform monotone, it is inversable and $\hat{\boldsymbol{F}}^{-1}$ is also Lipschitzian and uniform monotone. We can choose ν so that $\hat{\boldsymbol{G}}'$ is a contraction. For example, in the case of an isotropic media we have $\nu < 2\nu_{\min}$, namely $\mu > \mu_{\max}/2$. From the existence and uniqueness theorem it results that the function $\mathbf{M} \to \mathbf{H} = \hat{\boldsymbol{H}}(\mathbf{M})$ is well defined and from the orthogonality theorem it results that the function $\hat{\boldsymbol{H}}$ is nonexpansive,

$$||\hat{\boldsymbol{H}}(\mathbf{M}') - \hat{\boldsymbol{H}}(\mathbf{M}'')||_\mu \leq ||\mathbf{M}' - \mathbf{M}''||_\mu.$$

The iterative scheme corrects the magnetization $\mathbf{M}$ as an $\mathbf{H}$-function. It is also a Packard-Banach fixed point procedure to find the fixed point of the function $\hat{\boldsymbol{W}}' = \hat{\boldsymbol{G}}' \circ \hat{\boldsymbol{H}}$. Therefore:

The Second Theorem of Convergence. *We can choose μ so that the function $\hat{\boldsymbol{G}}'$ is a contraction and then the Polarization Method using $\boldsymbol{H}$-correction is convergent.*

Attention. The choise $\mu = \mu_0$ do not ensure the contractivity of the function $\hat{\boldsymbol{G}}'$ and the convergence of the $\mathbf{H}$-correction iterative method.

Errors.

Some important advantages may be pointed out:

- the convergence of the iterative scheme is sure;
- for each iteration we have the same permeability μ;
- the errors in comparison with the exact solution $(\mathbf{B}^*, \mathbf{H}^*)$ can be easily evaluated for n-th iteration

$$||\mathbf{B}^* - \mathbf{B}^{(n)}||_\nu \leq \frac{1}{1-\theta}||\Delta\mathbf{I}^{(n)}||_\nu; \tag{7}$$

$$||\mathbf{H}^* - \mathbf{H}^{(n)}||_\mu \leq \frac{1}{1-\theta}||\Delta\mathbf{I}^{(n)}||_\nu; \tag{8}$$

in subdomains Ω_{air} with air we have

$$(9) \qquad \int_{\Omega_{air}} \nu(\mathbf{B}^* - \mathbf{B}^{(n)})^2 d\Omega \leq \frac{\theta}{2\sqrt{1-\theta}} ||\Delta\mathbf{I}^{(n)}||_n u,$$

the error at the n-th iteration is placed to the constitutive relation $\hat{\boldsymbol{F}}$

$$(10) \qquad ||\mathbf{H}^{(n)} - \hat{\boldsymbol{F}}(\mathbf{B}^{(n)})||_\mu = ||\Delta\mathbf{I}^{(n)}||_\nu.$$

The Overrelaxation

The convergence of the above iterative procedure may be improved using the overrelaxation described in [4]. From relations (7), (8), (9), (10) it results that we need a very small value for $||\Delta\mathbf{I}^{(n)}||_\nu$ in order to have acceptable errors. If $\mathbf{I}^{(k)} = \hat{\boldsymbol{W}}(\mathbf{I}^{(k-1)})$, then we seek $\mathbf{I}^{(k)} = \mathbf{I}^{(k-1)} + \omega(\mathbf{I}^{(k)} - \mathbf{I}^{(k-1)})$ so that $h(\omega) = ||\hat{\boldsymbol{W}}(\mathbf{I}^{(k)}) - \mathbf{I}^{(k)}||_\nu^2$ is as small as possible. The numerical overrelaxation procedure has the following steps:

1) $\mathbf{B}^{(k)} = \hat{\mathbf{B}}(\mathbf{I}^{(k-1)})$
2) $\mathbf{I}^{(k)} = \hat{\boldsymbol{G}}(\mathbf{B}^{(k)})$
3) $\mathbf{B}^{(k+1)} = \hat{\boldsymbol{B}}(\mathbf{I}^{(k)})$
4) $h(1) = ||\hat{\boldsymbol{G}}(\mathbf{B}^{(k+1)}) - \mathbf{I}^{(k)}||_\nu^2$,

and if $h(1) < \varepsilon$ where ε is fixed, we stop the iteration, otherwise:

5) $\Delta\mathbf{I}^{(k)} = \mathbf{I}^{(k)} - \mathbf{I}^{(k+1)}$, $\Delta\mathbf{B}^{(k+1)} = \mathbf{B}^{(k+1)} - \mathbf{B}^{(k)}$.
6) The solving of the equation

$$\frac{1}{2}h'(\omega) = \left\langle \left.\frac{d\hat{\boldsymbol{G}}}{d\mathbf{B}}\right|_{\mathbf{B}^{(k)}+\omega\Delta\mathbf{B}^{(k+1)}} \left(\Delta\mathbf{B}^{(k+1)} - \Delta\mathbf{I}^{(k)}\right), \right.$$

$$\left. \hat{\boldsymbol{G}}\left(\mathbf{B}^{(k)} + \omega\Delta\mathbf{B}^{(k+1)}\right) - \left(\mathbf{I}^{(k-1)} + \omega\Delta\mathbf{I}^{(k)}\right)\right\rangle = 0$$

7) $\mathbf{I}^{(k+1)} = \hat{\boldsymbol{G}}(\mathbf{B}^{(k)} + \omega\Delta\mathbf{B}^{(k+1)})$, go to 1).

$\mathbf{B}$ and $\mathbf{I}$ are numerically defined by the values in a finite number of subdomains. Because the function $\hat{\boldsymbol{G}}$ is made by the local function $\mathbf{g}$, it results that $\hat{\mathbf{G}}$ and $\dfrac{d\hat{\boldsymbol{G}}}{d\mathbf{B}}$ can be easily calculated. Equation (14) is solved by the secant method, a number of 3 to 5 iterations being enough. The overrelaxation factors ω have values between 1,2 and 50.

1.3 The Solving of the Linear Field Problem by the Finite Element Method

For each iteration we have to calculate the magnetic field $(\mathbf{B}_T, \mathbf{H}_T)$ which verifies the equations: $\nabla \times \mathbf{H}_T = \mathbf{J}$, $\nabla\mathbf{B}_T = \rho$, $\mathbf{B}_T = \mu\mathbf{H}_T + \mathbf{I}_T$. We can obtain easily a magnetic field $(\mathbf{B}_s, \mathbf{H}_s)$ so that $Curl\mathbf{H}_s = \mathbf{J}$ and $Div\mathbf{B}_s = \rho$ (For example, using Biot-Savart-Laplace and Coulomb formulae). Also we can build a magnetic field $(\mathbf{B}_A, \mathbf{H}_A)$ having $Curl\mathbf{H}_A = \mathbf{0}$ and $Div\mathbf{B}_A = 0$ and the boundary conditions $\mathbf{H}_{tA} = \mathbf{f} - \mathbf{H}_{ts}$, and $B_{nA} = g - B_{ts}$ (For example using a scalar potential for $\mathbf{H}_A$ and a vector potential for $\mathbf{B}_A$). The magnetic fields $(\mathbf{B}_s, \mathbf{H}_s)$ and $(\mathbf{B}_A, \mathbf{H}_A)$ do not have any constitutive restrictions. It remain to compute the magnetic field $(\mathbf{B}, \mathbf{H}) = (\mathbf{B}_T, \mathbf{H}_T) - (\mathbf{B}_s, \mathbf{H}_s) - (\mathbf{B}_A, \mathbf{H}_A)$ having zero boundary conditions and zero sources. It verifies the equations

$$\nabla \times \mathbf{H} = \mathbf{0}, \quad \nabla \mathbf{B} = 0, \quad \mathbf{B} + (-\mu\mathbf{H}) = \mathbf{I}, \tag{11}$$

where $\mathbf{I} = \mathbf{I}_T + \mathbf{B}_s + \mathbf{B}_a - \mu(\mathbf{H}_s + \mathbf{H}_A)$.

Let $\boldsymbol{L}'$ be the closure of the space of the fields $\mathbf{H}$ which have the boundary condition $\mathbf{H}_t = \mathbf{0}$ on the surface S'. Let $\boldsymbol{L}''$ be the closure of the space of the fields $\mathbf{B}$ which have the boundary condition $B_n = 0$ on the surface S''. From the theorem of orthogonality we have

$$\langle \mathbf{B}, \mathbf{H} \rangle = \langle \mathbf{B}, \mu\mathbf{H} \rangle_\nu = 0.$$

Hence it results that $\boldsymbol{L}', \boldsymbol{L}''$ are orthogonal in $\boldsymbol{L}^2(\Omega)$ and $\mu\boldsymbol{L}', \boldsymbol{L}''$ are orthogonal in $\boldsymbol{L}^2_\nu(\Omega)$. From relation (11) it results that the solving of the magnetic field problem $(\mathbf{B}, \mathbf{H})$ consists in the decomposition of the polarization $\mathbf{I}$ in the spaces $\mu\boldsymbol{L}'$ and $\boldsymbol{L}''$ [5]. This is the same with the minimzation of the distance $d^2(\mathbf{X}) = ||\mathbf{I} - \mathbf{X}||^2_\nu$, or of the functional

$$\mathcal{F}(\mathbf{X}) = -2 < \mathbf{I}, \mathbf{X} >_\nu + ||\mathbf{X}||^2_\nu \tag{12}$$

in the space $\boldsymbol{L}''$, when we obtain $\mathbf{X} = \mathbf{B}$ or in the space $\mu\boldsymbol{L}'$, when we obtain $\mathbf{X} = -\mu\mathbf{H}$. When $\mathbf{X} \in \mu\boldsymbol{L}'$, we can use the formulation in scalar potential $\mathbf{X} = -\mu grad\Phi$, with the boundary condition $\Phi = 0$ on the surface S'. When $\mathbf{X} \in \boldsymbol{L}''$, we can use the formulation in vector potential $\mathbf{X} = curl\mathbf{A}$, with the boundary condition $A_t = 0$ on S''.

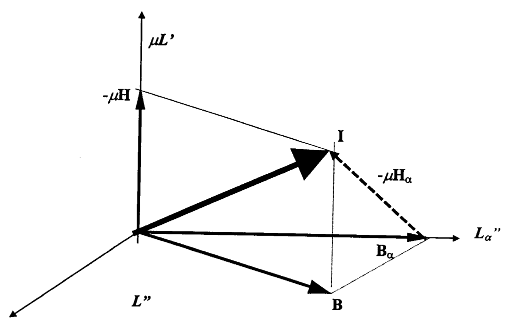

Fig. 3

In FEM the minimization of the functional (12) is done in finite subspace $\boldsymbol{L}''_\alpha$ of $\boldsymbol{L}''$ (fig. 3) or in subspace $\boldsymbol{L}'_\beta$ of $\mu\boldsymbol{L}'$. The component $\mathbf{B}_\alpha$ of the polarization $\mathbf{I}$ on the finite subspace $\boldsymbol{L}''_\alpha$ is the same as the component of $\mathbf{B}$ on this subspace. Therefore the FEM numerical computation of the approximate solution $\mathbf{B}_\alpha$ results as a composed function $\hat{\boldsymbol{P}}_\alpha \circ \hat{\boldsymbol{B}}$, where $\hat{\boldsymbol{P}}_\alpha(\mathbf{B}) = \mathbf{B}_\alpha$. The numerical scheme of the polarization method is

$$... \rightarrow \mathbf{I}^{(n-1)} \xrightarrow{\hat{\boldsymbol{B}}} \mathbf{B}^{(n)} \xrightarrow{\hat{\boldsymbol{P}}_\alpha} \mathbf{B}_\alpha^{(n)} \xrightarrow{\hat{\boldsymbol{G}}} \mathbf{I}^{(n)} \rightarrow ... \quad .$$

Because $\hat{\boldsymbol{P}}_\alpha$ and $\hat{\boldsymbol{B}}$ are nonexpansive, the above method leads to the Picard-Banach sequence of the contractive function $\hat{\boldsymbol{G}} \circ \hat{\boldsymbol{P}}_\alpha \circ \hat{\boldsymbol{B}}$.

Convergence Theorem of FEM and B-correction. *If the polarization $\boldsymbol{I}$ is corrected by $\boldsymbol{B}$ and the functional (12) is minimized in subspace $\boldsymbol{L}''_\alpha$, then the numerical approximation of the polarization method is convergent.*

The dual numerical scheme of the $\mathbf{H}$-corrected polarization method is

$$... \rightarrow \mathbf{M}^{(n-1)} \xrightarrow{\hat{\boldsymbol{H}}} \mathbf{H}^{(n)} \xrightarrow{\hat{\boldsymbol{P}}_\beta} \mathbf{H}_\beta^{(n)} \xrightarrow{\hat{\boldsymbol{G}}'} \mathbf{M}^{(n)} \rightarrow ... \, ,$$

where the component $\mathbf{H}_\beta$ of $\mathbf{H}$ on the finite subspace $\boldsymbol{L}'_\beta$ is $\mathbf{H}_\beta = \hat{\boldsymbol{P}}_\beta(\mathbf{H})$.

Convergence Theorem of FEM and H-correction. *If the magnetization $\boldsymbol{M}$ is corrected by $\boldsymbol{H}$ and the functional (12) is minimized in subspace $\boldsymbol{L}'_\beta$, then the numerical approximation of the polarization method is convergent.*

2 Eddy current problems

2.1 The Equations of the Quasistationary Electromagnetic Field

The conducting ferromagnetic bodies Ω_k move in the air Ω_0 of the domain Ω with the known speed $\mathbf{v}$. The electromagnetic field verifies the following equation:

(in Ω) $$\nabla\mathbf{B} = 0, \quad \mathbf{H} = \hat{\boldsymbol{F}}(\mathbf{B}),$$

(in Ω_k) $$rot\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \quad rot\mathbf{H} = \mathbf{J}, \quad \mathbf{J} = \sigma\mathbf{E},$$

(in Ω_0) $$rot\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \quad rot\mathbf{H} = \mathbf{J}_0,$$

where the current density is imposed. The equations are written in the local frame of the bodies and of the air.

On the body surfaces the passing conditions are:

$$J_n = 0, \quad B_n]_\Omega = B_n]_{\Omega_0}, \quad \mathbf{H}_t]_\Omega = \mathbf{H}_t]_{\Omega_0}.$$

The last relation results by neglecting the surface current and the Hertzian current.

The boundary conditions are the same as the stationary field boundary conditions.

The initial condition is $\mathbf{B}]_{t=0} = \mathbf{B}_i$.

The Uniqueness Theorem. *We have at the most one electromagnetic field which verifies the above equations, passing conditions, boundary conditions and initial condition.*

2.2 The Method of Polarization

Let Ω_F be the domain of the ferromagnetic bodies. Let $\boldsymbol{L}$ and $\boldsymbol{L}_\nu$ be the Hilbert spaces having inner products $< \mathbf{u}, \mathbf{v} >= \int_0^t \int_{D_F} \mathbf{u}\ \mathbf{v}\ dv\ d\tau$ and $< \mathbf{u}, \mathbf{v} >_\nu =< \mathbf{u}, \nu\mathbf{v} >$, respectively, where $\nu > 0$. So how we have shown at the stationary magnetic field, the Lipschitzian and uniform monotone constitutive relation $\mathbf{H} = \hat{\boldsymbol{F}}(\mathbf{B})$ is replace by

$$\mathbf{H} = \nu\mathbf{B} - \mathbf{M}, \tag{13}$$

where

$$\mathbf{M} = \nu\mathbf{B} - \hat{\boldsymbol{F}}(\mathbf{B}) = \hat{\boldsymbol{G}}(\mathbf{B}).$$

The iterative procedure for the treatment of the non-linearity consists in the following two steps, for arbitrary $\mathbf{M}^{(0)}(k \geq 1)$ [6], [7], [8]:

1) For a given magnetization $\mathbf{M}^{(k-1)}$, we compute the quasistationary field defined by equations:

$$\nabla \times \mathbf{E}^{(k)} = -\partial\mathbf{B}^{(k)}/\partial t \tag{14}$$

$$\nabla \times \mathbf{H}^{(k)} = \mathbf{J}^{(k)} \tag{15}$$

$$\mathbf{J}^{(k)} = \sigma\mathbf{E}^{(k)} + \mathbf{J}_0 \tag{16}$$

$$\mathbf{B}^{(k)} = \mu_0(\mathbf{H}^{(k)} + \mathbf{M}^{(k)}). \tag{17}$$

2) We correct $\mathbf{M}$, using function $\hat{\boldsymbol{G}}$:

$$\mathbf{M}^{(k)} = \hat{\boldsymbol{G}}(\mathbf{B}^{(k)}). \tag{18}$$

If

$$\varepsilon^{(k)} = ||\mathbf{M}^{(k)} - \mathbf{M}^{(k-1)}||_\mu$$

is small enough, we stop the iterations.

The function $\mathbf{M} \to \mathbf{B} = \hat{\boldsymbol{W}}(\mathbf{M})$, defined as the solution of equations (14) ,..., (17), is non-expansive, (Appendix A):

$$||\mathbf{B}' - \mathbf{B}''||_\nu = ||\, \hat{\boldsymbol{W}}(\mathbf{M}') - \hat{\boldsymbol{W}}(\mathbf{M}'')||_\nu \leq ||\mathbf{M}' - \mathbf{M}''||_\mu. \tag{19}$$

The iterative procedure leads to the fixed point $\mathbf{M}$ of the contractive function $\hat{\boldsymbol{G}} \circ \hat{\boldsymbol{W}} : \boldsymbol{L}_\mu \to \boldsymbol{L}_\mu$.

The linear convergence of Picard-Banach iteration can be efficiently accelerated by the overrelaxation methods described in §1.

2.3 Eddy-Current Integral Formulation for Computing the Linear Electromagnetic Field

If $\mathbf{B} = \nabla \times \mathbf{A}$ then the Faraday law is

$$\underset{\bullet}{\mathbf{E}} = -(\mathbf{A} + grad\, V) \tag{20}$$

where $\underset{\bullet}{\mathbf{E}}$ is the time integral of the electric field

$$\underset{\bullet}{\mathbf{E}} = \int_0^t \mathbf{E}(\tau) d\tau.$$

We choose in relation (13) $\nu = \nu_0 = \dfrac{1}{\mu_0}$ and from Ampere theorem we have

$$\nabla \times \nabla \times \mathbf{A} = \mu_0 (\mathbf{J} + \mathbf{J}_0 + \nabla \times \mathbf{M})$$

and

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_{\Omega_C} \frac{\mathbf{J}}{r} d\nu + \frac{\mu_0}{4\pi} \int_{\Omega_0} \frac{\mathbf{J}_0}{r} d\nu + \frac{\mu_0}{4\pi} \int_{\Omega_F} \frac{\mathbf{M} \times \mathbf{r}}{r^3} d\nu, \tag{21}$$

where Ω_C is the conducting domain, and Ω_F is the region of ferromagnetic materials. From relation (20) and (21) we obtain the integral equation of eddy current:

$$\rho \underset{\bullet}{\mathbf{J}} + \frac{\mu_0}{4\pi} \int_{\Omega_C} \frac{\mathbf{J}}{r} \mathbf{d}\nu + gradV = -\frac{\mu_0}{4\pi} \int_{\Omega_0} \frac{\mathbf{J}_0}{r} \mathbf{d}\nu - \frac{\mu_0}{4\pi} \int_{\Omega_F} \frac{\mathbf{M} \times \mathbf{r}}{r^3} \mathbf{d}\nu \tag{22}$$

Numerical Solving of the Eddy Current Equation

Condition $\nabla \mathbf{J} = 0$ is assured by introducing the electric potential T such as

$$\nabla \times \mathbf{T} = \mathbf{J}. \tag{23}$$

On the boundary $\partial \Omega_C$ we have the condition

$$\mathrm{n\ J} = \mathbf{n} \nabla \times \mathbf{T} = 0.$$

Obviously, relations (23) and (24) cannot define uniquely the potential $\mathbf{T}$. We must add a gauge condition for $\mathbf{T}$.

Let $\mathbf{N}_k$ be n_C functions defined on Ω_C and having $\nabla \times \mathbf{N}_k$ linear independent. We consider

$$\mathbf{T} = \sum_{k=1}^{n_C} \alpha_k(t) \mathbf{N}_k. \tag{25}$$

Therefore

$$\mathbf{J} = \sum_{k=1}^{n_C} \alpha_k(t) \nabla \times \mathbf{N}_k. \tag{26}$$

Using $\nabla \times \mathbf{N}_k$ as shape and weight function, the Galerkin method applied to equation (22) leads to the following linear system of ordinary differential equations

$$L\,\dot{I} + R\underset{\bullet}{I} = U + V,$$

where

$$I = (\alpha_1, \alpha_2, \alpha_3)^T,$$

$$R_{kj} = \int_{\Omega_C} \rho(\nabla \times \mathbf{N}_k)(\nabla \times \mathbf{N}_j)d\nu,$$

$$L_{kj} = \frac{\mu_0}{4\pi}\int_{\Omega_C}\int_{\Omega_C} \frac{1}{r}(\nabla \times \mathbf{N}_k)(\nabla \times \mathbf{N}_j)d\nu d\nu,$$

$$U_k = -\frac{\mu_0}{4\pi}\int_{\Omega_C} \nabla \times \mathbf{N}_k \int_{\Omega_C} \frac{\mathbf{J}_0}{r}d\nu d\nu,$$

$$V_k = -\frac{\mu_0}{4\pi}\int_{\Omega_C} (\nabla \times \mathbf{N}_k) \int_{\Omega_F} \frac{\mathbf{M} \times \mathbf{r}}{r^3}d\nu d\nu.$$

In the numerical approximation $\mathbf{M}$ is supposed to be uniform in every element ω_p of the ferromagnetic domain Ω_F. In this case above relation becomes

$$V_k = \sum_{p=1}^{n_F} \mathbf{a}_{kp}\mathbf{M}_p,$$

where n_F is the number of subdomains in ferromagnetic bodies and

$$\mathbf{a}_{kp} = \frac{\mu_0}{4\pi}\int_{\Omega_C} (\nabla \times \mathbf{N}_k) \times \int_{\omega_p} \frac{\mathbf{r}}{r^3}d\nu d\nu.$$

In the discrete edge element approximation, the uniqueness of $\mathbf{T}$ is assured by introducing tree-cotree decomposition of the graph made by the edges and the nodes of finite element mesh [9], [10], [11], [12], [13]. In particular, loop integrals of $\mathbf{T}$ uniquely define $\mathbf{J}$-fluxes across the faces linked with the loops but the edges values of $\mathbf{T}$ cannot be uniquely obtained from the loop integrals of $\mathbf{T}$. To do that, in acoord with basic circuit theory, we may define for the tree edges any value with the condition that on the cotree edges we enforce the values required to satisfy the analogue of Kirchhoff voltage law. In particular, the edges values of the tree can all be set to zero. Thus, for any values of $\mathbf{J}$-fluxes across the faces, edges values of $\mathbf{T}$ can always be found which are zero on the tree edges. Obviously, the set of these values is unique. This technique is the discrete analogue of the gauge condition $\mathbf{T}\ \mathbf{w} = 0$, where the field lines of $\mathbf{w}$ are given by the edges of the tree.

To impose the boundary condition (22), a boundary tree must be firstly defined. For simply connected domains, the edges values of $\mathbf{T}$ belonging to the boundary cotree must be zero.

For multiply connected domains we cannot annul all $\mathbf{T}$-integrals laying on the boundary. We have loops surrounding the surface $\partial\Omega_C$ and their $\mathbf{T}$-integrals cannot be enforced to be zero. It can be shown that in this case we must add a number of additional degress of freedom and associated shape functions $\mathbf{N}'$, equal to the genus n_B of the multiply connected domain [14],

$$\mathbf{T} = \sum_{j=1}^{n_I} \alpha_j(t)\mathbf{N}_j + \sum_{k=1}^{n_B} i_k(t)\mathbf{N}'_k,$$

where n_I is the number of the cotree edges inside the domain Ω_C and n_B is the number of the boundary sets of edges. In this case

$$I = (\alpha_1, \alpha_2, ..., \alpha_n, i_1, i_2, ..., i_{n_B}).$$

Computation of B.

Ferromagnetic bodies are divided in n_F subdomains. In each subdomain ω_i we take the average value of $\mathbf{B}$

$$\tilde{\mathbf{B}}_i = \frac{1}{\nu_i} \oint_{\partial\omega_i} (\mathbf{n} \times \mathbf{A}) dS,$$

where $\mathbf{A}$ is given by relation (21). Having assumed $\mathbf{M}$ constant in each element, we obtain

$$\tilde{\mathbf{B}}_i = \frac{1}{\nu_i} \sum_{k=1}^{n_C} \beta_{ik} \times \mathbf{J}_k - \frac{1}{\nu_i} \sum_{p=1}^{n_F} \overline{\overline{\gamma}}_{ip}\, \mathbf{M}_p + \mathbf{B}_0,$$

where

$$\beta_{ik} = \oint_{\partial\omega_i} \int_{\omega_k} \frac{\mathbf{n}_i}{r} d\nu_k dS_i$$

$$\overline{\overline{\gamma}}_{ip} = \oint_{\partial\omega_i} \oint_{\partial\omega_p} \frac{\mathbf{n}_p ; \mathbf{n}_i - \mathbf{n}_p \mathbf{n}_i}{r} dS_p dS_i.$$

Here ; is the dyadic product and $\mathbf{B}_0$ is given by the imposed current density. In each ferromagnetic subdomains ω_i, the magnetization $\mathbf{M}_i$ is corrected by average value $\tilde{\mathbf{B}}_i$. The numerical approximation of $\mathbf{B}$ by its average value is nonexpansive (Appendix B) and the convergence of the polarization method is ensured.

Appendix A. Nonexpansivity of the function $\hat{\mathbf{W}}$

Let $(\Delta\mathbf{B}, \Delta\mathbf{H}, \Delta\mathbf{E}, \Delta\mathbf{J})$ be the difference of the electromagnetic fields yields by $\mathbf{M}'$ and $\mathbf{M}''$. With $\underset{\bullet}{\mathbf{E}} = \int_0^t \mathbf{E} d\tau$ we have

$$\int_0^t \int_{\partial\Omega} (\Delta\mathbf{H} \times \Delta\underset{\bullet}{\mathbf{E}}) \mathbf{n} dS d\tau = \int_0^t \int_{\Omega} \Delta\mathbf{B} \Delta\mathbf{H} d\nu d\tau + \frac{1}{2} \int_{\Omega} \sigma \Delta\underset{\bullet}{\mathbf{E}}^2 d\nu,$$

where Ω is whole domain of the electromagnetic field. For boundary conditions correctly imposed, the left hand side of the above equality is zero and using relation (13) it results

$$\int_0^t \int_{\Omega} \Delta\mathbf{B}(\nu\Delta\mathbf{B} - \Delta\mathbf{M}) d\nu d\tau = ||\Delta\mathbf{B}||_{\nu}^2 - < \Delta\mathbf{B}, \mu\Delta\mathbf{M} >_{\nu} \leq 0,$$

and

$$||\Delta\mathbf{B}||_{\nu} \leq ||\Delta\mathbf{M}||_{\mu}.$$

Appendix B. Convergence of the numerical procedure.

The scheme of the numerical procedure is

$$... \to \mathbf{M}^{(k-1)} \xrightarrow{\hat{\boldsymbol{W}}} \mathbf{B}^{(k)} \xrightarrow{\hat{\boldsymbol{A}}} \tilde{\mathbf{B}}^{(k)} \xrightarrow{\hat{\boldsymbol{G}}} \mathbf{I}^{(k)} \to ... ,$$

where the functions $\hat{\boldsymbol{W}}$ and $\hat{\boldsymbol{G}}$ was above defined. In numerical procedures we need the average function $\hat{\boldsymbol{A}}$ which is nonexpansive. Indeed

$$\int_{\omega_k} \nu \tilde{\mathbf{B}}^2 dS = \sigma(\omega_k)\nu\Delta\tilde{\mathbf{B}}^2 = \sigma(\omega_k)\nu\frac{1}{\sigma^2(\omega_k)}\left(\int_{\omega_k} 1\Delta\mathbf{B}dS\right)^2 \leq$$

$$\leq \frac{1}{\sigma(\omega_k)}\nu \int_{\omega_k} 1dS \int_{\omega_k} \Delta\mathbf{B}^2 dS = \int_{\omega_k} \nu\Delta\mathbf{B}^2 dS,$$

where $\sigma(\omega_k)$ is the volume (or surface) of the subdomain ω_k. It follows that the composed function $\hat{\boldsymbol{G}} \circ \hat{\mathbf{A}} \circ \hat{\boldsymbol{W}}$ is a contraction and the iterative numerical procedure is convergent.

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