A Convex Polygon as a Discrete Plane Curve

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Abstract

In this paper we examine a convex polygon as a discrete substitute of a plane curve. We introduce a polygon with constant length of a diagonal as a counterpart of an oval with constant width. Moreover we define a convex polygon with constant perimeter of a special class circumscribed polygons.

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1 Introduction

In papers [1,3,4,6] applications of Fourier series to plane curves are presented . Plane curves examined in these papers are expressed by the following formulas:

$$(1.1) \qquad t \longmapsto z(t) = \int_0^t f(s)e^{is}ds \quad or \quad z(t) = \int_0^t k(s)f(s)e^{iK(s)}ds,$$

where f is a periodic function. The representation of considered curves is associated with the integral. Therefore we search for a geometrical domain associated with a finite sum instead of an integral. The geometrical domain is included in the class of all convex polygons in the plane. To define a representation of a convex polygon we imitate formula (1.1). Therefore we consider a periodic sequence instead of a periodic function. Next we introduce a discrete Fourier series for a periodic sequence as follows:

Let $x_1, x_2, x_3, \ldots$ be a periodic sequence of real numbers with the period n,i.e.:

$$x_{v+n} = x_v, v = 0, 1, 2, \ldots.$$

Then we apply a known trigonometrical interpolative polynomial

$$y(t) = a_0 + \sum_{j=1}^{n-1} [a_j \cos \frac{2\pi jt}{n} + b_j sin \frac{2\pi jt}{n}],$$

where

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$$a_0 = \frac{1}{n}\sum_{\mu=0}^{n-1} x_\mu, a_j = \frac{1}{n}\sum_{\mu=0}^{n-1} x_\mu \cos\frac{2\pi}{n} j\mu, \qquad b_j = \frac{1}{n}\sum_{\mu=0}^{n-1} x_\mu \sin\frac{2\pi}{n} j\mu.$$

The trigonometrical interpolative polynomial satisfies the condition:

$$y(v) = x_v, \quad v = 0, 1, \ldots n-1.$$

If we substitute instead the continuous variable $t \in (-\infty, +\infty)$ the dicsrete variable $v = 0, 1, 2, \ldots,$ then we obtain

$$x_v = a_0 + \sum_{j=1}^{n-1}[a_j \cos\frac{2\pi j v}{n} + b_j sin\frac{2\pi j v}{n}]. \tag{1.2}$$

In the sequel we call the formula (1.2) *a discrete Fourier series* of a periodic sequence $\{x_v\}$. We apply the discrete Fourier series to an n-polygon in the plane. The n-polygon is a polygon with n sides having the same interior angles equal to $2\pi - \frac{2\pi}{n}$, see [2,1]. In the paper n-polygon is "a discrete curve".

With reference to formula (1.1) we recall the following relations. There exists a strict correspondence between a property of curve (1.1) and a property of the function f. For example the following are known:

Lemma A.(see [5,1]). *A curve (1.1) is closed iff Fourier coefficients A_1, B_1 of f vanish, i.e.: $A_1 = B_1 = 0$.*

Lemma B.(see [6]). *If a closed curve represented by (1.1) is a curve with constant width ,then Fourier coefficients A_{2n}, B_{2n} of f vanish, i.e.:$A_{2n} = B_{2n} = 0$.*

Lemma C.(see [2,1]). *If a closed curve represented by (1.1) is a curve with constant perimeter of a circumscribed m-polygon, then the Fourier coefficients $A_{mj}, B_{mj}, \quad j = 1, 2, 3, \ldots$ vanish, i.e.:$A_{mj} = B_{mj} = 0$.*

Remark A.

If f is a constant function (different from zero), then equation (1.1) forms a circle. This means that in this case all Fourier coefficients of f vanish with the exception of A_0.

In a discrete domain, n-polygon is represented as the sum

$$k \longmapsto z_k = \sum_{v=0}^{k} x_v e^{i\frac{2\pi v}{n}}, \tag{1.3}$$

where $\{x_v\}$ is a sequence and $k = 0, 1, \ldots n-1$.

There exists a correspondence between a property of an n-polygon and a property of a sequence $\{x_v\}$. At the discrete domain a counterpart of a curve with constant width is a $2n$-polygon with constant diagonal (see p.7).

For $2n$-polygon with constant diagonal the following counterpart of the Barbier theorem is satisfied:

$$L = \pi\frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}} d, \tag{1.4}$$

where L denotes the perimeter of $2n$-polygon and d is the length of a constant diagonal.

A counterpart of a curve with constant perimeter of a circumscribed m-polygon is an mn-polygon with constant perimeter of a *b-circumscribed* m-polygon defined as follows:

Let P be a convex polygon with vertices $w_1, w_2, \dots, w_n, \quad n > 2$. To circumscribe a polygon with k sides $(3 \leq k \leq n)$ on polygon P, we arbitrary choose vertices

$$w_{i_1}, w_{i_2}, \dots, w_{i_k}.$$

Next we draw a straight line $l_{i_1}, l_{i_2}, \dots, l_{i_k}$ through vertices $w_{i_1}, w_{i_2}, \dots, w_{i_k}$. We consider only straight lines $l_{i_1}, l_{i_2}, \dots, l_{i_k}$ passing through the outside angles of polygon P. The point of intersection of successive straight lines $l_{i_s}, l_{i_{s+1}}, \quad s = 1, 2, \dots, k-1$ is a vertex of the circumscribed polygon. We call the circumscribed polygon *b-circumscribed* on polygon P if and only if all straight lines $l_{i_1}, l_{i_2}, \dots, l_{i_k}$ are bisectrices of the outside angles of the polygon P.

A property of a plane curve (represented by (1.1)) and "a discrete theory plane curve " are connected with the main result of the paper.

Perimeter $2\pi r$ of a circle with radius r can be obtained as the limit of perimeters of well-shaped regular polygons circumscribed on the circle. The above-mentioned idea and the counterpart of the Barbier theorem suggest that perimeter πd for an oval with constant width d can be obtained in the similarly way. To reach this aim we prove the following:

Theorem 1.1 *Every $2n$-polygon circumscribed on an oval with constant width δ is a $2n$-polygon with constant diagonal equal to*

$$\frac{\delta}{\cos \frac{\pi}{2n}}.$$

Theorem 1.2 *Every mn-polygon circumscribed on an oval with constant perimeter l of a circumscribed m-polygon is mn-polygon with constant perimeter $\frac{l}{\cos \frac{\pi}{mn}}$ of b-circumscribed m-polygon.*

2 Properties of a periodic sequence

A periodic sequence has some properties similar to a property of a periodic function. Therefore we recall (see [6,1]) those properties of a periodic function concerning of the discrete domain. Let f be 2π-periodic function having uniformly convergent Fourier series,

$$f(t) = \frac{1}{2}A_0 + \sum_{n=0}^{\infty}[A_n \cos(nt) + B_n \sin(nt)].$$

Theorem A. *Expression $f(t) + f(t+\pi)$ is a constant function iff Fourier coefficients $A_{2j}, B_{2j}, \quad j = 1, 2, \dots$ vanish.*

Theorem B. *Expression*

$$f(t)+f(t+\frac{2\pi}{m})+f(t+2\frac{2\pi}{m})+\ldots+f\left(t+(m-1)\frac{2\pi}{m}\right)$$

is a constant function iff Fourier coefficients $A_{mj}, B_{mj}, \quad j=1,2,\ldots$ *vanish.*

Theorems A and B have the following counterparts at the discrete domain:

Let x_v be a $2n$-periodic sequence,i.e.: $x_{v+2n}=x_v, \quad v=0,1,2,\ldots$. In this case sequence x_v has the discrete Fourier sum in the form

$$x_v=a_0+\sum_{j=1}^{2n-1}[a_j\cos\frac{\pi jv}{n}+b_j\sin\frac{\pi jv}{n}].$$

We prove two following lemmas:

Lemma 2.1 *If* $\{x_v\}$ *is a* $2n$*-periodic sequence, then the discrete Fourier sum of* x_v+x_{v+n} *has the form*

$$x_v+x_{v+n}=2a_0+2\sum_{l=1}^{n-1}[a_{2l}\cos\frac{2\pi lv}{n}+b_{2l}\sin\frac{2\pi lv}{n}].$$

Moreover

Lemma 2.2 *If*

$$x_v=a_0+\sum_{j=1}^{2n-1}[a_j\cos\frac{\pi jv}{n}+b_j\sin\frac{\pi jv}{n}]$$

is the discrete Fourier sum of $2n$*-periodic sequence* $\{x_v\}$*, then the sequence* $\{x_v+x_{v+n}\}$ *is a constant function if and only if* $a_{2l}=b_{2l}=0$ *for* $l=1,2,\ldots,n-1$.

Proof. To prove the lemma we verify that if $x_v+x_{v+n}=c, \quad v=0,1,\ldots$, then $a_{2l}=b_{2l}=0, \quad l=1,2,\ldots,n-1$. Indeed we have

$$a_{2l}=\frac{1}{2n}\sum_{\mu=0}^{2n-1}x_\mu\cos(2l\frac{\pi}{n}\mu)=$$

$$=\frac{1}{2n}(x_0+x_1\cos\frac{2\pi l}{n}+x_2\cos\frac{4\pi l}{n}+\ldots+x_n\cos\frac{2\pi l}{n}n+x_{n+1}\cos\frac{4\pi l}{n}(n+1)+\ldots)=$$

$$=\frac{1}{2n}((x_0+x_n)+(x_1+x_{1+n})\cos\frac{2\pi l}{n}+(x_2+_{2+n})\cos\frac{4\pi l}{n}+\ldots)=$$

$$=\frac{c}{2n}(1+\cos\frac{2\pi l}{n}+\cos\frac{4\pi l}{n}+\ldots)=0,$$

because

$$1+e^{i\frac{\pi}{n}}+\ldots+e^{i\frac{(n-1)\pi}{n}}=0$$

hence sum

$$\sum_{v=0}^{n-1}\cos\frac{v\pi}{n}=0$$

vanishes. Similarly we compute that $b_{2l=0}, l = 1, 2, \ldots, n-1$. So lemmas 1 and 2 are strict counterparts of the relation between Fourier coefficients of a 2π-periodic function $f(t)$ and the function $f(t) + f(t+\pi) \equiv C$. Theorem B has the following discrete counterpart:

Lemma 2.3 *If $\{x_v\}$ is a $m \cdot n$-periodic sequence and the discrete Fourier sum*

$$x_v = a_0 + \sum_{j=1}^{m \cdot n-1} [a_j \cos \frac{2\pi j v}{m \cdot n} + b_j \sin \frac{2\pi j v}{m \cdot n}]$$

is given, then the sequence $x_v + x_{v+n} + x_{v+2n} + \ldots + x_{v+(m-1)\cdot n}$ has the discrete Fourier sum in the form

$$x_v + x_{v+n} + x_{v+2n} + \ldots + x_{v+(m-1)\cdot n} =$$
$$= m a_0 + m \sum_{l=1}^{n-1} [a_{ml} \cos \frac{2\pi l v}{n} + b_{ml} \sin \frac{2\pi l v}{n}].$$

Moreover

Lemma 2.4 *If the discrete Fourier sum of $m \cdot n$-periodic sequence is given, then the sequence $x_v + x_{v+n} + x_{v+2n} + \ldots + x_{v+(m-1)\cdot n}$ is constant if and only if*

$$a_{ml} = b_{ml} = 0, \quad l = 1, 2, \ldots, n-1.$$

3 Convex polygons

Let x_v be an n-periodic sequence of real numbers . In this section we consider a polygon line represented by (1.3):

$$k \longmapsto z_k = \sum_{v=0}^{k} x_v e^{i \frac{2\pi v}{n}}.$$

The correspondence (1.3) describes the polygon line whenever $n > 2$. This means that for each fixed sequence x_v points z_k, $k = 0, 1, 2, \ldots$ are vertices of the polygon line in the Euclidean plane, see Fig.1.

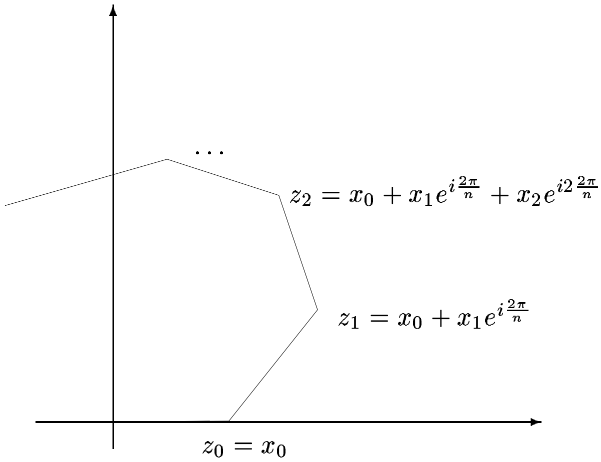


Fig.1 Polygon line z_k.

Obviously the value $x_k, \quad k = 0, 1, \ldots$ is equal to the distance between points z_{k-1} and z_k. Polygon line with vertices z_k becomes a convex polygon if we assume that:

(a) $\quad x_v > 0,$

(b) $\quad a_1 = b_1 = 0.$

Indeed, applying assumptions (a) and (b), we easy compute that

$$z_{k+n} = \sum_{v=0}^{k+n} x_v e^{i\frac{2\pi v}{n}} = z_k + \sum_{v=k+1}^{k+n} x_v e^{i\frac{2\pi v}{n}}.$$

Next we analyse the sum

$$S = \sum_{v=k+1}^{k+n} x_v e^{i\frac{2\pi v}{n}} = \sum_{v=k+1}^{k+n} x_v \cos\frac{2\pi v}{n} + i \sum_{v=k+1}^{k+n} x_v \sin\frac{2\pi v}{n}.$$

By the periodicity of sequence x_v we have:

$$\sum_{v=k+1}^{k+n} x_v \cos\frac{2\pi v}{n} =$$

$$= x_{k+1}\cos\frac{2\pi(k+1)}{n} + x_{k+2}\cos\frac{2\pi(k+2)}{n} + \ldots +$$

$$+ x_n \cos\frac{2\pi n}{n} + x_{n+1}\cos\frac{2\pi(n+1)}{n} + \ldots + x_{k+n}\cos\frac{2\pi(k+n)}{n} =$$

$$= x_0 + x_1\cos\frac{2\pi}{n} + \ldots x_k\cos\frac{2\pi k}{n} + x_{k+1}\cos\frac{2\pi(k+1)}{n} + \ldots + x_{n-1}\cos\frac{2\pi(n-1)}{n} = na_1.$$

Similarly we compute that

$$\sum_{v=k+1}^{k+n} x_v \sin\frac{2\pi v}{n} = nb_1.$$

Finally we obtain

(3.5) $$z_{k+n} = z_k + n(a_1 + ib_1).$$

Equality (3.5) implies the following:

Lemma 3.1 *The polygon line with vertices z_k is closed if and only if the coefficients a_1 and b_1 of the discrete Fourier sum of sequence x_v vanish.*

The above-mentioned lemma is the strict counterpart of Lm.A.

A polygon line with vertices z_k becomes a well-shaped regular polygon with n sides whenever x_v is a constant sequence. Therefore, comparing Remark A with formula (1.3), we state that a well-shaped regular polygon is a discrete counterpart of a circle.

4 On n-polygons with constant diagonal

In this section we examine 2n-polygons represented by formula

(4.6) $$z_k = \sum_{v=0}^{k} x_v e^{i\frac{\pi v}{n}},$$

where $n \geq 2$ and the sequence x_v satisfies the following conditions:

(i) $x_v > 0,$

(ii) $x_{v+2n} = x_v \quad v = 0, 1, \ldots,$

(iii) $x_v + x_{v+n} = c, \quad v = 0, 1, \ldots ,$

(iv) $a_1 = b_1 = 0.$

Then the sector between points z_k and z_{k+n} is a diagonal of the polygon.
Such a diagonal is called $\frac{1}{2}$-*diagonal* of $2n$-polygon because the number all vertices of the polygon between z_k and z_{k+n} is equal the number all vertices of the polygon between z_{k+n} and z_k.

Now we prove the main result of the paper.

Theorem 4.1 *If vertices of a $2n$-polygon are determined by formula (4.6) and the sequence x_v satisfies conditions (i)-(iv), then all $\frac{1}{2}$-diagonals of the polygon have the same length.*

Proof. We consider a $2n$-polygon represented by equation (4.6). Let p_k denote a $\frac{1}{2}$-diagonal of the polygon. We put

$$\mathsf{T}_k = e^{i\frac{\pi k}{n}} e^{i\frac{\pi}{2n}} = e^{i\frac{(2k+1)\pi}{2n}} \quad \text{and} \quad \mathsf{N}_k = i\mathsf{T}_k.$$

The vectors $\mathsf{T}_k, \mathsf{N}_k$ are parallel to bisectrices of outside and inside angle at vertices z_k, $k = 0, 1, \dots$ of $2n$-polygon, respectively. Vectors $\mathsf{T}_k, \mathsf{N}_k$ establish a basis for $k = 0, 1, \dots$. Therefore

$$p_k = D_k \mathsf{T}_k - d_k \mathsf{N}_k,$$

where $d_k = [p_k, \mathsf{T}_k]$ and $D_k = [p_k, \mathsf{N}_k]$ are determinants of two pairs of vectors p_k, T_k and p_k, N_k, respectively. Now we obtain the discrete Fourier sum of d_k and D_k. First

$$\begin{aligned} d_k = [p_k, \mathsf{T}_k] = \sum_{v=k+1}^{k+n} x_v [e^{i\frac{\pi v}{n}}, e^{i\frac{(2k+1)\pi}{2n}}] = \\ = \sum_{v=k+1}^{k+n} x_v \sin(\frac{(2k+1)\pi}{2n} - \frac{\pi v}{n}) = \\ = \sum_{v=k+1}^{k+n} x_v \sin\frac{(2k-2v+1)\pi}{2n}. \end{aligned}$$

On the other hand we have the following formula

$$x_v = a_0 + \sum_{l=1}^{n-1} [a_{2l+1} \cos\frac{(2l+1)v\pi}{n} + b_{2l+1} \sin\frac{(2l+1)v\pi}{n}]$$

and we insert it into the formula d_k . Hence we obtain

$$\begin{aligned} d_k = a_0 \sum_{v=k+1}^{k+n} \sin\frac{(2k-2v+1)\pi}{2n} + \\ + \sum_{l=1}^{n-1} \left(a_{2l+1} \sum_{v=k+1}^{k+n} \cos\frac{(2l+1)v\pi}{n} \sin\frac{(2k-2v+1)\pi}{2n} + \right. \\ \left. + b_{2l+1} \sum_{v=k+1}^{k+n} \sin\frac{(2l+1)v\pi}{n} \sin\frac{(2k-2v+1)\pi}{2n} \right) = \\ = -\frac{a_0}{\sin\frac{\pi}{2n}}. \end{aligned}$$

Similarly we compute that $D_k = 0$. So we have

$$p_k = \frac{a_0}{\sin\frac{\pi}{2n}} \mathsf{T}_k.$$

This means that every $\frac{1}{2}$-diagonal of the $2n$-polygon has the same length equal to

$$|p_k| = \frac{a_0}{\sin\frac{\pi}{2n}}.$$

□

Let L be a perimeter of $2n$-polygon with constant diagonal and let d be a length of an $\frac{1}{2}$-diagonal. Then we have the following counterpart of the Barbier's formula:

$$L = \pi \frac{\sin \frac{\pi}{2n}}{\frac{\pi}{2n}} d \tag{4.7}$$

because

$$a_0 = \frac{1}{2n}(x_0 + x_1 + \ldots + x_{2n-1}) = \frac{1}{2n} L.$$

The relation $D_k = 0$ means that

Corollary 4.1 *Each $\frac{1}{2}$-diagonal of $2n$-polygon determined by a sequence x_v satisfying conditions (i)-(iv) is a bisectrix of an inside angle of the polygon.*

To define a $2n$-polygon with constant diagonal by formula (4.6) we need a sequence x_v satisfying conditions (i)-(iv). This means that we solve the linear system $n+1$ equations with $2n$ unknown quantities. We solve these equations for $n = 3, 4$. The sequence $x_0 = d, x_1 = m - d, x_2 = d, x_3 = m - d, x_4 = d, x_5 = m - d$ determines 6-polygon with constant diagonal by formula (4.6) for fixed numbers $d > 0$ and $m - d > 0$. Then

$$\begin{aligned}
z_0 &= d, \\
z_1 &= d + (m-d)e^{i\frac{\pi}{3}}, \\
z_2 &= z_1 + de^{i2\frac{\pi}{3}}, \\
z_3 &= z_2 + (m-d)e^{i\pi}, \\
z_4 &= z_3 + de^{i4\frac{\pi}{3}}, \\
z_5 &= z_4 + (m-d)e^{i5\frac{\pi}{3}}.
\end{aligned}$$

To define a 8-polygon with constant diagonal we apply the following sequence.

$$\begin{aligned}
x_0 &= a, \\
x_1 &= \tfrac{1}{2}(a+e) + e\tfrac{1}{\sqrt{2}} - c\tfrac{1}{\sqrt{2}}, \\
x_2 &= c, \\
x_3 &= \tfrac{1}{2}(a+e) + a\tfrac{1}{\sqrt{2}} - c\tfrac{1}{\sqrt{2}}, \\
x_4 &= e, \\
x_5 &= \tfrac{1}{2}(a+e) - e\tfrac{1}{\sqrt{2}} + c\tfrac{1}{\sqrt{2}}, \\
x_6 &= a + e - c, \\
x_7 &= \tfrac{1}{2}(a+e) - a\tfrac{1}{\sqrt{2}} + c\tfrac{1}{\sqrt{2}},
\end{aligned}$$

where a, c, e are arbitrary numbers changed such that $x_v > 0, v = 0, 1, \ldots, 7$.

Now we present a simple method of defining a $2n$-polygon with constant diagonal. Let f be a 2π-periodic real positive function such that

$$f(t) + f(t+\pi) = C \quad for \quad all \quad t.$$

Then the Fourier series of f has the form (see[8]):

$$f(t) = \frac{1}{2}A_0 + \sum_{j=0}^{\infty}[A_{2j+1}\cos((2j+1)t) + B_{2j+1}\sin((2j+1)t)].$$

Moreover we assume that the series is uniformly convergent to f. Keeping the above-mentioned notions we prove the following lemma:

Lemma 4.1 *For each fixed t the sequence*

$$x_v = f(t + v\frac{\pi}{n}), \quad v = 0, 1, \dots$$

determines the $2n$-polygon with constant diagonal by formula (4.6) .

Proof. Conditions (i) and (ii) are obvious. We verify the remaining relations.
(iii)

$$x_v + x_{v+n} = f(t + v\frac{\pi}{n}) + f(t + (v+n)\frac{\pi}{n}) = C,$$

(iv)

$$2na_1 = \sum_{v=0}^{n-1} f(t + \frac{v\pi}{n}) \cos\frac{v\pi}{n} =$$

$$= \sum_{v=0}^{n-1} \left(\frac{1}{2}A_0 + \sum_{j=1}^{\infty} [A_{2j+1} \cos((2j+1)(t + \frac{v\pi}{n})) + B_{2j+1} \sin((2j+1)(t + \frac{v\pi}{n})) \right) \cos\frac{v\pi}{n} =$$

$$= \frac{1}{2}A_0 \sum_{v=0}^{n-1} \cos\frac{v\pi}{n} + \sum_{j=1}^{\infty} \left(A_{2j+1} \sum_{v=0}^{n-1} \cos((2j+1)(t + \frac{v\pi}{n})) \cos\frac{v\pi}{n} + \right.$$

$$\left. + B_{2j+1} \sum_{v=0}^{n-1} \sin((2j+1)(t + \frac{v\pi}{n})) \cos\frac{v\pi}{n} \right) = 0.$$

To verify that the sums:

$$\sum_{v=0}^{n-1} \cos((2j+1)(t + \frac{v\pi}{n})) \cos\frac{v\pi}{n},$$

$$\sum_{v=0}^{n-1} \sin((2j+1)(t + \frac{v\pi}{n})) \cos\frac{v\pi}{n},$$

vanish, we apply simple trigonometric relations and we successively compute that

$$\sum_{v=0}^{n-1} \cos((2j+1)(t + v\frac{\pi}{n}) + \frac{v\pi}{n}) =$$

$$= \frac{\sin(\frac{(j+1)\pi}{n} - 2jt - t)}{2\sin\frac{(j+1)\pi}{n}} - \frac{\sin((j(\frac{1}{n} - 4) + \frac{1}{n} - 4)\pi - 2jt - t)}{2\sin((\frac{j}{n} + \frac{1}{n})\pi)} = 0$$

$$\sum_{v=0}^{n-1} \cos((2j+1)(t + v\frac{\pi}{n}) - \frac{v\pi}{n}) =$$

$$= \frac{\sin(\frac{j\pi}{n} - 2jt - t)}{2\sin(\frac{j\pi}{n})} - \frac{\sin(j(\frac{1}{n} - 4)\pi - 2jt - t)}{2\sin(\frac{j\pi}{n})\sin(\frac{v\pi}{n} + (2j+1)(t + \frac{v\pi}{n}))} = 0$$

$$\sum_{v=0}^{n-1} \sin((2j+1)(t+v\frac{\pi}{n})+\frac{v\pi}{n}) =$$

$$= \frac{\cos((\frac{j}{n}+\frac{1}{n})\pi - 2jt - t)}{2\sin((\frac{j}{n}+\frac{1}{n})\pi))} - \frac{\cos((j(\frac{1}{n}-4)+\frac{1}{n}-4)\pi - 2jt - t)}{2\sin((\frac{j}{n}+\frac{1}{n})\pi)} = 0$$

$$\sum_{v=0}^{n-1} \sin((2j+1)(t+v\frac{\pi}{n})-\frac{v\pi}{n}) =$$

$$= \frac{\cos(\frac{j\pi}{n} - 2jt - t)}{2\sin(\frac{j\pi}{n})} - \frac{\cos(j(\frac{1}{n}-4)\pi - 2jt - t)}{2\sin(\frac{j\pi}{n}j)} = 0.$$

To verify the above-mentioned equalities the computer programm "Derive" was used. Therefore $a_1 = 0$. Similarly we compute that $b_1 = 0$.

4.1 On $2n$-polygons circumscribed on an oval with constant width

In this subsection we prove the Th.1.1. i.e.:
All $2n$-polygons circumscribed on an oval with constant width δ are $2n$-polygons with constant diagonal equal to

$$\frac{\delta}{\cos\frac{\pi}{2n}}.$$

Proof. Let an oval in arc length parametrization be represented by equation

$$s \longmapsto z(s) = x(s) + iy(s).$$

We will denote a curvature,tangent and normal vectors at point $z(s)$ by $k(s), T_s, N_s$, respectively. Moreover we define $K(s) = \int_0^s k(t)dt$. Now we apply function $\varphi(s) = K^{-1}(K(s)+\frac{\pi}{n})$, where K^{-1} is an inverse function of K. Denoting $\varphi^v(s) = \underbrace{\varphi(\varphi\ldots\varphi}_{v-times}(s)\ldots))$ we easy observe that $\varphi^n = K^{-1}(K(s)+\pi)$. Obviously

$$|z(s) - z(\varphi^n(s))| = |z(\varphi(s) - z(\varphi^{n+1}(s))| = \delta, \quad see \quad Fig.2.$$

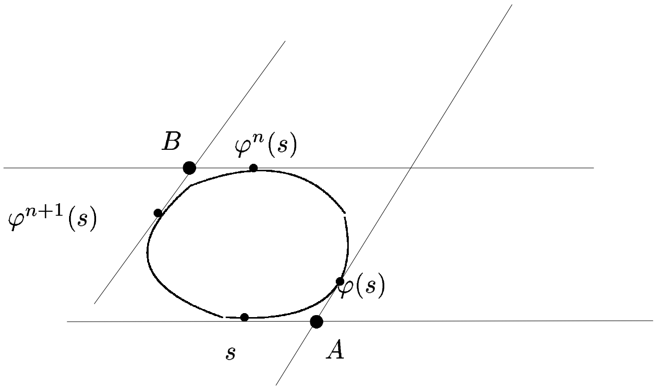


Fig.2. $2n$-polygon circumscribed on an oval.

Next we consider the following expressions

$$d_v = [x(\varphi^v(s)) - x(\varphi^{v+1}(s)), T_{\varphi^v(s)}], \quad v = 0, 1, \ldots, 2n-1,$$

$$D_v = [x(\varphi^v(s)) - x(\varphi^{v+1}(s)), N_{\varphi^v(s)}], \quad v = 0, 1, \ldots 2n-1.$$

Applying the same considerations as in [6 p.373] we solve the following system of equations:

$$z(\varphi^v(s)) + \xi_v T_{\varphi^v(s)} = z(\varphi^{v+1}(s)) + \eta_v T_{\varphi^{v+1}(s)}, \quad v = 0, 1, \ldots.$$

Hence we obtain the points

$$A \quad : z(s) + [-D_0 - d_0 \cot \frac{\pi}{n}]T_s,$$
$$B \quad : z(\varphi^n(s) + [-D_n - d_n \cot \frac{\pi}{n}](-T_s).$$

Now we compute the length of the diagonal AB:

$$|AB| = |z(s) - z(\varphi^n(s) + [(-D_n - D_0) + (-d_n - d_0) \cot \frac{\pi}{n}]T_s|,$$

but

$$z(s) - z(\varphi^n(s)) = -\delta N_s$$
$$D_n + D_0 = -\delta \sin \frac{\pi}{n}$$
$$d_n + d_0 = \delta(1 - \cos \frac{\pi}{n}).$$

Inserting these relations we express the length $|AB|$ as follows

$$|AB| = | - \delta N_s - \delta[\sin \frac{\pi}{n} + (1 - \cos \frac{\pi}{n}) \cot \frac{\pi}{n}]T_s| =$$

$$= \delta| - N_s + \tan\frac{\pi}{2n} \quad T_s| = \frac{\delta}{\cos\frac{\pi}{2n}}.$$

This implies that a perimeter of $2n$-polygons (circumscribed on the oval) tends to $\pi\delta$. Indeed we have

$$\pi\left(\frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}}\right)\frac{\delta}{\cos\frac{\pi}{2n}} \longmapsto \quad \pi\delta \quad as \quad n \longmapsto \infty.$$

5 On m-polygons circumscribed on an $m \cdot n$-polygon

The results of this section are discrete counterparts of theorems presented in papers [1,3,4].

In the section we examine $m \cdot n$-polygons represented by the formula

$$(5.8) \qquad z_k = \sum_{v=0}^{k} x_v e^{i\frac{2\pi v}{m \cdot n}},$$

where $n \geq 2, \quad m \geq 3$ and a sequence x_v satisfies the following conditions:

$1^o \quad x_v > 0,$

$2^o \quad x_{v+m\cdot n} = x_v, \quad v = 0, 1, \ldots,$

$3^o \quad x_v + x_{v+n} + x_{v+2n} + \ldots + x_{v+(m-1)n} = c, \quad v = 0, 1, \ldots,$

$4^o \quad a_1 = b_1 = 0.$

We consider an m-polygon b-circumscribed on an $m \cdot n$-polygon. For a fixed integer k we draw bisectrices of outside angles in vertices

$$z_k, z_{k+n}, z_{k+2n}, \ldots, z_{k+(m-1)n}.$$

This m-polygon b-circumscribed on $m \cdot n$-polygon has vertices defined as a point of intersection of two successive bisectrices passing through vertices $z_{k+jn}, z_{k+(j+1)n}$. Keeping notions as before we show

Theorem 5.1 *All m-polygons b-circumscribed on an $m \cdot n$-polygon have the same perimeter whenever the sequence x_v satisfies conditions $1^o - 4^o$.*

Proof. To prove the theorem we denote vectors parallel to bisectrices of inside and outside angels at vertex z_k of the polygon by N_k and T_k,respectively. Applying Fig.3 we easy observe that

$$\mathsf{T}_k = e^{i\frac{2\pi k}{m\cdot n}} e^{i\frac{\pi}{m\cdot n}} = e^{i\frac{(2k+1)\pi}{m\cdot n}} \quad \text{and} \quad \mathsf{N}_k = iT_k.$$

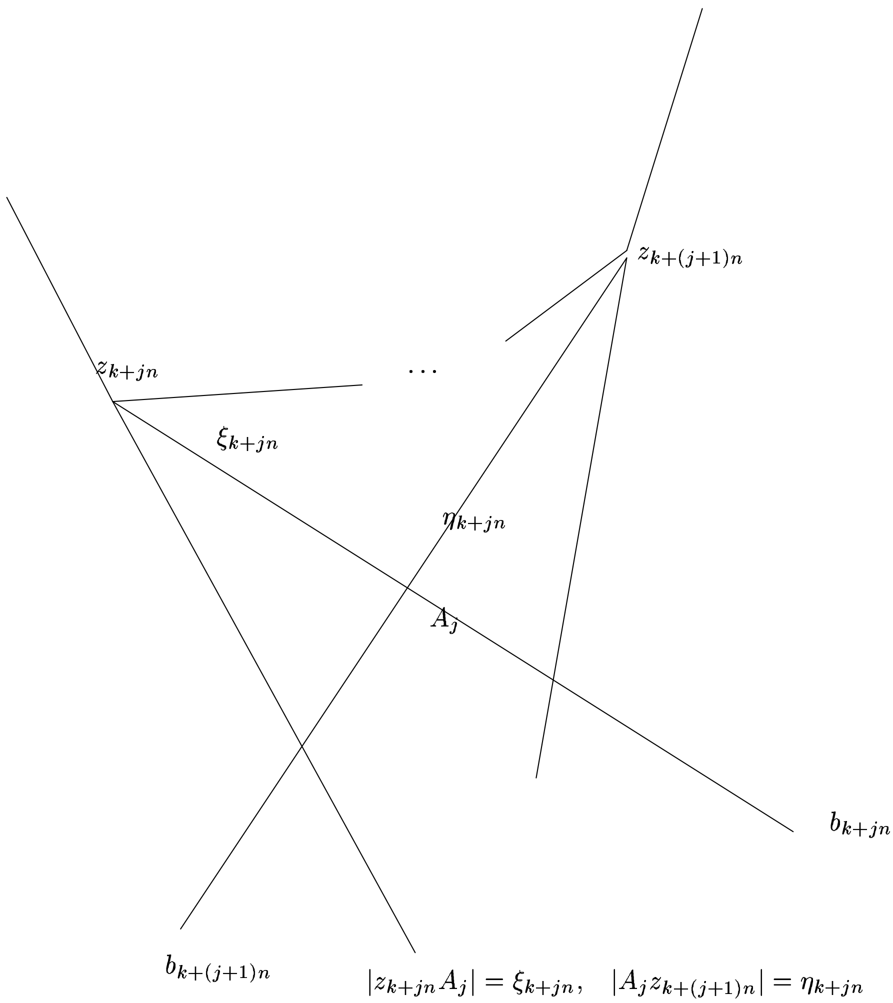
To compute the perimeter of b-circumscribed m-polygon we use the following vectors

$$\mathsf{T}_{k+jn} = \varepsilon^j \mathsf{T}_k, \quad \text{and} \quad \mathsf{N}_{k+jn} = i\mathsf{T}_{k+jn}, j = 1, 2, \ldots m-1,$$

where $\varepsilon = \cos\frac{2\pi}{m} + i\sin\frac{2\pi}{m}$. Next we solve the following system of equations

$$z_{k+jn} + \xi_{k+jn}\varepsilon^j \mathsf{T}_k = z_{k+(j+1)n} + \eta_{k+jn}\varepsilon^{j+1}\mathsf{T}_k, \quad j = 0, 1, 2, \ldots m-1.$$

The geometrical meaning of the above-mentioned equations is illustrated in Fig.3



$|z_{k+jn}A_j| = \xi_{k+jn}, \quad |A_j z_{k+(j+1)n}| = \eta_{k+jn}$

Fig.3.

Solving these equations we obtain

$$\eta_{k+jn} = \frac{[z_{k+(j+1)n} - z_{k+jn}, \varepsilon^j \mathsf{T}_k]}{\sin \frac{2\pi}{m}}, \quad \xi_{k+jn} = \frac{[z_{k+(j+1)n} - z_{k+jn}, \varepsilon^{j+1} \mathsf{T}_k]}{\sin \frac{2\pi}{m}}.$$

Let L_k denote a perimeter of b-circumscribed m-polygon . The Fig.3 suggests that

$$L_k = \sum_{v}^{m-1} (\xi_{k+vn} - \eta_{k+vn}).$$

Inserting all formulas on ξ_p, η_q we obtain

$$L_k = \frac{1}{\sin \frac{2\pi}{m}} \sum_{v=0}^{m-1} ([z_{k+(v+1)n} - z_{k+vn}, \varepsilon^{v+1} \mathsf{T}_k] - [z_{k+(v+1)n} - z_{k+vn}, \varepsilon^{v} \mathsf{T}_k]) =$$

$$= \frac{1}{\sin\frac{2\pi}{m}} \sum_{v=0}^{m-1} ([\varepsilon^{m-v-1}(z_{k+(v+1)n} - z_{k+vn}), \mathsf{T}_k] - [\varepsilon^{m-v}(z_{k+(v+1)n} - z_{k+vn}), \mathsf{T}_k]) =$$

$$= \frac{1}{\sin\frac{2\pi}{m}}[(2 - \varepsilon - \frac{1}{\varepsilon}) \sum_{v=0}^{m-1} \varepsilon^{m-v} z_{k+vn}, \mathsf{T}_k],$$

but $2 - \varepsilon - \frac{1}{\varepsilon} = 2 - 2Re(\varepsilon) = 4\sin^2\frac{\pi}{m}$. Hence putting

$$p_k = \sum_{v=0}^{m-1} \varepsilon^{m-v} z_{k+vn}$$

we express L_k as follows

$$L_k = 2[p_k, T_k] \tan\frac{\pi}{m}.$$

Now introducing notions $d_k = [p_k, T_k]$ and $D_k = [p_k, N_k]$ we express vector p_k as follows

$$p_k = D_k T_k - d_k N_k.$$

In conclusion we show that discrete Fourier sums of sequences d_k and D_k have the form

$$d_k = [p_k, \mathsf{T}_k] = \left[\sum_{j=0}^{m-1} \varepsilon^{m-j} \sum_{v=0}^{k+jn} x_v e^{i\frac{2\pi v}{mn}}, e^{i\frac{(2k+1)\pi}{mn}}\right] =$$

$$= \sum_{j=0}^{m-1} \sum_{v=0}^{k+jn} x_v \sin\frac{(2k+1-2v+2jn)\pi}{mn} =$$

$$= \sum_{j=0}^{m-1} \sum_{v=0}^{k+jn} \left(a_0 + \sum_{s=1}^{m-1} \sum_{l=1}^{n-1} \left[a_{ml+s} \cos\frac{2\pi(ml+s)v}{mn} + \right.\right.$$

$$\left.\left. + b_{ml+s} \sin\frac{2\pi(ml+s)v}{mn}\right]\right) \sin\frac{(2k-2v+1+jn)\pi}{mn} =$$

$$= a_0 \sum_{j=0}^{m-1} \sum_{v=0}^{k+jn} \sin\frac{(2k-2v+2nj+1)\pi}{mn} +$$

$$+ \sum_{s=1}^{m-1} \sum_{l=1}^{n-1} a_{ml+s} \sum_{j=0}^{m-1} \sum_{v=0}^{k+jn} \cos\frac{2\pi(ml+s)v}{mn} \sin\frac{(2k-2v+2nj+1)\pi}{mn} +$$

$$+ \sum_{s=1}^{m-1} \sum_{l=1}^{n-1} b_{ml+s} \sum_{j=0}^{m-1} \sum_{v=0}^{k+jn} \sin\frac{2\pi(ml+s)v}{mn} \sin\frac{(2k-2v+2nj+1)\pi}{mn} = \frac{ma_0}{2\sin\frac{\pi}{mn}}.$$

Similarly we compute that $D_k = 0$. Hence we finally obtain

$$L_k = 2[p_k, \mathsf{T}_k] \tan\frac{\pi}{m} = 2\frac{ma_0}{2\sin\frac{\pi}{mn}} \tan\frac{\pi}{m}.$$

This means that all b-circumscribed m-polygons have the same perimeter independent of index k.

□

Moreover we observe that every vector $p_k = -d_k \mathsf{N}_k$ has the same length equal to $\frac{ma_0}{2\sin\frac{\pi}{mn}}$. We put $d = |p_k|$, then we express perimeter L of mn-polygon with a constant perimeter of b-circumscribed m-polygon by the following relation

$$L = \frac{2\pi}{m}\left(\frac{\sin\frac{\pi}{mn}}{\frac{\pi}{mn}}\right)d, \tag{5.9}$$

because $mnL = x_0 + x_1 + \ldots + x_{mn-1}$. Tending to infinity with n we obtain

$$L = \frac{2\pi d}{m}.$$

Formula (5.9) is a discrete counterpart of Th.1.[3] and becomes formula (1.4) for $m = 2$. To define an mn-polygon with a constant perimeter of a b-circumscribed m-polygon we apply a 2π-periodic positive function f such that

$$\sum_{v=0}^{m-1} f(t + v\frac{2\pi}{m}) \equiv C.$$

We assume that function f has uniformly convergent Fourier series and this series has a form

$$f(t) = \frac{1}{2}A_0 + \sum_{l=1}^{\infty}[A_l\cos(lt) + B_l\sin(lt)],$$

where $A_{mj} = B_{mj} = 0, \quad j = 1, 2, \ldots$,see [4,1]. Putting

$$x_v = f(t + \frac{2\pi v}{mn}), \quad v = 0, 1, \ldots$$

we obtain (for a fixed variable t) a sequence which satisfies conditions 1^o–4^o. Therefore a mn-polygon represented by equation

$$z_k = \sum_{v=0}^{k} x_v e^{i\frac{2\pi v}{mn}}$$

has the constant perimeter of each m-polygon b-circumscribed on it.

5.1 On an oval with constant perimeter of a circumscribed m-polygon and on an mn-polygon

In subsection 4.1 we proved that every $2n$-polygon circumscribed on an oval with constant width d is the polygon with a constant diagonal equal to $\frac{d}{\cos\frac{\pi}{2n}}$. In this subsection we prove Theorem 1.2. i.e.:

All m-polygons b-circumscribed on an mn-polygon which is circumscribed on an oval with a constant perimeter of a circumscribed m-polygon have the same perimeter equal to

$$\frac{l}{\cos\frac{\pi}{mn}},$$

where l denotes the length of m-polygon circumscribed on this oval.

Proof. We keep notion as before. Let $z(s)$ be an oval with a constant perimeter of a circumscribed m-polygon. Putting $\varphi(s) = K^{-1}(K(s) + \frac{2\pi}{mn})$ we easy observe that $\varphi^{mn}(s) = s+L$ and that mn-polygon circumscribed on the oval is tangent (to the oval) at points $z(\varphi^v(s))$, $v = 0, 1, 2, \ldots$. Then vertices of an mn-polygon circumscribed on the oval are expressed as follows:

$$z(\varphi^v(s)) + \xi_v T_{\varphi^v(s))}, \quad v = 0, 1, 2, \ldots,$$

where

$$\xi = -D_v - d_v \cot\frac{2\pi}{mn}, \quad D_v = [z(\varphi^v(s)) - z(\varphi^{v+1}(s)), N_{\varphi^v(s))}]$$

and $d_v = [z(\varphi^v(s)) - z(\varphi^{v+1}(s)), T_{\varphi^v(s))}]$. Now we consider m-polygon b-circumscribed on mn-polygon, see Fig.4.

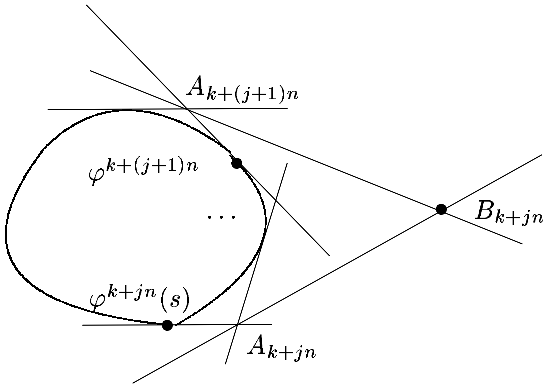


Fig.4. m-polygon b-circumscribed on mn-polygon which is circumscribed on the oval.

We denote by T_{k+jn} the tangent vector at point $z(\varphi^{k+jn}(s))$ for fixed k and $j = 0, 1, \ldots, m-1$. To compute the perimeter of m-polygon b-circumscribed on mn-polygon we solve the following system of equations (Fig.3 and Fig.4):

$$z(\varphi^{k+jn}(s)) + \xi_{k+jn}T_{k+jn} + \xi^1_{k+jn}T_{k+jn}e^{i\frac{\pi}{mn}} =$$

$$= z(\varphi^{k+(j+1)n}(s)) + \xi_{k+(j+1)n}T_{k+(j+1)n} + \eta^1_{k+jn}T_{k+(j+1)n}e^{i\frac{\pi}{mn}},$$

where $j = 0, 1, \ldots m-1$.

Moreover the length of sectors $|z(\varphi^{k+jn}(s))A_{k+jn}|$ and $|z(\varphi^{k+(j+1)n}(s)A_{k+(j+1)n}|$ is denoted by ξ_{k+jn} and $\xi_{k+(j+1)n}$,respectively. The length of sectors $|a_{k+jn}B_{k+jn}|$ and $|B_{k+jn}A_{k+(j+1)n}|$ is denoted by ξ^1_{k+jn} and $\eta^1_{k+(j+1)n}$, respectively. Moreover vectors $T_{k+jn}e^{i\frac{\pi}{mn}}$ and $T_{k+(j+1)n}e^{i\frac{\pi}{mn}}$ are parallel to bisectrices of outside angles of mn-polygon. Obviously perimeter l_k of m-polygon b-circumscribed on mn-polygon is equal to

$$l_k = \sum_{j=0}^{m-1} (\xi^1_{k+jn} - \eta^1_{k+jn}).$$

At first we compute ξ^1_{k+jn}

$$\xi^1_{k+jn} = \frac{1}{-[T_{k+jn}e^{i\frac{\pi}{mn}}, T_{k+(j+1)n}e^{i\frac{\pi}{mn}}]} \Big([z(\varphi^{k+jn}(s)) - z(\varphi^{k+(j+1)n}(s)), T_{k+(j+1)n}e^{i\frac{\pi}{mn}}] +$$

$$+ [\xi_{k+jn}T_{k+jn} - \xi_{k+(j+1)n}T_{k+(j+1)n}, T_{k+(j+1)n}e^{i\frac{\pi}{mn}}] \Big) =$$

$$= \frac{1}{-\sin\frac{2\pi}{mn}} \Big([z(\varphi^{k+jn}(s)) - z(\varphi^{k+(j+1)n}(s)), T_{k+(j+1)n}e^{i\frac{\pi}{mn}}] +$$

$$+ \xi_{k+jn} \sin\frac{(2n+1)\pi}{mn} - \xi_{k+(j+1)n} \sin\frac{\pi}{mn} \Big)$$

and

$$\eta^1_{k+jn} = \frac{1}{\sin\frac{2\pi}{mn}} \Big([z(\varphi^{k+jn}(s)) - z(\varphi^{k+(j+1)n}(s)), T_{k+(j+1)n}e^{i\frac{\pi}{mn}}] +$$

$$+ \xi_{k+jn} \sin\frac{\pi}{mn} + \xi_{k+(j+1)n} \sin\frac{(2n-1)\pi}{mn} \Big).$$

Inserting relation $T_{k+jn} = \varepsilon^j T_k, \quad \varepsilon^m = 1, \varepsilon \neq 1$ we obtain

$$W = \sum_{j=0}^{m-1} ([z(\varphi^{k+jn}(s)) - z(\varphi^{k+(j+1)n}(s)), T_{k+(j+1)n}e^{i\frac{\pi}{mn}}] +$$

$$+ - [z(\varphi^{k+jn}(s)) - z(\varphi^{k+(j+1)n}(s)), T_{k+(j+1)n}e^{i\frac{\pi}{mn}}]) =$$

$$= [\bar{\varepsilon}p_k - p_k - p_k + \varepsilon p_k, T_k e^{i\frac{\pi}{mn}}],$$

where

$$p_k = \sum_{j=0}^{m-1} \varepsilon^{m-j} z(\varphi^{k+jn}(s)).$$

By (3.1)[4,p.374] $p_k = -dN_k$ (where d denotes n-width of the oval) we obtain

$$W = (\bar{\varepsilon} + \varepsilon - 2)[p_k, T_k e^{i\frac{\pi}{mn}}] = -4d\sin^2\frac{\pi}{m}\cos\frac{\pi}{mn}.$$

It is easy to observe that

$$\sum_{j=0}^{m-1} \xi_{k+jn} = \sum_{j=0}^{m-1} \xi_{k+(j+1)n}.$$

The sum

$$\sum_{j=0}^{m-1} \xi_{k+jn}$$

is equal to

$$\sum_{j=0}^{m-1} \xi_{k+jn} = -\sum_{j=0}^{m-1}([z(\varphi^{k+jn}(s)) - z(\varphi^{k+jn+1}(s)), N_{k+jn}]+$$

$$+[z(\varphi^{k+jn}(s)) - z(\varphi^{k+jn+1}(s)), T_{k+jn}] \cot \frac{2\pi}{mn}) =$$

$$= -\sum_{j=0}^{m-1}([\varepsilon^{m-j} z(\varphi^{k+jn}(s)) - \varepsilon^{m-j} z(\varphi^{k+jn+1}(s)), N_k]+$$

$$+[\varepsilon^{m-j} z(\varphi^{k+jn}(s)) - \varepsilon^{m-j} z(\varphi^{k+jn+1}(s)), T_k] \cot \frac{2\pi}{mn}) =$$

$$= -([p_k - p_{k+1}, N_k] + [p_k - p_{k+1}, T_k] \cot \frac{2\pi}{mn} =$$

$$= d(\sin \frac{2\pi}{mn} - \tan \frac{\pi}{mn} \cos \frac{2\pi}{mn}).$$

Finally we obtain perimeter l_k as the following expression

$$l_k = \frac{-4d \sin^2 \frac{\pi}{m} \cos \frac{\pi}{mn}}{-\sin \frac{2\pi}{m}} +$$

$$+\frac{1}{-\sin \frac{2\pi}{m}} \left(\sum_{j=0}^{m-1} \xi_{k+jn} (\sin \frac{(2n+1)\pi}{mn} - \sin \frac{\pi}{mn})+ \right.$$

$$\left. + \sum_{j=0}^{m-1} \xi_{k+(j+1)n} (-\sin \frac{\pi}{mn} - \sin \frac{(2n-1)}{mn}) \right) =$$

$$= 2d \tan \frac{\pi}{m} (\cos \frac{\pi}{mn} + \sin \frac{\pi}{mn} (\sin \frac{2\pi}{mn} - \tan \frac{\pi}{mn} \cos \frac{2\pi}{mn})) =$$

$$= 2d \tan \frac{\pi}{m} \frac{1}{\cos \frac{\pi}{mn}}.$$

But by [4,p.373] $l = 2d \tan \frac{\pi}{m}$ is equal to the perimeter of m-polygon circumscribed on an oval . Therefore

$$l_k = \frac{l}{\cos \frac{\pi}{mn}}.$$

This means that all m-polygons b-circumscribed on mn-polygon have the same perimeter. Moreover if n tends to infinity then perimeter l_k tends to the perimeter of m-polygon circumscribed on an oval. □

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