

Geometrical and Topological Structure of Magnetic Fields Generated by Rectilinear Wires

Cristian Dumitrescu

Abstract

Real phenomena on Plasma Physics or Controlled Thermonuclear Fusion, some problems on geophysics etc., determined important studies about the phase portraits of magnetic vector fields as well as of the dynamics of a particle along magnetic lines. These reasons imposed the study initiated by Sabba Ştefănescu on magnetic vector fields generated by piecewise rectilinear configurations wandered through electrical current. The ideas of Sabba Ştefănescu were reconsidered by the research team coordinated by Constantin Udrişte. As member of this team I realised the present paper.

Section 1 refines the theory of Sabba Ştefănescu on magnetic fields produced by piecewise rectilinear circuits. Section 2 describes the magnetic lines approximation via the Lie Transformation. Section 3 studies general geometrical and topological properties of the magnetic lines and surfaces. Section 4 and 5 refers to the magnetic traps and the Lorentz-Udrişte World-Force Law.

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Magnetic vector fields. Lines and magnetic surfaces. The energy of a magnetic vector field

This section describes the magnetic vector fields obtained from usual transformations (translations, rotations, isometries) of a given magnetic vector field and proves that any Biot-Savart-Laplace magnetic vector field is solenoidal and locally irrotational. Then it defines the magnetic lines and surfaces, the energy of a magnetic vector field and characterizes the critical points of the energy of a stationary magnetic vector field.

Biot-Savart-Laplace vector fields

Let $c : I(\subset \mathbf{R}) \rightarrow \mathbf{R}^3$ be a \mathbf{C}^1 curve modelling the electric wire $\gamma = \text{Im}c \subset \mathbf{R}^3$, whose transversal section is negligible, wandered by continuous current of constant intensity.

Let $\vec{v} = \frac{I}{\|\dot{c}\|} \dot{c}$ be the unit vector field, tangent to the curve (current density).

If $P = c(t) \in \gamma$ is arbitrary on γ and $M \in \mathbf{R}^3 \setminus \gamma$ is a point which does not belong to γ , then the vector field

$$(1) \quad \vec{F}_\gamma(M) = \int_\gamma \frac{\vec{v} \times \overrightarrow{PM}}{PM^3} d\tau_P$$

is called Biot-Savart-Laplace vector field (here $PM = \|\overrightarrow{PM}\|$, and $d\tau_P = \|\dot{c}(t)\| dt$).

If γ is an unbounded curve at its both ends, or a closed curve, then the vector field \vec{F}_γ and the magnetic field \vec{H}_γ created around the wire γ are connected by $\vec{H}_\gamma = \frac{1}{4} \pi \vec{F}_\gamma$.

Proposition. *If $\mathcal{T} : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is a translation, $\mathcal{R} : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ a rotation and $\mathcal{I} = \mathcal{T} \circ \mathcal{R}$ an isometry, then we have the following relations:*

- a) $\vec{F}_{\mathcal{T}\gamma}(\mathcal{T}M) = \vec{F}_\gamma(M)$, $(\forall) M \in \mathbf{R}^3 \setminus \gamma$;
- b) $\vec{F}_{\mathcal{R}\gamma}(\mathcal{R}M) = \mathcal{R}\vec{F}_\gamma(M)$, $(\forall) M \in \mathbf{R}^3 \setminus \gamma$;
- c) $\vec{F}_{\mathcal{I}\gamma}(\mathcal{I}M) = \mathcal{R}\vec{F}_\gamma(M)$, $(\forall) M \in \mathbf{R}^3 \setminus \gamma$.

□

Let $D \subset \mathbf{R}^3$ be an open and connected set with piecewise smooth boundary and \vec{J} be a \mathbf{C}^∞ vector field on $\bar{D} = D \cup \partial D$. We call Biot-Savart-Laplace vector field on \mathbf{R}^3 the field $\vec{F}(M)$

$$(2) \quad \vec{F}(M) = \int_D \frac{\vec{J}(P) \times \overrightarrow{PM}}{PM^3} dv_P,$$

where $P \in \bar{D}$, $M \in \mathbf{R}^3$ and $PM = \|\overrightarrow{PM}\|$.

Since the measure of ∂D is zero, the integral defining $\vec{F}(M)$, $M \in \bar{D}$ is an improper integral of the first type (both of the first and of the second type) if the domain D is bounded (unbounded).

If \bar{D} is wandered by a current of intensity $\vec{J}(P)$, then the magnetic field \vec{H} generated in \mathbf{R}^3 by this current can be calculated by $\vec{H} = \frac{1}{4\pi} \vec{F}$.

Moreover, we remark that on $\mathbf{R}^3 \setminus \partial D$ the vector field $\vec{H}(M)$ is of class \mathbf{C}^∞ and on ∂D the field is continuous.

Proposition. *The B.S.L. magnetic vector field is solenoidal since it admits the potential vector*

$$\vec{A}(M) = \frac{1}{4\pi} \int_D \frac{\vec{J}(P)}{PM} dv_P.$$

Remarks.

a) Suppose ∂D is of class \mathbf{C}^1 and bounded, and \vec{J} is solenoidal with ∂D field surface (that is $(\vec{n}(P), \vec{J}(P)) = 0$, where $\vec{n}(P)$ is the normal unit vector field on ∂D).

Then the field $\vec{A}(M)$ is solenoidal since the following relations hold,

$$\operatorname{div} \vec{A}(M) = -\frac{1}{4\pi} \int_D \frac{(\vec{n}(P), \vec{J}(P))}{PM} dv_P + \frac{1}{4\pi} \int_D \frac{\operatorname{div} \vec{J}(P)}{PM} dv_P = 0.$$

Also, when $\operatorname{div} \vec{A}(M) = 0$ the following relations hold,

$$\operatorname{rot} \vec{H}(M) = \nabla_M \operatorname{div} \vec{A}(M) - \Delta_M \vec{A}(M) = -\Delta_M \vec{A}(M).$$

According to these relations, we have

$$\operatorname{rot} \vec{H}(M) = \begin{cases} \vec{0}, & (\forall) M \in \mathbf{R}^3 \setminus \bar{D} \\ \vec{J}(M), & (\forall) M \in \bar{D}; \end{cases}$$

so the B.S.L. magnetic vector field is not irrotational but its restriction on $\mathbf{R}^3 \setminus \bar{D}$ is irrotational, hence admits a local scalar potential.

b) When the vector field is constant on \bar{D} , the magnetic vector field \vec{H} is biscalar.

Lines and magnetic surfaces.

The energy of a magnetic vector field

Let $\vec{H} = H_x \vec{i} + H_y \vec{j} + H_z \vec{k}$ be magnetic vector field on \mathbf{R}^3 and $M(x, y, z) \in \mathbf{R}^3$ an arbitrary point with $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ its position vector. We call *magnetic line* starting from $M_0(x_0, y_0, z_0)$, at the moment $t = 0$, any oriented curve $\vec{r} = \vec{r}(t)$, $t \in (-\varepsilon, \varepsilon) \subset \mathbf{R}$ solution of the Cauchy problem

$$\frac{d\vec{r}}{dt} = \vec{H}(\vec{r}), \quad \vec{r}(0) = \vec{r} = x_0 \vec{i} + y_0 \vec{j} + z_0 \vec{k}.$$

We call *magnetic surface* passing through the curve $\beta : (a, b) \rightarrow \mathbf{R}^3$, the surface Σ of equation $h(x, y, z) = c$, solution of the Cauchy problem

$$(\vec{H}, \nabla h) = 0; \quad h(\beta(u)) = h(\beta(0)), \quad (\forall) u \in (a, b).$$

Definition. If $\vec{H} = H_x \vec{i} + H_y \vec{j} + H_z \vec{k}$ is a magnetic field defined on an open set $D \subset \mathbf{R}^3$, then the function

$$f : D \rightarrow \mathbf{R}, \quad f = \frac{1}{2}(H_x^2 + H_y^2 + H_z^2),$$

is called *the energy* of \vec{H} ; the functions $\frac{1}{2}H_x^2$, $\frac{1}{2}H_y^2$, $\frac{1}{2}H_z^2$, are called *partial energies* of the magnetic vector field \vec{H} .

Proposition. If

$$\operatorname{div} \vec{H} = 0, \quad \operatorname{rot} \vec{H} = \vec{0},$$

then the critical points of the energy (critical points of partial energies) are minimum points or saddle points only. □

If we consider the derivative along a field line of H , we get

$$\frac{d^2 x}{dt^2} = \frac{\partial H_x}{\partial x} H_x + \frac{\partial H_x}{\partial y} H_y + \frac{\partial H_x}{\partial z} H_z = \frac{\partial f}{\partial x}$$

$$(*) \quad \begin{aligned} \frac{d^2 y}{dt^2} &= \frac{\partial H_y}{\partial x} H_x + \frac{\partial H_y}{\partial y} H_y + \frac{\partial H_y}{\partial z} H_z = \frac{\partial f}{\partial y} \\ \frac{d^2 z}{dt^2} &= \frac{\partial H_z}{\partial x} H_x + \frac{\partial H_z}{\partial y} H_y + \frac{\partial H_z}{\partial z} H_z = \frac{\partial f}{\partial z}. \end{aligned}$$

These equations show that any magnetic field line is solution of a conservative differential system with three degrees of freedom and potential $-f$; the motion of a particle along a magnetic line is contained within the motion of a particle in the force field of potential $-f$.

In the phase space $(x, y, z, u, v, w) \in \mathbf{R}^6$, the preceding system can be written as a Hamiltonian system as follows:

$$\begin{aligned} \frac{dx}{dt} &= -u, & \frac{dy}{dt} &= -v, & \frac{dz}{dt} &= -w, \\ \frac{du}{dt} &= -\frac{\partial f}{\partial x}, & \frac{dv}{dt} &= -\frac{\partial f}{\partial y}, & \frac{dw}{dt} &= -\frac{\partial f}{\partial z}. \end{aligned}$$

The Hamiltonian (mechanical energy) of this system has the form

$$\mathcal{H}(x, y, z, u, v, w) = \frac{1}{2}(u^2 + v^2 + w^2) - f(x, y, z),$$

and represents a first integral of our system (the total energy holds on). Therefore, if at the initial moment the total energy is \mathcal{H} , then the whole trajectory described by (*) is contained within the domain (potential well) characterised by the inequality

$$-f(x, y, z) \leq \mathcal{H}.$$

Lie Transformation of magnetic dynamical systems. Magnetic lines approximations using Lie Transformation

This section states and applies Lie Transformation to approximate the magnetic field line. As an illustration, we started with Biot-Savart-Laplace integral on an arch of parabola in \mathbf{R}^3 . We approximate the scalar function $\|\vec{PM}\|^{3/2}$ by a polynomial function up to the second degree and then we transform the system associated to the magnetic lines in order to apply the Lie Transformation. Using an appropriate choice of the principal functions, we determined the solution of the associated dynamical system.

Basic properties on Lie transformation theory for dynamical systems

Let be given the functions of class C^∞

$$\hat{f}_{i,n} : \mathbf{R}^M \rightarrow \mathbf{R}, \quad i = 1, 2, \dots, M; \quad n = 0, 1, 2, \dots, N.$$

We consider the dynamical system

$$\frac{dx_i}{dt} = \dot{x}_i = \sum_{n=0}^N \frac{\varepsilon^n}{n!} \hat{f}_{i,n}(x) \stackrel{not}{=} \hat{f}_i(x, \varepsilon), \quad i = 1, 2, \dots, M, \quad t \in I,$$

where $\varepsilon \in \mathbf{R}$ is parameter (*perturbation parameter*). If we consider the state vector

$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_M(t) \end{pmatrix}, \quad t \in I$$

and the functions

$$\hat{f}(x, \varepsilon) = \begin{pmatrix} \hat{f}_1(x, \varepsilon) \\ \vdots \\ \hat{f}_M(x, \varepsilon) \end{pmatrix}, \quad \hat{f}_n(x) = \begin{pmatrix} f_{1,n}(x) \\ \vdots \\ f_{M,n}(x) \end{pmatrix}, \quad n = 0, 1, 2, \dots, N,$$

then we can write this system in vectorial form

$$(1) \quad \frac{dx}{dt} = \dot{x} = \sum_{n=0}^N \frac{\varepsilon^n}{n!} \hat{f}_n(x) \stackrel{not}{=} \hat{f}(x, \varepsilon), \quad t \in I.$$

By analogy, let us consider the dynamical system

$$\frac{dy_i}{dt} = \dot{y}_i = \sum_{n=0}^N \frac{\varepsilon^n}{n!} \check{g}_{i,n}(y) \stackrel{not}{=} \check{g}_i(y, \varepsilon), \quad i = 1, 2, \dots, M, \quad t \in I \subset \mathbf{R},$$

written in vectorial form

$$(2) \quad \frac{dy}{dt} = \dot{y} = \sum_{n=0}^N \frac{\varepsilon^n}{n!} \check{g}_n(y) \stackrel{not}{=} \check{g}(y, \varepsilon), \quad t \in I.$$

Definition. The system (2) is called *the dual* of (1) by Lie Transformation if for any solution $y(t)$ of (2) and for any solution $x(t)$ of (1) there exist the functions

$$\check{x}_n(y_1, \dots, y_M) : \mathbf{R}^M \rightarrow \mathbf{R}^M, \quad n = 1, 2, \dots, N,$$

such that the following relation hold good

$$(3) \quad x(t) = y(t) + \sum_{n=1}^N \frac{\varepsilon^n}{n!} \check{x}_n(y(t)) \stackrel{not}{=} \check{x}(y, \varepsilon), \quad t \in I.$$

In this condition, the relation $x(t)$ is called the *Lie Transformation* of N -th order and $y(t)$ is called the *inverse (dual)* of the Lie Transformation of N -th order.

Proposition. a) If

$$\check{x}(y, \varepsilon) = y + \frac{\varepsilon}{1!} \check{x}_1(y) + \dots + \frac{\varepsilon^N}{N!} \check{x}_N(y) = x$$

is the Lie Transformation given by (3), then

$$\tilde{x}_1(y) = -\left. \frac{\partial \hat{y}}{\partial \varepsilon} \right|_{\varepsilon=0}.$$

b) If the vectorial function

$$w = \left. \frac{\partial \hat{y}}{\partial \varepsilon} \right|_{x=\tilde{x}(y,\varepsilon)}$$

admits the representation

$$w = w_1(y) + \frac{\varepsilon}{2!}w_2(y) + \dots + \frac{\varepsilon^{N-1}}{(N-1)!}w_N(y),$$

and

$$x = y + \frac{\varepsilon}{1!}\tilde{x}_1(y) + \dots + \frac{\varepsilon^N}{N!}\tilde{x}_N(y) = \tilde{x}(y, \varepsilon)$$

is the Lie Transformation of N -th order, then we have

$$\begin{aligned} \tilde{x}_1 &= -w_1, \\ \tilde{x}_2 &= w_1 \frac{\partial w_1}{\partial y} - w_2, \\ \tilde{x}_3 &= -3w_1 \left(\frac{\partial w_1}{\partial y} \right)^2 + 3w_1 \frac{\partial w_2}{\partial y} - 5w_1^2 \frac{\partial^2 w_1}{\partial y^2} - w_2 \frac{\partial w_1}{\partial y} + w_2 + \frac{4}{3}w_3 + w_1^3. \end{aligned}$$

□

Proposition. Let be given the dynamical systems in vectorial form

$$(1) \quad \frac{dx}{dt} = \dot{x} = \sum_{n=0}^N \frac{\varepsilon^n}{n!} \hat{f}_n(x)$$

and

$$(2) \quad \frac{dy}{dt} = \dot{y} = \sum_{n=0}^N \frac{\varepsilon^n}{n!} \hat{g}_n(y).$$

If (2) is the dual of (1), and

$$x = y + \sum_{n=1}^N \frac{\varepsilon^n}{n!} \tilde{x}_n(y) = \tilde{x}(y, \varepsilon)$$

is the Lie Transformation of N -th order, then if denote

$$\check{f}_n(y) = \hat{f}_n(\tilde{x}(y, \varepsilon)), \quad n = 0, 1, 2, \dots, N,$$

we get

$$\check{f}_0 = \check{g}_0,$$

$$\begin{aligned}\check{f}_1 &= \check{g}_1 + \check{g}_0 \frac{\partial \check{x}_1}{\partial y}, \\ \check{f}_2 &= \check{g}_2 + 2\check{g}_1 \frac{\partial \check{x}_1}{\partial y} + \check{g}_0 \frac{\partial \check{x}_2}{\partial y}, \\ \check{f}_3 &= \check{g}_3 + 3\check{g}_2 \frac{\partial \check{x}_1}{\partial y} + 3\check{g}_1 \frac{\partial \check{x}_2}{\partial y} + \check{g}_0 \frac{\partial \check{x}_3}{\partial y} \quad \text{etc.}\end{aligned}$$

□

If we define the operator $\mathcal{L}_f(\bullet) = \left(f \frac{\partial}{\partial y}(\bullet) \right)$, then the relations of this Proposition can be written in the following form

$$\begin{aligned}\hat{f}_0 &= \check{g}_0, \\ \check{f}_N &= \check{g}_N + \sum_{n=1}^N \binom{N}{n} \mathcal{L}_{\check{g}_{N-1}}(\check{x}_n), \quad N \geq 1.\end{aligned}$$

If we consider the operator of Nayfeh

$$L_i = \mathcal{L}_{w_i}(\bullet) - \mathcal{L}_{(\bullet)}(w_i), \quad i = 1, 2, \dots, N,$$

then we can rewrite the preceding relations as

$$(4) \quad \begin{aligned}\check{g}_0 &= \hat{f}_0 = \hat{f}_0 \Big|_{x=y}, \\ \check{g}_N &= \hat{f}_N \Big|_{x=y} - L_N(\check{g}_0) - \sum_{n=1}^{N-1} L_{N-n} \left(\hat{f}_n \Big|_{x=y} + \check{g}_n \right), \quad N \geq 1.\end{aligned}$$

We conclude that if we know w_1, \dots, w_N , then we can determine step by step the functions $\check{g}_0, \check{g}_1, \dots, \check{g}_N$. In this case, we can write the transformed Lie system and we can find its solution. Moreover, using w_1, \dots, w_N we can find step by step the functions $\check{x}_1, \check{x}_2, \check{x}_3$ etc. In this manner, we can write the expression of the Lie transformation $x(t)$. As a conclusion, we remark that is very important to determine w_1, \dots, w_N such that the dual system to be more convenient. In this regard, we recommend to write \hat{f}_n , $n = 1, 2, \dots, N$ as a sum of two function as follows.

Let us suppose the decomposition of the form

$$\hat{f}_n = \hat{f}_n^P + \hat{f}_n^S, \quad n = 1, 2, \dots, N.$$

Then (4) and (5) lead to

$$\begin{aligned}L_1(\check{g}_0) &= \hat{f}_1^S \Big|_{x=y}, \\ L_N(\check{g}_0) + \sum_{n=1}^{N-1} L_{N-n} \left(\hat{f}_n^S \Big|_{x=y} + \check{g}_n \right) &= \hat{f}_N^S \Big|_{x=y}, \quad N \geq 2.\end{aligned}$$

Since we shall use polynomial approximations for the components of \vec{H} , we shall consider the Bolotin type system

$$\dot{z}_i = \sum_{j=1}^M a_{ij} z_j + \sum_{j,k=1}^M b_{ijk} z_j z_k + \sum_{j,k,l=1}^M c_{ijkl} z_j z_k z_l + \dots, \quad i = 1, 2, \dots, M,$$

where $a_{ij}, b_{ijk}, c_{ijkl}, \dots$ are constant coefficients and M represents the dimension of the state vector.

By the transformation $z = \varepsilon x$ we get the system

$$(6) \quad \dot{x}_i = \sum_{j=1}^M a_{ij} x_j + \frac{\varepsilon}{1!} \sum_{j,k=1}^M B_{ijk} x_j x_k + \frac{\varepsilon^2}{2!} \sum_{j,k,l=1}^M C_{ijkl} x_j x_k x_l + \dots,$$

$$i = 1, 2, \dots, M.$$

We consider the matrix $A = (a_{ij})_{i,j=1,2,\dots,M}$ and define the vectorial functions

$$\hat{f}_1(x) = \begin{pmatrix} \sum_{j,k=1}^M B_{1jk} x_j x_k \\ \vdots \\ \sum_{j,k=1}^M B_{Mjk} x_j x_k \end{pmatrix}, \quad \hat{f}_2(x) = \begin{pmatrix} \sum_{j,k,l=1}^M C_{1jkl} x_j x_k x_l \\ \vdots \\ \sum_{j,k,l=1}^M C_{Mjkl} x_j x_k x_l \end{pmatrix} \quad \text{etc.}$$

Then, the system (6) can be written as

$$(7) \quad \dot{x} = Ax + \frac{\varepsilon}{1!} \hat{f}_1(x) + \frac{\varepsilon^2}{2!} \hat{f}_2(x) + \dots$$

Consequently the dual system associated to (7) becomes

$$\dot{y} = Ay + \frac{\varepsilon}{1!} \check{g}_1(y) + \frac{\varepsilon^2}{2!} \check{g}_2(y) + \dots$$

If suppose $\check{g}_n = \hat{f}_n^P \Big|_{x=y}$, $n = 1, 2, \dots, N$, then this system is written

$$(8) \quad \dot{y} = Ay + \frac{\varepsilon}{1!} \hat{f}_1^P \Big|_{x=y} + \frac{\varepsilon^2}{2!} \hat{f}_2^P \Big|_{x=y} + \dots$$

The vector functions w_1, \dots, w_N corresponding to (8) satisfy the following relations

$$A \left(w_1 - y \frac{\partial w_1}{\partial y} \right) = \hat{f}_1^S \Big|_{x=y},$$

$$A \left(w_N - y \frac{\partial w_N}{\partial y} \right) + \sum_{n=1}^{N-1} L_{N-n} \left(\hat{f}_n \Big|_{x=y} \right) = \hat{f}_N^S \Big|_{x=y}, \quad N \geq 2.$$

Magnetic field lines approximation using Lie Transformation

We shall apply the preceding properties to approximate the magnetic lines of the Biot-Savart-Laplace field generated by current through an arch of parabola.

Let us consider the arch of parabola

$$x = \frac{t^2}{2p} = x(t); \quad y = 0 = y(t); \quad z = t = z(t); \quad t \in [0, T].$$

The components of the field $\vec{F}_\gamma = F_x \vec{i} + F_y \vec{j} + F_z \vec{k}$ have the expressions

$$\begin{aligned} F_x &= -y \int_{[0, T]} \frac{dt}{PM^3}, \\ F_y &= x \int_{[0, T]} \frac{dt}{PM^3} - \frac{z}{p} \int_{[0, T]} \frac{tdt}{PM^3} + \frac{1}{2p} \int_{[0, T]} \frac{t^2 dt}{PM^3}, \\ F_z &= \frac{y}{p} \int_{[0, T]} \frac{tdt}{PM^3}, \\ PM^3 &= \|\vec{PM}\|^3 = \left[\left(x - \frac{t^2}{2p} \right)^2 + y^2 + (z - t)^2 \right]^{3/2}. \end{aligned}$$

Let be given the function $\psi(x, y, z) = (x^2 + y^2 + z^2)^\alpha$, $\alpha \in \mathbf{R}$, of class \mathbf{C}^∞ on $\mathbf{R}^3 \setminus \{(0, 0, 0)\}$. We approximate this functions as follows

$$\begin{aligned} \psi(x, y, z) &\approx \psi(0, 1, 0) + \frac{1}{1!} d\psi((0, 1, 0); (x, y - 1, z)) + \\ &+ \frac{1}{2!} d^2\psi((0, 1, 0); (x, y - 1, z)) = 1 + 2\alpha(y - 1) + \alpha x^2 + (2\alpha^2 - \alpha)(y - 1)^2 + \alpha z^2. \end{aligned}$$

If consider $\alpha := -\frac{3}{2}$, $x := -\frac{t^2}{2p}$ and $z := -t$, we have

$$\frac{1}{PM^3} \approx -\frac{3}{2}x^2 + 6y^2 - \frac{3}{2}z^2 + \frac{3t^2}{2p}x - 15y + 3tz - \frac{3}{8p^2}t^4 - \frac{3}{2}t^2 + 10,$$

and, using this relation, we approximate the integrals from F_x, F_y, F_z .

If rewrite $x = z_1$, $y = z_2$ and $z = z_3$, then the system which define the field lines is of the form

$$(1) \quad \dot{z}_i = F_{z_i} = \alpha_i + \sum_{j=1}^3 a_{ij} z_j + \sum_{j,k=1}^3 b_{ijk} z_j z_k + \sum_{j,k,l=1}^3 c_{ijkl} z_j z_k z_l, \quad i = 1, 2, 3.$$

If denote by z^C the solution of $\dot{z}_i = \alpha_i$, $i = 1, 2, 3$ (hence $z_i^C = \alpha_i t + \beta_i$, $i = 1, 2, 3$), and by z^L the solution of the system

$$(2) \quad \dot{z}_i = \sum_{j=1}^3 a_{ij} z_j + \sum_{j,k=1}^3 b_{ijk} z_j z_k + \sum_{j,k,l=1}^3 c_{ijkl} z_j z_k z_l, \quad i = 1, 2, 3,$$

then $z = z^C + z^L$ is the solution of (1).

If we use the transformation $z = \varepsilon x$, then (2) becomes

$$(3) \quad \dot{x} = Ax + \frac{\varepsilon}{1!} \hat{f}_1(x) + \frac{\varepsilon^2}{2!} \hat{f}_2(x)$$

where $A = (a_{ij})$,

$$(4) \quad A = \begin{pmatrix} 0 & \frac{3T^5}{4p^2} + \frac{T^3}{2} + 10T & 0 \\ \frac{3T^5}{4p^2} - \frac{T^3}{2} + 10T & -\frac{5T^3}{2p} & \frac{T^6}{16p^3} + \frac{3T^4}{16p^3} - \frac{5T^2}{p} \\ 0 & -\frac{T^6}{16p^3} - \frac{3T^4}{8p} + \frac{5T^2}{p} & 0 \end{pmatrix},$$

while $\hat{f}_1(x)$ and $\hat{f}_2(x)$ have the form

$$(5) \quad \hat{f}_1(x) = \begin{pmatrix} 15Tx_2^2 - \frac{T^3}{2p}x_1x_2 - \frac{3T^2}{2}x_1x_2 \\ \frac{T^3}{4p}x_1^2 + \frac{T^3}{p}x_2^2 - \frac{5T^3}{4p}x_3^2 - 15x_1x_2 + \frac{15T^2}{2p}x_2x_3 + \left(\frac{3T^2}{2} - \frac{3T^4}{8p^2}\right)x_3x_1 \\ -\frac{15T^2}{2p}x_2^2 + \frac{3T^4}{8p^2}x_1x_2 + \frac{T^3}{p}x_2x_3 \end{pmatrix},$$

$$\hat{f}_2(x) = \begin{pmatrix} -6Tx_2^3 + \frac{3T}{2}x_1^2x_2 + \frac{3T}{2}x_1x_3^2 \\ -\frac{3T}{2}x_1^3 + \frac{3T^2}{4p}x_2^3 + \frac{3T^2}{4p}x_1^2x_3 + 6x_1x_2^2 - \frac{3T^2}{2}x_1x_3^2 - \frac{3T}{p}x_2^2x_3 \\ \frac{3T^2}{p}x_2^3 - \frac{3T^2}{4p}x_1^2x_2 - \frac{3T^2}{4p}x_2x_3^2 \end{pmatrix}.$$

In order to integrate (3), we set $\hat{f}_1^P = 0$ and $\hat{f}_2^P = 0$, such that the associated system is $\dot{y} = Ay$, and w_1 is solution of equation

$$A \left(w_1 - y \frac{\partial w_1}{\partial y} \right) = \hat{f}_1 \Big|_{x=y}.$$

These lead to

$$w_1 = \begin{pmatrix} -\frac{1}{a_{12}} f_{1,1} \Big|_{x=y} \\ \frac{a_{21}}{a_{12}a_{22}} f_{1,1} \Big|_{x=y} + \frac{a_{23}}{a_{32}a_{22}} f_{3,1} \Big|_{x=y} - \frac{1}{a_{22}} f_{2,1} \Big|_{x=y} \\ -\frac{1}{a_{32}} f_{3,1} \Big|_{x=y} \end{pmatrix},$$

where $f_{i,1}$, $i = 1, 2, 3$ are the components of \hat{f}_1 .

By analogy, the equation

$$A \left(w_2 - y \frac{\partial w_2}{\partial y} \right) + L_1(\hat{f}_1|_{x=y}) = \hat{f}_2^S|_{x=y},$$

leads to

$$w_2 = \begin{pmatrix} -\frac{1}{a_{12}}(-f_{1,1}|_{x=y} + L_{1,1}) \\ \frac{a_{21}}{a_{12}a_{22}}(-f_{1,2}|_{x=y} + L_{1,1}) + \frac{a_{23}}{a_{32}a_{22}}(f_{3,2}|_{x=y} + L_{3,1}) - \frac{1}{a_{22}}(f_{2,2}|_{x=y} + L_{2,1}) \\ -\frac{1}{a_{32}}(f_{3,2}|_{x=y} + L_{3,1}) \end{pmatrix}.$$

Therefore $\check{x}_1 = -w_2$, and the function \check{x}_2 follows from the condition

$$\check{x}_2 = \left(w_1 \frac{\partial}{\partial y} \right) w_1 - w_2.$$

Taking into account these considerations, we may write the relation of (3) in the form

$$x = y + \frac{\varepsilon}{1!} \check{x}_1 + \frac{\varepsilon^2}{2!} \check{x}_2.$$

Finally, in our hypotheses for $\frac{1}{PM^3}$, the field lines in this example have the parametric equations

$$z(t) = z^C(t) + z^L(t),$$

that is

$$z(t) = \begin{pmatrix} 0 \\ -\frac{3T^8}{112p^3} - \frac{3T^5}{20} + \frac{5T^3}{3p} \\ 0 \end{pmatrix} \cdot t + \varepsilon x(t).$$

Geometrical properties of magnetic lines and surfaces

In this section, we study geometrical and topological characteristics of magnetic lines and surfaces. First we define the *angular* configurations and *spatial* configurations whose magnetic vector field will be studied in the sequel. We evidentiate the magnetic vector field and its vector/scalar potential for angular configurations in space. These results are used in further for studying the symmetries of these fields and the associated phase portraits. We state a geometrical study of magnetic lines. These geometrical properties are illustrated by numerical simulation of phase portraits. In the last section, we study the open magnetic lines.

Angular configurations. Piecewise rectilinear configurations in space. Equivalent configurations

A union of semilines with the same origin X , $\gamma = \gamma^- \cup \gamma^+$, where the semilines $\gamma^- = (X^-X]$, $\gamma^+ = [XX^+)$ represent electric wires wandered through the current \vec{I} whose intensity $I = \|\vec{I}\|$ is constant on γ^- and γ^+ respectively, is called *elementary configuration* (fig. 3.1.1.)



Fig. 3.1.1

The union

$$\Gamma = \bigcup_{i=1}^n \gamma_i, \quad \gamma_i = \bigcup_{j=1}^m \gamma_{ij},$$

where the family $(\gamma_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$ is a set of straight lines, semilines or segments (wires in \mathbf{R}^3) is called *piecewise rectilinear* configuration (network) in space if it satisfies the following properties:

P1) if γ_{ij} is a segment, then its edges are in contact with the extremities of other segments or semilines, γ_{kl} ; each semiline γ_{ij} has its finite edge in contact with the edge of a segment or of a semiline, γ_{kl} ;

P2) each circuit γ_i ,

$$\gamma_i = \bigcup_{j=1}^m \gamma_{ij},$$

is closed at finite distance or at infinity, and is wandered by a constant current \vec{I}_i on each rectilinear piecewise γ_{ij} ;

P3) at each constant point, the second Kirchoff law holds an. This axiom ensure that the associated magnetic field admits scalar potential.

For any rectilinear circuit, γ_{ij} , we denote $\vec{I}_{ij} = \vec{I}_i \Big|_{\gamma_{ij}}$ the local current and by $\vec{J}_i = \frac{1}{\|\vec{I}_i\|} \cdot \vec{I}_i$ and $\vec{J}_{ij} = \frac{1}{\|\vec{I}_{ij}\|} \cdot \vec{I}_{ij}$ respectively the corresponding unit vectors.

Taking into account these assumptions, the "fictitious" magnetic field generated around γ_{ij} is given by the Biot-Savart-Laplace law

$$\vec{H}_{\gamma_{ij}}(M) = I_{ij} \int_{\gamma_{ij}} \frac{\vec{J}_{ij} \times \vec{PM}}{PM^3} d\tau_P, \quad (\forall) M \in \mathbf{R}^3 \setminus \gamma_{ij},$$

where P is arbitrary point on γ_{ij} , and $I_{ij} = \|\vec{I}_{ij}\|$.

The magnetic vector field associated to the whole network Γ is given by

$$\vec{H}_{\Gamma}(M) = \sum_{i=1}^n \vec{H}_{\gamma_i}(M), \quad (\forall) M \in \mathbf{R}^3 \setminus \Gamma.$$

Two piecewise rectilinear configuration in space Γ and Γ^* are called *equivalent* iff they generate equal magnetic vector fields, that is

$$\vec{H}_{\Gamma}(M) = \vec{H}_{\Gamma^*}(M), \quad (\forall) M \in \mathbf{R}^3 \setminus (\Gamma \cup \Gamma^*).$$

Magnetic field associated to an elementary angular spatial configuration

The background given above allows to state the following basic result

Theorem. *Let be given the spatial configuration $\Gamma = \Gamma_{12} \cup \Gamma_{34}$, $\Gamma_{12} = \gamma_1 \cup \gamma_2$, $\Gamma_{34} = \gamma_3 \cup \gamma_4$, Γ_{12} and Γ_{34} being elementary angular configuration, with vertices $X_{12}(-x^0, 0, -z^0)$ and $X_{34}(x^0, 0, z^0)$, wandered by unitary electric current (fig.3.3.4.).*

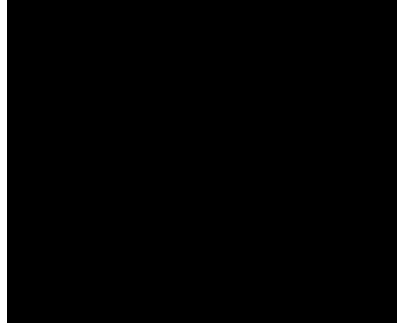


Fig.3.3.4

Then

1) the associated magnetic vector field

$$\vec{H}_{\Gamma}(M) = H_x(M) + H_y(M) + H_z(M), \quad M = M(x, y, z) \in \mathbf{R}^3 \setminus \Gamma,$$

has the components

$$\begin{aligned} H_x(M) &= -y \left[\frac{s_1}{r_1(r_1 - \tau_1)} - \frac{s_2}{r_1(r_1 - \tau_2)} + \frac{s_3}{r_2(r_2 - \tau_3)} - \frac{s_4}{r_2(r_2 - \tau_4)} \right], \\ H_y(M) &= \frac{us_1 - wc_1}{r_1(r_1 - \tau_1)} - \frac{us_2 - wc_2}{r_1(r_1 - \tau_2)} + \frac{\bar{u}s_3 - \bar{w}c_3}{r_2(r_2 - \tau_3)} - \frac{\bar{u}s_4 - \bar{w}c_4}{r_2(r_2 - \tau_4)}, \\ H_z(M) &= y \left[\frac{c_1}{r_1(r_1 - \tau_1)} - \frac{c_2}{r_1(r_1 - \tau_2)} + \frac{c_3}{r_2(r_2 - \tau_3)} - \frac{c_4}{r_2(r_2 - \tau_4)} \right], \end{aligned}$$

where

$$s_i = \sin \alpha_i, \quad c_i = \cos \alpha_i, \quad i \in \{1, \dots, 4\}, \quad u = x + x^0, \quad w = z + z^0,$$

$$\bar{u} = x - x^0, \quad \bar{w} = z - z^0, \quad r_1 = \sqrt{u^2 + y^2 + w^2}, \quad r_2 = \sqrt{\bar{u}^2 + y^2 + \bar{w}^2}$$

and

$$\tau_i = \begin{cases} uc_i + ws_i, & i = 1, 2 \\ \bar{u}c_i + \bar{w}s_i, & i = 3, 4; \end{cases}$$

2) for any point $M = M(x, y, z) \in \mathbf{R}^3 \setminus \Gamma$, the vector potential of \vec{H}_Γ is

$$\vec{V}_\Gamma(M) = \left(\ln \frac{|r_1 - \tau_2|^{c_2} |r_2 - \tau_4|^{c_4}}{|r_1 - \tau_1|^{c_1} |r_2 - \tau_3|^{c_3}}, 0, \ln \frac{|r_1 - \tau_2|^{s_2} |r_2 - \tau_4|^{s_4}}{|r_1 - \tau_1|^{s_1} |r_2 - \tau_3|^{s_3}} \right),$$

3) the associated scalar potential to \vec{H}_Γ is

$$\begin{aligned} U_\Gamma(M) &= 2 \left(\operatorname{arctg} \frac{-w \sin \frac{\sigma_1}{2} - u \cos \frac{\sigma_1}{2} + r_1 \cos \frac{\Delta_1}{2}}{y \sin \frac{\Delta_1}{2}} + \right. \\ &\quad \left. + \operatorname{arctg} \frac{-\bar{w} \sin \frac{\sigma_2}{2} - \bar{u} \cos \frac{\sigma_2}{2} + r_2 \cos \frac{\Delta_2}{2}}{y \sin \frac{\Delta_2}{2}} \right), \end{aligned}$$

$$(\forall) M = M(x, y, z) \in \mathbf{R}^3 \setminus (xOz),$$

where $\sigma_1 = \alpha_1 + \alpha_2$, $\Delta_1 = \alpha_1 - \alpha_2$, $\sigma_2 = \alpha_3 + \alpha_4$, $\Delta_2 = \alpha_3 - \alpha_4$.

□

The symmetries of magnetic vector fields and of phase portraits

When the configuration γ is wandered by unitary electric current I and γ has some symmetries, then the associated magnetic field (and its equilibrium sets) has some symmetries properties too. Denote by $D = \mathbf{R}^3 \setminus \gamma$ and $\chi(D)$ the Lie algebra of vector fields of class \mathcal{C}^1 on D . Suppose $\gamma \subset xOz$.

Definition. Let $\vec{H}_\gamma \in \chi(D)$ be the magnetic vector field associated to γ .

- a) the image $\alpha(I)$ of a maximal field line $\alpha : I \subset \mathbf{R} \rightarrow \mathbf{R}^3$ is called *orbit* of \vec{H}_γ ;
- b) the set $\mathcal{P}(\vec{H}_\gamma)$ of all orbits is called *phase portraits* of the field \vec{H}_γ .

Definition. Let $\gamma_i, i = 1, 2$ be two configuration and $\vec{H}_{\gamma_i} \in \chi(D), D = \mathbf{R}^3 \setminus (\gamma_1 \cup \gamma_2), i = 1, 2$ be the associated magnetic vector fields. The phase portraits are said to be equivalent if there exists a homeomorphism $h : D \rightarrow D$ such that for all orbits in $\mathcal{P}(\vec{H}_{\gamma_1})$ exists a homeomorphic orbit in $\mathcal{P}(\vec{H}_{\gamma_2})$. We denote $\mathcal{P}(\vec{H}_{\gamma_1}) \sim \mathcal{P}(\vec{H}_{\gamma_2})$.

Theorem. Let Γ be rectilinear wire configuration wandered by \vec{I} who induces the magnetic field \vec{H}_Γ . Consider that the symmetry $S \subset \mathbf{R}^3$ (with determinant δ) acts in the same manner on the points and the fields of the algebra $\chi(D), D = \mathbf{R}^3 \setminus \Gamma$.

If $\vec{I} \circ S = \pm S \circ \vec{I}$, then $\vec{H}_\Gamma \circ S = \pm \delta S \circ \vec{H}_\Gamma$.

□

Magnetic flows and traps. Stationary and quasistationary magnetic vector fields

Flows. Magnetic traps

The theorem of differentiable mappings evidentiates the current T^t generated by the magnetic field \vec{H} .

Definition. An open and connected set U or its closure \bar{U} is called *magnetic trap* of the magnetic field \vec{H} iff for all $t \geq 0$ we have $T^t(\bar{U}) \subset \bar{U}$.

Obviously, any invariant domain with respect to T^t is magnetic trap.

In this conditions, we remark that a magnetic line starting from an interior point of U cannot reach ∂U , therefore this one cannot leave U .

Moreover, a magnetic line starting from an exterior point of U can enter to U .

Theorem. Let $U \subset \mathbf{R}^3$ be open and connected set, with ∂U closed and piecewise bounded. If the set $\bar{U} = U \cup \partial U$ is compact and U is magnetic trap of \vec{H} of class C^∞ , then the following properties hold:

- 1) $\Sigma = \partial U$ is a magnetic surface,
- 2) $T^t(\bar{U}) = \bar{U}, (\forall) t \geq 0$.

□

Proposition. Let \vec{n} be the (unit) normal vector field to ∂U , oriented to the exterior of U . Then, we have the properties

(1) if U is a magnetic trap of \vec{H} , then $(\vec{n}, \vec{H})|_{\partial U} \leq 0$;

(2) if $(\vec{n}, \vec{H})|_{\partial U} = 0$, then ∂U is a magnetic surface of \vec{H} ;

(3) if $(\vec{n}, \vec{H})|_{\partial U} > 0$, then $\mathbf{R}^3 \setminus U$ is a magnetic trap of \vec{H} ;

(4) if $(\vec{n}, \vec{H})|_{\partial U} < 0$, then U is a magnetic trap of \vec{H} .

□

Stationary magnetic vector fields

Definition. The magnetic vector field \vec{H} of class \mathbf{C}^n , $n \geq 1$, is called *stationary field* on the open set $D \subset \mathbf{R}^3$, if it is solenoidal and irrotational ($\operatorname{div} \vec{H} = 0$, and $\operatorname{rot} \vec{H} = \vec{0}$ for all $x \in D$).

If $\vec{H} = H_1 \vec{i}_1 + H_2 \vec{i}_2 + H_3 \vec{i}_3$ is stationary magnetic field on $D \subset \mathbf{R}^3$, then any magnetic line of H is trajectory of a potential dynamical system with three degrees of freedom of potential $V = -f$.

Theorem (Lorentz-Udriște World-Force Law). Let \vec{H} be stationary magnetic vector field on the open set $D \subset \mathbf{R}^3$ and let $f = \frac{1}{2} \|\vec{H}\|^2$ be its energy. Then any nonconstant trajectory of the dynamical system,

$$\frac{d^2 x^i}{dt^2} = \frac{\partial f}{\partial x^i}, \quad i = 1, 2, 3$$

is a reparametrized geodesic of Riemann-Jacobi manifold

$$(D \setminus \mathcal{E}, g_{ij} = (\mathcal{H} + f)\delta_{ij}, \quad i, j = 1, 2, 3),$$

where \mathcal{E} is the set of all zeros of $\vec{H} = H_1 \vec{i}_1 + H_2 \vec{i}_2 + H_3 \vec{i}_3$ and \mathcal{H} is the total energy. \square

Definition. Let (M, g) be Riemannian manifold. Any surfaces generated by the motion of a geodesic α along a given curve β is called *ruled surface* in (M, g) .

The curve β may be chosen such that to be orthogonal to the magnetic lines. In this respect, β can be magnetic line of the solenoidal field $\vec{v} = \operatorname{grad} h \times \vec{H}$, with h solution of p.d.e

$$H_1 \frac{\partial h}{\partial x^1} + H_2 \frac{\partial h}{\partial x^2} + H_3 \frac{\partial h}{\partial x^3} = 0.$$

Corollary. The magnetic surfaces are ruled surfaces of the Riemann manifold

$$(D \setminus \mathcal{E}, g_{ij}).$$

On the Riemann-Jacobi manifold

$$(D \setminus \mathcal{E}, g_{ij}),$$

the Gauss curvature K of a magnetic surface cannot be positive. \square

Quasistationary magnetic vector fields

Definition. The field \vec{H} is called *quasistationary field* or *static field* on the open set $D \subset \mathbf{R}^3$ if it satisfies the equations of Maxwell

$$\operatorname{rot} \vec{H} = \vec{J}, \quad \operatorname{div} \vec{H} = 0.$$

Theorem. Any magnetic line of a magnetic quasistationary field

$$\vec{H} = H_1 \vec{i}_1 + H_2 \vec{i}_2 + H_3 \vec{i}_3,$$

is trajectory of a nonpotential conservative dynamical system with three degrees of freedom, and total energy

$$\mathcal{H} = \frac{1}{2} \delta_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} - f(x^1, x^2, x^3).$$

□

Theorem (Lorentz-Udriște World-Force Law). Any nonconstant trajectory of the system

$$\frac{d^2 x^i}{dt^2} = \frac{\partial f}{\partial x^i} + \left(\frac{\partial H_i}{\partial x^j} - \frac{\partial H_j}{\partial x^i} \right) \frac{dx^i}{dt}, \quad i = 1, 2, 3 \quad (*)$$

with energy \mathcal{H} , is a reparametrized horizontal geodesic of Riemann-Jacobi-Lagrange manifold

$$(D \setminus \mathcal{E}, g_{ij} = (\mathcal{H} + f)\delta_{ij}, \quad N_j^i = \Gamma_{jk}^i y^k + F_j^i, \quad i, j, k = 1, 2, 3),$$

where Γ_{jk}^i is the metric connection and

$$F_j^i = g^{ih} \left(\frac{\partial H_j}{\partial x^h} - \frac{\partial H_h}{\partial x^j} \right).$$

Any magnetic surface is ruled surface of the Riemann-Jacobi-Lagrange manifold

$$(D \setminus \mathcal{E}, g_{ij}, N_j^i).$$

□

Remarks. 1) The vector field of components

$$X_i = \left(\frac{\partial H_i}{\partial x^j} - \frac{\partial H_j}{\partial x^i} \right) \frac{dx^i}{dt}, \quad i = 1, 2, 3,$$

is orthogonal on the solution of (*). Therefore, it does not produce a dissipation of energy along a trajectory of (*).

2) If a magnetic line passes the boundary ∂D , then

a) the portion of the line contained into D is trajectory of a nonpotential conservative dynamical system of the second order

b) the portion of the line exterior to D (not included into $\mathbf{R}^3 \setminus \bar{D}$) is trajectory of a potential (conservative) dynamical system of the second order.

Generalization [5]

The study of magnetic vector fields associated to rectilinear wires leads to difficult problems related by the wide set of configurations as well as by the nature of differential systems modelling them. It appears the necessity to state a global theory in order to get general laws. This was the main goal of our thesis.

The preceding results can be generalized by considering a differentiable manifold M and a (0,2) - tensor g symmetric on M .

We suppose that for all $(\forall)x \in M$, $\text{index } g(x) = \dot{I}$ and denote by ∇ the induced connection by the metric tensor field g . If X is vector field of class \mathbf{C}^∞ on M , then the kinematic differential system defined on X is

$$(*) \quad \frac{dx}{dt} = X(x).$$

Using the metric tensor g and the vector field X on M , then the generalized energy is $f : M \rightarrow \mathbf{R}$, $f = \frac{1}{2}g(X, X)$.

Also the (1,1) exterior tensor field

$$T = \nabla X - g^{-1} \otimes g(\nabla X)$$

allows us to generalize the above remarks.

Taking into account these assumptions, the World-Force Law of Lorentz-Udriște becomes

1) Any nonconstant trajectory of the dynamical system

$$\frac{\nabla dx}{dt} = \text{grad}f,$$

corresponding to a constant value of the Hamiltonian \mathcal{H} is a reparametrized geodesic of the Riemann-Jacobi manifold

$$(M \setminus \mathcal{E}, \quad h = (\mathcal{H} + f)g).$$

2) Any nonconstant trajectory of the dynamical system

$$\frac{\nabla dx}{dt} = \text{grad}f + T \left(\frac{dx}{dt} \right),$$

corresponding to a constant value of the Hamiltonian \mathcal{H} is a reparametrized horizontal geodesic of the Riemann-Jacobi-Lagrange manifold

$$(M \setminus \mathcal{E}, \quad h = (\mathcal{H} + f)g, \quad N_j^i = \Gamma_{jk}^i y^k + T_j^i, \quad i, j, k = 1, 2, \dots, n),$$

where Γ_{jk}^i are local components of the connection ∇ and

$$T_j^i = \nabla_j X^i - g^{il} g_{kj} \nabla_l X^k$$

are local components of the exterior tensor field T (with g_{ij} are the local components of the metric g).

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University Politehnica of Bucharest
Department of Mathematics I
Splaiul Independenței 313
77206 Bucharest, Romania