

Complete Minimal Hypersurfaces in a Locally Symmetric Space

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Abstract

In this paper we shall consider the Riemannian manifold under some supposed curvature conditions and investigate some properties about a complete minimal hypersurface in a locally symmetric space. In particular, we obtain characterizations of a totally geodesic minimal hypersurface in a locally symmetric space.

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1 Introduction

Let $M^m(c)$ be an m -dimensional Riemannian manifold of constant curvature c which is called a *space form* and M be a hypersurface of $M^m(c)$. Many differential geometers have studied M from various points of view. For example, the study of hypersurfaces with constant mean curvature of $M^m(c)$ was initiated by Nomizu and Smyth [4], who proved some results. Later, Okumura [5] and Hasanis [2] characterized totally umbilical hypersurfaces of $M^m(c)$, $c > 0$, under a certain condition given by an inequality between the length of the second fundamental form and the mean curvature. Moreover, Cheng and Nakagawa [1] also generalized their results.

In consideration of these subjects, it seems interesting to the authors to investigate some properties about complete hypersurfaces with constant mean curvature in a locally symmetric space. This is closely relevant to the generalization of the so-called *Bernstein problem*.

The purpose of this paper is to obtain the characterization of totally geodesic hypersurfaces in a locally symmetric space under some supposed curvature conditions and also to characterize complete minimal hypersurfaces in the same one.

2 Preliminaries

We begin with recalling basic properties of hypersurfaces of a Riemannian manifold. Let (M', g') be an $(n + 1)$ -dimensional Riemannian manifold with the Riemannian metric g' . Throughout this paper, manifolds are always assumed to be connected and the differentiability of any geometric object is assumed to be of class C^∞ . For any point x in M' we choose a local field of orthonormal frames $\{e_A\} = \{e_0, e_1, \dots, e_n\}$ around x adapted the Riemannian metric g' . We denote $\{\omega_A\} = \{\omega_0, \omega_1, \dots, \omega_n\}$ by the dual frames on M' which consists of 1-forms on M' such that $\omega_A(e_B) = \delta_{AB}$. Here and in the sequel, the following convention on the range of indices is used, unless otherwise stated :

$$\begin{aligned} A, B, \dots &= 0, 1, \dots, n, \\ i, j, \dots &= 1, \dots, n. \end{aligned}$$

Associated with the frame field $\{e_A\}$, there exist 1-forms $\{\omega_{AB}\}$. They are usually called *connection forms* on M' so that they satisfy the structure equations of M' :

$$(2.1) \quad d\omega_A + \sum_A \omega_{AB} \wedge \omega_B = 0, \quad \omega_{AB} + \omega_{BA} = 0,$$

$$(2.2) \quad d\omega_{AB} + \sum_c \omega_{BC} \wedge d\omega_{CB} = \Omega'_{AB},$$

$$(2.3) \quad \Omega'_{AB} = -\frac{1}{2} \sum_{C,D} R'_{ABCD} \omega_C \wedge \omega_D,$$

where $\Omega' = (\Omega'_{AB})$ (resp. R'_{ABCD}) denotes the curvature form (resp. the components of the Riemannian curvature tensor R') of M' .

Now, relative to the frame field chosen above, the Ricci tensor S' of M' can be expressed as follows :

$$(2.4) \quad S' = \sum_{A,B} S'_{AB} \omega_A \otimes \omega_B,$$

where $S'_{AB} = \sum_C R'_{CABC} = S'_{BA}$. The scalar curvature r' is also given by

$$(2.5) \quad r' = \sum_A S'_{AA}.$$

The components R'_{ABCDE} of the covariant derivative of the Riemannian curvature tensor R' are obtained by

$$(2.6) \quad \begin{aligned} &\sum_E R'_{ABCDE} \omega_E = dR'_{ABCD} \\ &- \sum_E (R'_{EBCD} \omega_{EA} + R'_{AECD} \omega_{EB} + R'_{ABED} \omega_{EC} + R'_{ABCE} \omega_{ED}). \end{aligned}$$

By $M^m(c)$ we denote an m -dimensional Riemannian manifold of constant curvature c , which is called a *space form*. Then the Riemannian curvature tensor R'_{ABCD} of $M^m(c)$ is given by

$$(2.7) \quad R'_{ABCD} = c(\delta_{AD}\delta_{BC} - \delta_{AC}\delta_{BD}).$$

Next, let (M', g') be an $(n+1)$ -dimensional Riemannian manifold and let M be an n -dimensional hypersurface of M' . We choose a local field of orthonormal frames $\{e_A\} = \{e_0, e_j\} = \{e_0, e_1, \dots, e_n\}$ on a neighborhood of M' in such a way that restricted to M , e_1, \dots, e_n are tangent to M and e_0 is normal to M . With respect to this frame field, let $\{\omega_A\} = \{\omega_0, \omega_j\}$ be its dual frame field. Then the Riemannian metric tensor g' of M' is given by $g' = \sum_A \omega_A \otimes \omega_A$. The connection forms on M' are denoted by ω_{AB} . Restricting these forms to the submanifold M , we have

$$(2.8) \quad \omega_0 = 0,$$

and the induced Riemannian metric tensor g of M is given by $g = \sum_j \omega_j \otimes \omega_j$. Then $\{e_j\}$ is a local field of orthonormal frames with respect to this metric and $\{\omega_j\}$ is a local dual frame field due to $\{e_j\}$, which consists of 1-forms on M . Moreover, $\omega_1, \dots, \omega_n$ are linearly independent, and they are said to be *canonical 1-forms* on M . It follows from (2.8) and the Cartan lemma that the exterior derivative of (2.8) gives rise to

$$(2.9) \quad \omega_{0i} = \sum_j h_{ij} \omega_j, \quad h_{ij} = h_{ji}.$$

The quadratic form $\alpha = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j \otimes e_0$ with values in the normal bundle is called the *second fundamental form* of the submanifold M . From the structure equations of M' it follows that the structure equations for M are similarly given by

$$(2.10) \quad d\omega_i + \sum_j \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$(2.11) \quad \begin{aligned} d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} &= \Omega_{ij}, \\ \Omega_{ij} &= \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l. \end{aligned}$$

For the Riemannian curvature tensor R and R' of M and M' , respectively, it follows from (2.3) and (2.11) that we have the Gauss equation

$$(2.12) \quad R_{ijkl} = R'_{ijkl} + (h_{il}h_{jk} - h_{ik}h_{jl}).$$

The components of the Ricci tensor S and the scalar curvature r of M are given by

$$(2.13) \quad S_{ij} = \sum_k R'_{kijk} + h h_{ij} - h_{ij}^2,$$

$$(2.14) \quad r = \sum_{j,k} R'_{kjjk} + h^2 - h_2,$$

where $h = \sum_i h_{ii}$, $h_{ij}^2 = \sum_r h_{ir}h_{rj}$ and $h_2 = \sum_j h_{jj}^2$.

Now, the components h_{ijk} of the covariant derivative of the second fundamental form α on M are given by

$$(2.15) \quad \sum_k h_{ijk}\omega_k = dh_{ij} - \sum_k (h_{kj}\omega_{ki} + h_{ik}\omega_{kj}).$$

Using (2.1) ~ (2.3), and (2.9) ~ (2.11), it follows from the coefficients of $\omega_j \wedge \omega_k$ that

$$(2.16) \quad h_{ijk} - h_{ikj} = R'_{0ijk}.$$

Similarly the components h_{ijkl} of the covariant derivative of h_{ijk} can be defined by

$$(2.17) \quad \sum_l h_{ijkl}\omega_l = dh_{ijk} - \sum_l (h_{ljk}\omega_{li} + h_{ilk}\omega_{lj} + h_{ijl}\omega_{lk}).$$

On the other hand, differentiating (2.15) exteriorly and using the properties $d^2 = 0$, (2.10) and (2.11) we see

$$\begin{aligned} & \sum_k (dh_{ijk} \wedge \omega_k - h_{ijk} \sum_l \omega_{kl} \wedge \omega_l) \\ &= - \sum_k \left\{ dh_{kj} \wedge \omega_{ki} - h_{jk} \left(\sum_l \omega_{kl} \wedge \omega_{li} + \frac{1}{2} \sum_{l,m} R_{kil m} \omega_l \wedge \omega_m \right) \right. \\ & \quad \left. + dh_{ik} \wedge \omega_{kj} - h_{ik} \left(\sum_l \omega_{kl} \wedge \omega_{lj} + \frac{1}{2} \sum_{l,m} R_{kjl m} \omega_l \wedge \omega_m \right) \right\}. \end{aligned}$$

Substituting (2.15) and (2.17) into the above equation, we have the following Ricci formula for the second fundamental form on M

$$(2.18) \quad h_{ijkl} - h_{ijlk} = - \sum_r (h_{ir}R_{rjkl} + h_{jr}R_{rikl})$$

from the coefficient of $\omega_k \wedge \omega_l$.

Let us denote the covariant derivative of R'_{ABCD} , as a curvature tensor of M' , by $R'_{ABCD;E}$. Then, restricting on M , $R'_{0ijk;l}$ is given by

$$(2.19) \quad R'_{0ijk;l} = R'_{0ijkl} - R'_{0i0k}h_{jl} - R'_{0ij0}h_{kl} + \sum_m R'_{mijk}h_{ml},$$

where R'_{0ijkl} denotes the covariant derivative of R'_{0ijk} as a tensor on M so that

$$(2.20) \quad \begin{aligned} \sum_l R'_{0ijkl}\omega_l &= dR'_{0ijk} - \sum_l R'_{0ljk}\omega_{li} \\ &\quad - \sum_l R'_{0ilk}\omega_{lj} - \sum_l R'_{0ijl}\omega_{lk}. \end{aligned}$$

For the sake of brevity, a tensor h_{ij}^m and a function h_m on M for any integer $m(\geq 2)$ are introduced as follows :

$$(2.21) \quad h_{ij}^m = \sum_{i_1, \dots, i_{m-1}} h_{ii_1} h_{i_1 i_2} \cdots h_{i_{m-1} j},$$

$$(2.22) \quad h_m = \sum_i h_{ii}^m.$$

Here we introduce a fundamental property for the generalized maximal principle due to Omori [6] and Yau [7] and quote the following in order to prove our results.

Theorem 2.1. *Let M be an n -dimensional complete Riemannian manifold whose Ricci curvature is bounded from below. Let f be a C^2 -function bounded from above on M , then for any $\varepsilon > 0$, there exists a point p in M at which it satisfies*

$$\sup f - \varepsilon < f(p), \quad |\text{grad } f(p)| < \varepsilon \quad \text{and} \quad \Delta f(p) < \varepsilon, \text{leqno}(2.23)$$

where Δ is the Laplacian operator on M .

3 The Laplacian operator

In this section we shall consider the Laplacian of the second fundamental form. Let M be a hypersurface of an $(n + 1)$ -dimensional Riemannian manifold M' and let us denote by α the second fundamental form on M . Then the Laplacian Δh_{ij} of the components h_{ij} of α is defined by

$$(3.1) \quad \Delta h_{ij} = \sum_k h_{ijkk}.$$

From (2.16) and (2.18) it follows that

$$\begin{aligned} \Delta h_{ij} &= \sum_k h_{ikjk} + \sum_k R'_{0ijkk} \\ &= \sum_k h_{kijj} + \sum_k R'_{0ijkk} \\ &= \sum_k \{h_{kikj} - \sum_l (h_{il} R_{lkjk} + h_{kl} R_{lijj}) + R'_{0ijkk}\}. \end{aligned}$$

Replacing h_{kikj} with $h_{kkij} + R'_{0kikj}$ in the above equation, we get

$$\begin{aligned} \Delta h_{ij} &= \sum_k h_{kkij} + \sum_k (R'_{0ijkk} + R'_{0kikj}) \\ &\quad - \sum_{k,l} (h_{il} R_{lkjk} + h_{kl} R_{lijj}). \end{aligned}$$

Combining (2.12) and (2.19) with the above equation, we obtain

$$\begin{aligned}
\Delta h_{ij} &= \sum_k h_{kkij} + \sum_k (R'_{0ijk;k} + R'_{0kik;j}) \\
&+ \sum_k (h_{jk}R'_{0i0k} + h_{kk}R'_{0ij0} - \sum_l h_{kl}R'_{lijk}) \\
&+ \sum_k (h_{ij}R'_{0k0k} + h_{kj}R'_{0ki0} - \sum_l h_{jl}R'_{lki k}) \\
&- \sum_{k,l} (R'_{kjl} + h_{kl}h_{jk} - h_{kk}h_{jl})h_{il} \\
&- \sum_{k,l} (R'_{kji} + h_{kl}h_{ij} - h_{lj}h_{ik})h_{kl}
\end{aligned}$$

and hence we have

$$\begin{aligned}
\Delta h_{ij} &= \sum_k h_{kkij} + \sum_k (R'_{0ijk;k} + R'_{0kik;j}) \\
&+ (hR'_{0ij0} + \sum_k h_{ij}R'_{0k0k}) \\
(3.2) \quad &- \sum_{k,l} (2h_{kl}R'_{kijl} + h_{jl}R'_{lki k} + h_{il}R'_{kjl}) \\
&+ hh_{ij}^2 - h_2h_{ij}.
\end{aligned}$$

4 The curvature conditions

In this section we shall consider the Riemannian manifold under some supposed curvature conditions and estimate Δh_2 from below. Let M' be an $(n+1)$ -dimensional Riemannian manifold and let M be a hypersurface of M' . For a point x in M' let $\{e_0, e_1, \dots, e_n\}$ be a local field of orthonormal frames of M' on a neighborhood of x in such a way that, restricted to M , the vectors e_1, \dots, e_n are tangent to M and the other is normal to M . For linearly independent vectors u and v in the tangent space $T_x M'$ we denote by $K'(u, v)$ the sectional curvature of the section spanned by u and v in M' and by R' or $Ric'(u, u)$ the Riemannian curvature tensor in M' or the Ricci curvature in the direction of u in M' , respectively. Let us denote by ∇' the Riemannian connection on M' . We assume that the ambient space M' satisfies the following conditions :

$$(4.1) \quad Ric'(e_0, e_0) \leq c_1,$$

$$(4.2) \quad K'(e_i, e_j) \geq c_2 \text{ for any } i \text{ and } j \text{ such that } i \neq j$$

and

$$(4.3) \quad |\nabla' R'| \leq \frac{c_3}{n}$$

for some constant c_1, c_2 and c_3 .

When M' satisfies the above conditions (4.1), (4.2) and (4.3), it is said simply for M' to satisfy *the condition (*)*.

Remark 4.1. If $c_3 = 0$, then the ambient space M' is locally symmetric. Conversely if the ambient space M' is locally symmetric, then without loss of generality, we may regard as $c_3 = 0$.

Remark 4.2. If M' is a space form $M^{n+1}(c)$ of constant curvature c , then it satisfies the condition (*), where $\frac{c_1}{n} = c_2 = c$.

In the sequel, we assume that the ambient space M' satisfies the condition (*) and hypersurface M is minimal. Namely we assume that the function h vanishes. Then the Laplacian of the squared norm h_2 of the second fundamental form α of M is given by

$$\begin{aligned}\Delta h_2 &= \Delta\left(\sum_{i,j} h_{ij}h_{ij}\right) = 2\sum_{i,j,k} (h_{ijk}h_{ij})_k \\ &= 2\sum_{i,j,k} (h_{ijkk}h_{ij} + h_{ijk}h_{ijk}) \\ &= 2|\nabla\alpha|^2 + 2\sum_{i,j,k} h_{ijkk}h_{ij},\end{aligned}$$

where ∇_α is the covariant derivative of the second fundamental form α and $|\nabla\alpha|^2$ is the squared norm of $\nabla\alpha$. i.e., $|\nabla\alpha|^2 = \sum_{i,j,k} h_{ijkk}h_{ij}$. Hence from (3.1) and (3.2) it follows that

$$\begin{aligned}\Delta h_2 = 2|\nabla\alpha|^2 &+ 2\sum_{i,j} \left\{ \sum_k (R'_{0kik;j} + R'_{0ijk;k}) + \sum_k h_{ij}R'_{0k0k} \right. \\ &\left. - 2\sum_{k,l} (h_{kl}R'_{lij}k + h_{lj}R'_{lik} + h_{li}R'_{lkj}k) - h_2h_{ij} \right\} h_{ij}.\end{aligned}$$

Thus we have

$$(4.4) \quad \begin{aligned}\Delta h_2 = 2|\nabla\alpha|^2 &+ 2\left\{ \sum_{i,j,k} (R'_{0kik;j} + R'_{0ijk;k})h_{ij} + h_2\sum_k R'_{0k0k} \right. \\ &\left. - \sum_{k,l} 2(h_{ij}h_{kl}R'_{lij}k + h_{lj}^2R'_{lkj}k) - h_2^2 \right\}.\end{aligned}$$

On the other hand, since the matrix $H = (h_{ij})$ can be diagonalized, we can expressed by

$$(4.5) \quad h_{ij} = \lambda_i\delta_{ij},$$

where λ_i is the principal curvature on M . By definition, we see

$$h_2 = \sum_i \lambda_i^2 \geq \lambda_i^2$$

and hence we have

$$(4.6) \quad -\sqrt{h_2} \leq \lambda_i \leq \sqrt{h_2},$$

$$(4.7) \quad -h_2 \leq \lambda_i \lambda_j \leq h_2.$$

Now in order to prove our results, we need to estimate Δh_2 of (4.4) from below. First of all, we treat with the second term of (4.4). It is seen that

$$\begin{aligned} 2 \sum_{i,j,k} (R'_{0kik;j} + R'_{0ijj;k}) h_{ij} &= 2 \sum_{j,k} \lambda_j (R'_{0kj k;j} + R'_{0jj k;k}) \\ &\geq -2 \sum_{j,k} |\lambda_j| (|R'_{0kj k;j}| + |R'_{0jj k;k}|). \end{aligned}$$

So by (4.3) and (4.6) we have

$$(4.8) \quad \text{the second term of (4.4)} \geq -4c_3 \sqrt{h_2}.$$

Next, we consider the third term of (4.4). It is estimated as follows :

$$\begin{aligned} 2h_2 \sum_k R'_{0k0k} &= -2h_2 \sum_k R'_{0kk0} \\ &= -2h_2 \sum_k R'_{0kk0} \geq -2h_2 c_1, \end{aligned}$$

where we have used (4.1). Hence we have

$$(4.9) \quad \text{the third term of (4.4)} \geq -2c_1 h_2.$$

It is evident that if the ambient space M^l is a space form $M^{n+1}(c)$ of constant curvature c , then (4.9) also holds and hence

$$\text{the third term of (4.4)} \geq -2c n h_2.$$

Last, we estimate the fourth term of (4.4). We have by (4.2)

$$\begin{aligned} -4 \sum_{k,l} (h_{ij} h_{kl} R'_{lij k} + h_{ij}^2 R'_{lkjk}) &= -4 \sum_{j,k} (\lambda_j \lambda_k R'_{kjj k} - \lambda_j^2 R'_{kjj k}) \\ &= -4 \sum_{j,k} (\lambda_j \lambda_k - \lambda_j^2) R'_{kjj k} = 2 \sum_{j,k} (\lambda_j - \lambda_k)^2 R'_{kjj k} \\ &\geq 2c_2 \sum_{j,k} (\lambda_j - \lambda_k)^2. \end{aligned}$$

Accordingly we obtain

$$(4.10) \quad \text{the fourth term of (4.4)} \geq 4c_2 n h_2.$$

Thus, substituting (4.8), (4.9) and (4.10) into (4.4), we can prove the following

Lemma 4.1. *Let M' be an $(n + 1)$ -dimensional Riemannian manifold satisfying the condition (*) and let M be a minimal hypersurface of M' . Then we have*

$$(4.11) \quad \Delta h_2 \geq -4c_3\sqrt{h_2} + 2(2nc_2 - c_1)h_2 - 2h_2^2.$$

In particular, if $M' = M^{n+1}(c)$, then substituting $\frac{c_1}{n} = c_2 = c$ and $c_3 = 0$ into (4.11) we obtain

$$(4.12) \quad \Delta h_2 \geq 2(nc_2 h_2 - h_2^2).$$

5 Locally symmetric spaces

Let M' be an $(n + 1)$ -dimensional locally symmetric space which satisfies the condition (*) and let M be a complete minimal hypersurface of M' . For a point x in M let $\{e_0, e_1, \dots, e_n\}$ be a local field of orthonormal frames of M' around of x in such a way that, restricted to M , the vectors e_1, \dots, e_n are tangent to M and the other is normal to M . Then for two constants c_1 and c_2

$$Ric'(e_0, e_0) \leq c_1,$$

$$K'(e_i, e_j) \geq c_2 \text{ for any } i \text{ and } j \text{ such that } i \neq j.$$

Moreover, since M' is locally symmetric, we have $c_3 = 0$ and the inequalities (4.11) implies

$$(5.1) \quad \Delta f^2 \geq 2\{(2nc_2 - c_1) - nh_2\}f^2,$$

where f is the non-negative function defined by $f^2 = h_2$.

Theorem 5.1. *Let M' be a locally symmetric space which satisfies the condition (*) and let M be a complete minimal hypersurface. If it satisfies*

$$(5.2) \quad n \sup h_2 \leq 2nc_2 - c_1,$$

then M is totally geodesic.

Proof. We define a non-negative function f by $f^2 = h_2$. Then it is bounded by the assumption. Let $\lambda_1, \dots, \lambda_n$ be principal curvatures on M . It is easily seen that $f^2 = \sum_i \lambda_i^2$ and hence the function f vanishes identically on M if and only if $\lambda_i = 0$ for any indices i , i.e., M is totally geodesic.

First of all, we will show that the Ricci tensor is bounded from below. The Ricci tensor S_{ij} is expressed by

$$\begin{aligned} S_{ij} &= \sum_k (R'_{kijk} + h_{ij}h_{kk} - h_{ik}h_{jk}) \\ &\geq (n-1)c_2 - \lambda_i^2 \delta_{ij}. \end{aligned}$$

Since the principal curvatures satisfy $\lambda_i \lambda_j \geq -h_2$, we have

$$S_{ij} \geq (n-1)c_2 - h_2,$$

which yields the Ricci curvature of M is bounded from below under the assumption. This means that Theorem 2.1 due to Omori [5] and Yau [6] can be applied to the function f^2 . Therefore given any positive number ε there exists a point p in M at which f^2 satisfies (2.23). Accordingly, for a converge sequence $\{\varepsilon_m\}$ so that it converges to zero as m tends to infinity, there exists a point sequence $\{p_m\}$ so that the sequence $\{f(p_m)\}$ converges to f_0 by taking a subsequence, if necessary. By (2.23) we see

$$f(p_m) \rightarrow f_0 = \sup f.$$

By (5.1) we can estimate Δf^2 from the below, and hence by (2.23) we obtain

$$\varepsilon_m \geq \Delta f^2(p_m) \geq 2\{(2nc_2 - c_1) - nf^2(p_m)\}f^2(p_m),$$

from which it follows that we have

$$2\{(2nc_2 - c_1) - nf_0^2\}f_0^2 \leq 0$$

as m tends to infinity. Under this inequality we have

$$f_0 = 0 \quad \text{or} \quad nf_0^2 \geq (2nc_2 - c_1).$$

Under the assumption of the present theorem, the above restriction of the supremum of the function f yields that $f_0 = 0$, which implies f vanishes identically on M and hence M is totally geodesic. \square

Remark 5.1. An $(n + 1)$ -Euclidean space $M' = \mathbf{R}^{n+1}$ satisfies the condition $(*)$ with $c_1 = c_2 = c_3 = 0$. The complete hypersurface $M = S^{n-1}(c) \times \mathbf{R}^1$ in $M' = \mathbf{R}^{n+1}$ is not umbilic and it satisfies $h_2 = \frac{h^2}{n-1} > 0$. In this example, the condition (5.2) is equivalent to $\sup h_2 < \frac{h^2}{n}$.

6 Minimal hypersurfaces

Let M' be an $(n + 1)$ -dimensional Riemannian manifold which satisfies the condition $(*)$. That is, for any point x in M let $\{e_0, e_1, \dots, e_n\}$ be a local field of orthonormal frames of M' around of x in such a way that, restricted to M , the vectors e_1, \dots, e_n are tangent to M and the other is normal to M . Then for three constants c_1, c_2 and c_3 we have

$$\begin{aligned} Ric'(e_0, e_0) &\leq c_1, \\ k'(e_i, e_j) &\geq c_2 \quad \text{for any } i \text{ and } j \text{ such that } i \neq j, \\ |\nabla R'| &\leq \frac{c_3}{n}. \end{aligned}$$

Let M be a complete minimal hypersurface of M' . From the inequality (4.11) it follows that

$$(6.1) \quad \Delta f^2 \geq -4c_3 f + 2(2nc_2 - c_1)f^2 - 2f^4,$$

where f is the non-negative function defined by $f^2 = h_2$.

Theorem 6.1. *Let M' be an $(n+1)$ -dimensional Riemannian manifold which satisfies the condition (*) and let M be a complete minimal hypersurface. If the Ricci curvatures of M is bounded from below and if it satisfies*

$$(6.2) \quad (2nc_2 - c_1)^3 > 27c_3^2 > 0,$$

then there exists constants a_2 and a_3 depending only on c_1, c_2, c_3 and n such that

$$0 \leq \sup h_2 \leq a_2 \quad \text{or} \quad \sup h_2 \geq a_3.$$

Proof. Let $\lambda_1, \dots, \lambda_n$ be principal curvatures on M . Since the hypersurface M is minimal, the Ricci tensor S_{ij} is given by

$$\begin{aligned} S_{ij} &= \sum_k (R'_{kij k} - h_{ik} h_{jk}) \\ &= \sum_k R'_{kij k} - \lambda_i^2 \delta_{ij} \geq (n-1)c_2 - \lambda_i^2 \delta_{ij}. \end{aligned}$$

Here, we suppose that the squared norm h_2 is not bounded from above, then there exists at least one index j so that the principal curvature λ_j is not bounded, from this fact together with the above inequality it follows that the Ricci curvature of M is not bounded from below. Thus we have a contradiction. Therefore the squared norm h_2 is bounded from above, which implies Theorem 2.1 due to Omori [5] and Yau [6] can be applied to the function f^2 . Given any positive number ε there exists a point p in M , at which f^2 satisfies (2.23). Accordingly, for a converge sequence $\{\varepsilon_m\}$ so that it converges to zero as m tends to infinite, there exists a point sequence $\{p_m\}$ so that the sequence $\{f(p_m)\}$ converges to f_0 , by taking a subsequence, if necessary. By (2.23) we see

$$f(p_m) \rightarrow f_0 = \sup f.$$

By means of (6.1) we can estimate Δf^2 from the below, and hence by (2.23) we obtain

$$\varepsilon_m \geq \Delta f^2(p_m) \geq 2\{-2c_3 + (2nc_2 - c_1)f(p_m) - f^3(p_m)\}f(p_m),$$

which implies

$$(6.3) \quad \{-2c_3 + (2nc_2 - c_1)f_0 - f_0^3\}f_0 \leq 0$$

as m tends to infinity. On the other hand, the quadrant expression y is defined by

$$(6.4) \quad y = y(x) = -2\{x^3 - (2nc_2 - c_1)x + 2c_3\}x,$$

then we have

$$(6.5) \quad \begin{aligned} y' &= -4\{2x^3 - (2nc_2 - c_1)x + c_3\}, \\ y'' &= -4\{6x^2 - (2nc_2 - c_1)\}, \\ y''' &= -48x, \end{aligned}$$

which implies that y' has the relative maximum on the interval $(0, \infty)$, whose value is $4\left\{\frac{2}{3\sqrt{6}}(2nc_2 - c_1)^{3/2} - c_3\right\}$, which implies that if $2(2nc_2 - c_1)^3 > 27c_3^2$, then

$y' \left(\sqrt{\frac{2nc_2 - c_1}{6}} \right) > 0$. Thus under the assumption (6.2) we get $y' \left(\sqrt{\frac{2nc_2 - c_1}{6}} \right) > 0$. This yields that there exist two positive roots b_2 and b_3 ($b_2 < b_3$) of the algebraic equation $y'(x) = 0$. Thus from (6.4) it follows that

$$(6.6) \quad y(b_3) = \{(2nc_2 - c_1)b_3 - 3c_3\}b_3.$$

On the other hand, by the direct calculation, we have

$$y \left(\frac{3c_3}{2nc_2 - c_1} \right) = -6c_3^2 \frac{27c_3^2 - (2nc_2 - c_1)^3}{(2nc_2 - c_1)^4}.$$

So, under the assumption (6.2), the value is positive. Because $y'(b_3) = 0$ and $\lim_{x \rightarrow \infty} y'(x) = -\infty$ and $y'|_{[b_3, \infty)} \leq 0$, we obtain

$$\frac{3c_3}{2nc_2 - c_1} < b_3,$$

from which together with (6.6) it follows that

$$y(b_3) > 0.$$

By (6.4) we see $y(0) = 0$ and $\lim_{x \rightarrow \infty} y(x) = -\infty$ and also by means of (6.5) we have $y'|_{[0, b_2)} < 0$, this yields that there exists two positive roots a_2 and a_3 of the equation $y(x) = 0$. By (6.3) and (6.4) we obtain

$$0 \leq f_0 = \sup f \leq a_2 \quad \text{or} \quad f_0 = \sup f \geq a_3.$$

It completes the proof. \square

Remark 6.1. The values a_2 and a_3 are two positive roots of the cubic equation

$$(6.7) \quad x^3 - 2(2nc_2 - c_1)x + 2c_3 = 0$$

and hence we can require concretely the values. They are depended on the constant c_1, c_2, c_3 and n .

Owing to Theorem 5.1, Theorem 6.1 and Remark 6.1, the following property is easily verified.

Corollary 6.2. *Let M' be an $(n + 1)$ -dimensional locally symmetric space which satisfies the condition (*) and let M be a complete minimal hypersurface. If it satisfies*

$$(6.8) \quad \sup h_2 < 2nc_2 - c_1,$$

then M is totally geodesic.

Proof. By the assumption (6.8) the Ricci curvature on M is bounded from below, because we see

$$S_{ij} \geq (n - 1)c_2 - \lambda_j^2 \delta_{ij} \geq (n - 1)c_2 - h_2$$

On the other hand, since M' is locally symmetric, we see $c_3 = 0$, from which it follows that the algebraic equation (6.7) is reduced to

$$x^3 - 2(2nc_2 - c_1)x = 0.$$

So the supremum f_0 of the function $f = h_2$, i.e., the squared norm of the second fundamental form, satisfies

$$f_0^3 - 2(2nc_2 - c_1)f_0 = 0.$$

The assumption (6.8) implies $f_0 = 0$, which yields that M is totally geodesic.

It completes the proof. \square

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