Strong Morphisms of Groupoids

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Abstract

We reffer to the groupoids in the sense of Ehresmann. The aim of this paper is to give some various topics of strong morphisms of groupoids.

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Introduction

The concept of groupoid in the sense of Ehresmann is a natural generalization of the algebraic notion of groupoid introduced by H. Brandt in the paper: Uber eine Verallgemeinerung der Gruppenbegriffe, Math. Ann., 96 (1926), 360-366.

The notions of topological and differentiable groupoids has been introduced by Ehresmann in 1950 in his paper on connections (cf. [4])

Many authors have investigated the Lie groupoids (in particular, symplectic groupoids) in connection with their applications in differential geometry, symplectic geometry, Poisson geometry, quantum mechanics ergodic theory , geometric quantization and gauge theories (cf. [1], [11] - [15], [17] - [20]).

Recent applications of Lie groupoids endowed with supplementary structures have also contributed to a renewed interest in these studies.

In this paper we study a special case of groupoid morphism, namely: strong morphism of Ehresmann groupoids.

Other special morphisms of groupoids are the following:

- similar morphisms of Brandt groupoids (these morphisms are used in [7] for construct a cohomology theory of Brandt groupoids which extends the usual cohomology theory of groups;
- pullback, fibrewise injective (resp., surjective, bijective) and piecewise injective (resp., surjective, bijective) morphism of groupoids (for various topics concerning these special morphisms see [9]).

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1 Morphisms of groupoids

In this section we construct the category of Ehresmann groupoids and some important properties concerning the morphisms of groupoids are given.

Definition 1.1. ([15]) A *groupoid* (in the sense of Ehresmann) Γ over Γ_0 or *groupoid* with the base Γ_0, is a pair $(\Gamma;\Gamma_0)$ of sets equipped with:

(i) two surjections $\alpha,\beta:\Gamma\longrightarrow\Gamma_0$, called the *source* and the *target* map;

(ii) a (partial) composition law $\mu:\Gamma_{(2)}\longrightarrow\Gamma, (x,y)\longrightarrow\mu(x,y)=x\cdot y=xy$, with domain $\Gamma_{(2)}=\{(x,y)\in\Gamma\times\Gamma \quad | \quad \beta(x)=\alpha(y)\}$;

(iii) an injection $\epsilon:\Gamma_0\longrightarrow\Gamma, u\longrightarrow\epsilon(u)=\widetilde{u}$, called the *inclusion map*;

(iv) a map $i:\Gamma\longrightarrow\Gamma,\quad x\longrightarrow i(x)=x^{-1}$, called the *inversion map*.

These maps must satysfy the following algebraic axioms generalizing those of groups:

(G1) (*associative law*) For arbitrary $x,y,z\in\Gamma$ the triple product $(xy)z$ is defined iff $x(yz)$ is defined. In case either is defined, we have $\quad (xy)z=x(yz) \quad$; hence, the triple xyz is defined whenever $\beta(x)=\alpha(y)$ and $\beta(y)=\alpha(z)$.

(G2) (*identities*) For each $x\in\Gamma$ we have $(\epsilon(\alpha(x)),x)\in\Gamma_{(2)};(x,\epsilon(\beta(x))\in\Gamma_{(2)}$ and $\epsilon(\alpha(x))\cdot x=x\cdot\epsilon(\beta(x))=x$

(G3) (*inverses*) For each $x\in\Gamma$ we have $(x,i(x))\in\Gamma_{(2)};(i(x),x)\in\Gamma_{(2)}$ and $x\cdot i(x)=\epsilon(\beta(x)),\quad i(x)\cdot x=\epsilon(\alpha(x))\quad\triangle$

Every group G with e as unity, is a groupoid over $G_0=\{e\}$.

We denote a groupoid Γ over Γ_0 by $(\Gamma,\alpha,\beta,\epsilon,i,\mu;\Gamma_0)$ or $(\Gamma,\alpha,\beta;\Gamma_0)$ or $(\Gamma;\Gamma_0)$.

For each $u\in\Gamma_0$, the set $\Gamma_u=\alpha^{-1}(u)$ (resp. $\Gamma^u=\beta^{-1}(u)$) is called the *α-fibre* (resp. *β-fibre*) of Γ over $u\in\Gamma_0$ and if $u,v\in\Gamma$, we will write $\Gamma_u^v=\Gamma_u\cap\Gamma^v$.

A groupoid Γ over Γ_0 such that Γ_0 is a subset of Γ is called Γ_0-*groupoid* or *Brandt groupoid.*

We summarize some properties of these mappings obtained from definitions.

Proposition 1.1. Let Γ be a groupoid over Γ_0. The following assertions hold:

(i) $\alpha\circ\epsilon=\beta\circ\epsilon=Id_{\Gamma_0}$.

(ii) $\alpha(xy)=\alpha(x)\quad$ and $\quad\beta(xy)=\beta(y)\quad$ for all $\quad(x,y)\in\Gamma_{(2)}$.

(iii) $\epsilon(u)\cdot\epsilon(u)=\epsilon(u)\quad$ for each $\quad u\in\Gamma_0$.

(iv) Let $u,v\in\Gamma_0$. We have:

(a) if $(x,\epsilon(u))\in\Gamma_{(2)}$ such that $x\cdot\epsilon(u)=x$ then $\epsilon(u)=\epsilon(\beta(x))$.

(b) if $(\epsilon(v),x)\in\Gamma_{(2)}$ such that $\epsilon(v)\cdot x=x$ then $\epsilon(v)=\epsilon(\alpha(x))$.

(v) For all $x\in\Gamma\quad$ we have $\quad\beta(x^{-1})=\alpha(x)\quad$ and $\quad\alpha(x^{-1})=\beta(x)$

(vi) For $u\in\Gamma_0\quad$ we have $\quad(\epsilon(u))^{-1}=\epsilon(u)$.

(vii) $\alpha\circ i=\beta,\quad\beta\circ i=\alpha\quad$ and $\quad i\circ i=Id_\Gamma$.

(viii) For each $u\in\Gamma_0$, the set $\Gamma(u)=\alpha^{-1}(u)\cap\beta^{-1}(u)$ is a group under the restriction of the partial multiplication (this group is be called the *isotropy group at* u of the groupoid Γ).

(ix) In the case $\Gamma_0\subseteq\Gamma$, we have:

(a) $\epsilon(\Gamma_0)=\Gamma_0$.

(b) $\epsilon(u)=u,\quad$ for each $u\in\Gamma_0$. $\triangle$

In view of Proposition 1.1., the element $\epsilon(\alpha(x))$ (resp. $\epsilon(\beta(x))$) is the *left unit* (resp., *right unit*) of $x\in\Gamma$. The subset $\epsilon(\Gamma_0)$ is be called the *unity set* of Γ.

Definition 1.2. (a) A groupoid Γ over Γ_0 is said to be *transitive* if the map $\alpha\times\beta:\Gamma\longrightarrow\Gamma_0\times\Gamma_0$, given by $(\alpha\times\beta)(x)=(\alpha(x),\beta(x)),(\forall)x\in\Gamma$ is surjective.

(b) By *group bundle* we mean a groupoid Γ over Γ_0 such that $\alpha(x) = \beta(x)$ for each $x \in \Gamma$. Moreover, a group bundle is the union of its isotropy groups $\Gamma(u) = \alpha^{-1}(u), u \in \Gamma_0$ (here two elements may be composed iff they lie in the same fiber.) $\triangle$

If $(\Gamma, \alpha, \beta; \Gamma_0)$ is a groupoid over Γ_0, then $Is(\Gamma) = \{x \in \Gamma \mid \alpha(x) = \beta(x)\}$ is a group bundle, called the *isotropy group bundle* associated to Γ. It is easy to see that $\epsilon(\Gamma_0) \subseteq Is(\Gamma)$.

Proposition 1.2. If $(\Gamma, \alpha, \beta; \Gamma_0)$ is a groupoid, then the following assertions hold:

(i) (*cancellation law*) If $x \cdot z_1 = x \cdot z_2$ (resp., $z_1 \cdot x = z_2 \cdot x$) then $z_1 = z_2$.

(ii) If $(x, y) \in \Gamma_{(2)}$ then $(y^{-1}, x^{-1}) \in \Gamma_{(2)}$ and $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$.

(iii) The isotropy groups $\Gamma(\alpha(x))$ and $\Gamma(\beta(x))$ are isomorphic.

(iv) If Γ is transitive, then the isotropy groups of Γ are groups isomorphes.

Proof. (i) and (ii) These assertions follows from definitions.

(iii) We prove that the map $\varphi : \Gamma(\alpha(x)) \longrightarrow \Gamma(\beta(x)), a \longrightarrow \varphi(a) = x \cdot a \cdot x^{-1}$ is a isomorphism of groups.

(iv) It follows from (iii) and the fact that $\alpha \times \beta : \Gamma \longrightarrow \Gamma_0 \times \Gamma_0$ is surjective. $\triangle$

Example 1.1. (a) *Nul groupoid.* Any set B is a groupoid on itself with $\Gamma = \Gamma_0 := B$, $\alpha = \beta = \epsilon := id_B$ and every element is a unity, called it *the nul groupoid.* The multiplication is given by $x \cdot x := x$ for all $x \in B$.

(b) *Coarse groupoid.* If B is any non-empty set, then $B \times B$ is a groupoid over B with the rules:

$$\alpha(x, y) := x; \beta(x, y) := y; \epsilon(x) := (x, x), \quad i(x, y) := (y, x)$$

and

$$\mu((x, y), (y', z)) := (x, z) \quad \text{iff} \quad y = y'.$$

The unit set of this groupoid, called it *the coarse groupoid associated* to B, is the diagonal Δ_B of the cartesian product $B \times B$.

(c) *Trivial groupoid.* Let B be any non- empty set and $\mathcal{G}$ be a multiplicative group with e as unity. Construct a transitive groupoid Γ over B, called the *trivial groupoid on B with group* $\mathcal{G}$, in the following way:

$$\Gamma := B \times B \times \mathcal{G}; \; \Gamma_0 := B; \quad \alpha(a, b, x) := a; \quad \beta(a, b, x) := b; \quad \epsilon(b) := (b, b, e);$$

$$i(a, b, x) := (b, a, x^{-1}) \quad \text{and} \quad \mu((a, b, x), (b', c, y)) := (a, c, xy) \quad \text{iff} \quad b = b'.$$

For this groupoid we have

$$\epsilon(\Gamma_0) = \{(b, b, e) \mid b \in B\} \quad \text{and} \quad \Gamma(b) = \{(b, b, x) \mid x \in \mathcal{G}\},$$

which are identified with B resp. $\mathcal{G}$.

If $\mathcal{G} = \{e\}$, then we can identify $B \times B \times \mathcal{G}$ with the coarse groupoid associated to B.

(d) A *vector bundle* $E \xrightarrow{\pi} M$ is a group bundle on M. Here $\Gamma := E$ is the total space, $\Gamma_0 := M$ is the base space, $\alpha = \beta := \pi$ so that $\Gamma_{(2)} := \uplus_{x \in M} E_x \times E_x$ (E_x is the fibre at x) and the composition law is fibrewise addition.$\triangle$

Other examples of groupoids are the following: the *fundamental groupoid of a topological space* (see [6]), the *disjoint union* of a disjoint family of groupoids (see [9]) and the *action groupoid* (see [13]).

Definition 1.3. Let $(\Gamma,\alpha,\beta,\epsilon,i,\mu;\Gamma_0)$ and $(\Gamma',\alpha',\beta',\epsilon',i',\mu';\Gamma'_0)$ be two groupoids. A *morphism of groupoids* or *groupoid morphism* is a pair (f,f_0) of maps $f:\Gamma\longrightarrow\Gamma'$ and $f_0:\Gamma_0\longrightarrow\Gamma'_0$ such that the following two conditions are satisfied:

$$(1)\qquad f(\mu(x,y))=\mu'(f(x),f(y))\quad\text{for every}\quad(x,y)\in\Gamma_{(2)}$$

$$(2)\qquad \alpha'\circ f=f_0\circ\alpha\quad\text{and}\quad\beta'\circ f=f_0\circ\beta.\triangle$$

If $\Gamma_0=\Gamma'_0$ and $f_0=Id_{\Gamma_0}$, we say that f is a Γ_0**- morphism**.$\triangle$

Note that the condition (1) ensure that $(f(x),f(y))\in\Gamma'_{(2)}$, i.e. $\mu'(f(x),f(y))$ is defined whenever $\mu(x,y)$ is defined.

Applying Propositions 1.1 and 1.2 we obtain:

Proposition 1.3. The groupoids morphisms preserve unities and inverses, i.e. $f(\widetilde{u})=\widetilde{f_0(u)}$, $(\forall)u\in\Gamma_0$ and $f(x^{-1})=(f(x))^{-1}$, $(\forall)x\in\Gamma$; in other words, we have: $f\circ\epsilon=\epsilon'\circ f_0$ and $f\circ i=i'\circ f$ $\triangle$

Proposition 1.4. A pair $(f,f_0):(\Gamma;\Gamma_0)\longrightarrow(\Gamma';\Gamma'_0)$ is a groupoid morphism iff the following condition holds:

$$(3)\quad(\forall)(x,y)\in\Gamma_{(2)}\implies(f(x),f(y))\in\Gamma'_{(2)}\quad\text{and}\quad f(\mu(x,y))=\mu'(f(x),f(y))$$

Proof. The condition (3) is a consequence of Definition 1.4. and Prop.1.3.

Conversely, let $f:\Gamma\longrightarrow\Gamma'$ which satisfy (3) and we define the map $f_0:\Gamma_0\longrightarrow\Gamma'_0$ by $f_0(u)=\alpha'(f(\epsilon(u)))$, $(\forall)u\in\Gamma_0$. We prove that $\alpha'\circ f=f_0\circ\alpha$ and $\beta'\circ f=f_0\circ\beta$.

Indeed, since $(x,\epsilon(\beta(x)))\in\Gamma_{(2)}$ it follows that $(f(x),f(\epsilon(\beta(x))))\in\Gamma'_{(2)}$ and

$$f(x)\cdot f(\epsilon(\beta(x)))=f(x\cdot\epsilon(\beta(x)))=f(x);\ \text{but}\ f(x)\cdot\epsilon'(\beta'(f(x)))=f(x);$$

$$\implies\epsilon'(\beta'(f(x)))=f(\epsilon(\beta(x)))\implies\alpha'(\epsilon'(\beta'(f(x))))=\alpha'(f(\epsilon(\beta(x))))$$

and applying Prop.1.1. we obtain succesivelly

$$\beta'(f(x))=(f_0\circ\alpha)(\epsilon(\beta(x)))\implies\beta'(f(x))=f_0(\beta(x))$$

i.e. $\beta'\circ f=f_0\circ\beta$. Similarly we prove that $\alpha'\circ f=f_o\circ\alpha.\triangle$

Example 1.2. (a) If $(\Gamma,\alpha,\beta,\epsilon;\Gamma_0)$ is a groupoid, then $(Id_\Gamma,Id_{\Gamma_0})$ is a groupoid morphism.

(b) If $(f,f_0):(\Gamma,\Gamma_0)\to(\Gamma';\Gamma'_0)$ and $(g,g_0):(\Gamma',\Gamma'_0)\to(\Gamma'',\Gamma''_0)$ are groupoid morphisms, then the composition $(g,g_0)o(f,f_0):(\Gamma,\Gamma_0)\to(\Gamma'',\Gamma''_0)$ defined by $(g,g_0)\circ(f,f_0)=(g\circ f,g_0\circ f_0)$ is a groupoid morphism.$\triangle$

If $(f,f_0):(\Gamma;\Gamma_0)\longrightarrow(\Gamma';\Gamma'_0)$ is a groupoid morphism, then for every $u,v\in\Gamma_0$ we have:

$$f(\Gamma_u)\subseteq\Gamma'_{f_0(u)};\quad f(\Gamma^v)\subseteq(\Gamma')^{f_0(v)}\ \text{and}\ f(\Gamma_u^v)\subseteq(\Gamma')_{f_0(u)}^{f_0(v)}.$$

Then the restriction of f to $\Gamma_u,\Gamma^v,\Gamma_u^v$ respectively, defines the groupoid morphisms

$$\Gamma_u\longrightarrow\Gamma'_{f_0(u)};\ \Gamma^v\longrightarrow(\Gamma')^{f_0(v)},\Gamma_u^v\longrightarrow(\Gamma')_{f_0(u)}^{f_0(v)},$$

denoted by f_u,f^v and f_u^v.

Definition 1.4. A groupoid morphism $(f,f_0):(\Gamma;\Gamma_0)\longrightarrow(\Gamma';\Gamma'_0)$ is said to be *isomorphism* of groupoids if there exists a groupoid morphism $(g,g_0):(\Gamma';\Gamma'_0)\to(\Gamma,\Gamma_0)$

with $(g, g_0)o(f, f_0) = (id_\Gamma, id_{\Gamma_0})$ and $(f, f_0)o(g, g_0) = (id_{\Gamma'}, id_{\Gamma'_0})$. Two groupoids $(\Gamma; \Gamma_0)$ and $(\Gamma'; \Gamma'_0)$ are said to be **isomorphic** if there exists an isomorphism $(f, f_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma'_0)$.$\triangle$

Proposition 1.5. Let $(f, f_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma'_0)$ be a groupoid morphism. Then the following assertions hold:

(i) If f is injective (resp., surjective), then also is f_0.

(ii) (f, f_0) is an isomorphism iff the map f is bijective.

(iii) $f(Is(\Gamma)) \subseteq Is(\Gamma')$.

(iv) A groupoid morphism (f, f_0) such that f is surjective and f_0 is injective (in particular, every surjective Γ_0- morphism of groupoids) preserve the isotropy group bundles, i.e. $f(Is(\Gamma)) = Is(\Gamma')$.

Proof. (i) This follows imediately from Definition 1.3. and Proposition 1.4.

(ii) It is a consequence of Definition 1.4. and of the assertion (i).

(iii) Let $x' \in f(Is(\Gamma))$. Then $x' = f(x)$ with $x \in Is(\Gamma)$ and we have

$$\alpha'(x') = \alpha'(f(x)) = f_0(\alpha(x)) = f_0(\beta(x)) = \beta'(f(x)) = \beta'(x'),$$

since $\alpha(x) = \beta(x)$; hence $x' \in Is(\Gamma')$. Therefore, $f(Is(\Gamma)) \subseteq Is(\Gamma')$.

(iv) It suffices to prove that $Is(\Gamma') \subseteq f(Is(\Gamma))$. Let $x' \in Is(\Gamma')$ i.e. $x' \in \Gamma'$ such that $\alpha(x') = \beta'(x')$. For $x' \in \Gamma'$ there exists $x \in \Gamma$ such that $x' = f(x)$, since f is surjective. Then $\alpha'(f(x)) = \beta'(f(x))$ and we obtain that $f_0(\alpha(x)) = f_0(\beta(x))$. Hence, $\alpha(x) = \beta(x)$, since f_0 is injective. Thus, $x \in Is(\Gamma)$ and $x' \in f(Is(\Gamma))$. Therefore, $Is(\Gamma') \subseteq f(Is(\Gamma))$.$\triangle$

Example 1.3. (a) Let $(\Gamma, \alpha, \beta, \epsilon; \Gamma_0)$ be a groupoid and $(\Gamma_0 \times \Gamma_0, \alpha', \beta', \epsilon'; \Gamma_0)$ the coarse groupoid associated to Γ_0.Then $\alpha \times \beta : \Gamma \longrightarrow \Gamma_0 \times \Gamma_0$, $(\alpha \times \beta)(x) = (\alpha(x), \beta(x))$ is a Γ_0- morphism of the groupoid Γ into the coarse groupoid $\Gamma_0 \times \Gamma_0$.

b) Let $(\Gamma, \alpha, \beta, \epsilon, i, \mu; \Gamma_0)$ be a groupoid over Γ_0 and X a set with the same cardinal as Γ_0, i.e. there exists a bijection φ from Γ_0 to X. Then Γ has a canonical structure of a groupoid over X, that is $(\Gamma, \alpha', \beta', \epsilon', i', \mu'; X)$ is a groupoid over X where $\alpha' := \varphi o \alpha; \beta' := \varphi o \beta; \epsilon' := \epsilon o \varphi^{-1}; i' := \varphi o i; \mu' := \mu$.Moreover, $(id_\Gamma, \varphi) : (\Gamma; \Gamma_0) \to (\Gamma; X)$ is an isomorphism of groupoids.$\triangle$

Example 1.4. (*the induced groupoid*) Let $(\Gamma, \alpha, \beta, \epsilon; \Gamma_0)$ be a groupoid, X an abstract set and $f : X \longrightarrow \Gamma_0$ a map from X to Γ_0. Then the set:

$$f^*(\Gamma) = \{(x, y, a) \in X \times X \times \Gamma \quad | \quad f(x) = \alpha(a), f(y) = \beta(a)\}$$

has a canonical structure of groupoid over X with respect to the following rules:

$$\alpha^*(x, y, a) := x; \beta^*(x, y, a) := y; \epsilon^*(x) := (x, x, \epsilon(f(x))); i^*(x, y, a) := (y, x, i(a)),$$

and

$$\mu^*((x, y, a), (y', z, b)) := (x, z, \mu(a, b))$$

iff $y = y'$ and $(a, b) \in \Gamma_{(2)}$.

The groupoid $(f^*(\Gamma), \alpha^*, \beta^*, \epsilon^*, \mu^*; \Gamma_0)$ is called the *induced groupoid* or the *inverse image* of Γ *under* f; it is denoted sometimes by $f^*(\Gamma)$.

If $f^*(\Gamma)$ is the induced groupoid of Γ under $f : X \longrightarrow \Gamma_0$ then $f^*_\Gamma : f^*(\Gamma) \longrightarrow \Gamma$ defined by $f^*_\Gamma(x, y, a) = a$ together with f define a groupoid morphism $(f^*_\Gamma, f) : (f^*(\Gamma); X) \longrightarrow (\Gamma; \Gamma_0)$ and it is called the *canonical morphism of an induced groupoid.*$\triangle$

2 Strong morphisms of groupoids

This section is dedicated to study of a particular type of groupoid morphisms, namely: the strong morphisms of groupoids. One of the most important results of strong morphisms is the correspondence theorem for subgroupoids (resp., for normal subgroupoids).

Definition 2.1. A *subgroupoid* of a groupoid $(\Gamma;\Gamma_0)$ is a pair $(\Gamma';\Gamma'_0)$ of subsets, where $\Gamma' \subseteq \Gamma, \quad \Gamma'_0 \subseteq \Gamma_0$ such that the following conditions are verified:

(i) $\alpha(\Gamma') \subseteq \Gamma'_0; \quad \beta(\Gamma') \subseteq \Gamma'_0$

(ii) for every $x, y \in \Gamma'$ such that the product $x \cdot y$ is defined implies that $x \cdot y \in \Gamma'$, i.e. Γ' is closed under the partial multiplication.

(iii) $(\forall) u \in \Gamma'_0 \quad \Longrightarrow \quad \epsilon(u) \in \Gamma'$

(iv) $(\forall) x \in \Gamma' \quad \Longrightarrow \quad x^{-1} \in \Gamma'$.

A subgroupoid $(\Gamma';\Gamma'_0)$ of $(\Gamma;\Gamma_0)$ is **wide** if $\Gamma'_0 = \Gamma_0$. $\triangle$

Definition 2.2. A *normal subgroupoid* of a groupoid $(\Gamma;\Gamma_0)$ is a wide subgroupoid N of Γ such that: for any $\lambda \in N$ and any $x \in \Gamma$ such that $\beta(x) = \alpha(\lambda) = \beta(\lambda)$ we have $x \cdot \lambda \cdot x^{-1} \in N$.$\triangle$

Example 2.1. (a) If $(\Gamma;\Gamma_0)$ is a groupoid, then $\epsilon(\Gamma_0) = \{\widetilde{u} \quad | \quad u \in \Gamma_0\}$ is a normal subgroupoid of Γ over Γ_0, called the *nul subgroupoid* of Γ.

(b) If $(\Gamma;\Gamma_0)$ is a groupoid, then $Is(\Gamma) = \bigcup_{u\in\Gamma_0} \Gamma_u^u$ is a normal subgroupoid of Γ over Γ_0, called the *inner subgroupoid* of Γ.

(c) The *kernel* of a groupoid morphism $(f.f_0) : (\Gamma;\Gamma_0) \longrightarrow (\Gamma';\Gamma'_0)$ defined by: $Kerf = \{x \in \Gamma \quad | \quad f(x) \in \epsilon'(\Gamma'_0)\}$ is a normal subgroupoid of Γ over Γ_0.$\triangle$

Proposition 2.1. Let $(f, f_0) : (\Gamma;\Gamma_0) \longrightarrow (\Gamma';\Gamma'_0)$ be a groupoid morphism. Then the following assertions hold:

(i) If $(\Omega';\Omega'_0)$ is a subgroupoid of $(\Gamma';\Gamma'_0)$, then $(f^{-1}(\Omega'); f_0^{-1}(\Omega'_0))$ is a subgroupoid of $(\Gamma;\Gamma_0)$.

(ii) If Ω' is a normal subgroupoid of Γ', then $f^{-1}(\Omega')$ is a normal subgroupoid of Γ such that $Kerf \subseteq f^{-1}(\Omega)$.

Proof. (i) We prove that $(f^{-1}(\Omega'); f_0^{-1}(\Omega'_0))$ satisfies the conditions of Definition 2.1.

- $\alpha(f^{-1}(\Omega')) \subseteq f_0^{-1}(\Omega'_0)$. Indeed, if $u \in \alpha(f^{-1}(\Omega'))$ it follows that $u = \alpha(x)$ with $x \in f^{-1}(\Omega')$. Then $f_0(u) = f_0(\alpha(x)) = \alpha'(f(x)) \in \Omega'_0$, since $f(x) \in \Omega'$ and $\alpha'(\Omega') \subseteq \Omega'_0$. Hence, $u \in f_0^{-1}(\Omega'_0)$. Similarly, $\beta(f^{-1}(\Omega')) \subseteq f_0^{-1}(\Omega'_0)$.

- Let $x, y \in f^{-1}(\Omega')$ such that $x \cdot y$ is defined, i.e. $\beta(x) = \alpha(y)$. It follows that $f(x), f(y) \in \Omega'$ and $\beta'(f(x)) = f_0(\beta(x)) = f_0(\alpha(y)) = \alpha'(f(y))$; hence, $f(x) \cdot f(y)$ is defined in Γ'. Then $f(x) \cdot f(y) \in \Omega'$, since Ω' is subgroupoid. Then $f(x \cdot y) \in \Omega'$, i.e. $x \cdot y \in f^{-1}(\Omega')$.Therefore, the condition (ii) of Definition 2.1. is verified.

- For every $u \in f_0^{-1}(\Omega'_0)$ we have $\epsilon(u) \in f^{-1}(\Omega')$. Indeed, $f_0(u) \in \Omega'_0$ and we have $\epsilon'(f_0(u)) \in \Omega'$, since Ω' is subgroupoid. Then $f(\epsilon(u)) \in \Omega'$, i.e. $\epsilon(u) \in f^{-1}(\Omega')$.

- For every $x \in f^{-1}(\Omega')$, we have $x^{-1} \in f^{-1}(\Omega')$. Indeed, from $f(x) \in \Omega'$ follows $(f(x))^{-1} \in \Omega'$, since Ω' is subgroupoid. Then $f(x^{-1}) \in \Omega'$, i.e. $x^{-1} \in f^{-1}(\Omega')$.

(ii) In view of (i) follows that $f^{-1}(\Omega';\Gamma_0)$ is a subgroupoid of $(\Gamma;\Gamma_0)$.

Let $\lambda \in f^{-1}(\Omega')$ and $x \in \Gamma$ such that $\beta(x) = \alpha(\lambda) = \beta(\lambda)$ and we prove that $x \cdot \lambda \cdot x^{-1} \in f^{-1}(\Omega')$.

Indeed, we have $f(\lambda) \in \Omega'$ and $\beta'(f(x)) = f_0(\beta(x)) = f_0(\alpha(\lambda)) = \alpha'(f(\lambda))$ and $\beta'(f(x)) = f_0(\beta(x)) = f_0(\beta(\lambda)) = \beta'(f(\lambda))$. From $f(\lambda) \in \Omega', \quad \beta'(f(x)) = \alpha'(f(\lambda)) =$

$\beta'(f(\lambda))$ and the fact that Ω' is normal in Γ' follows $f(x)\cdot f(\lambda)\cdot(f(x))^{-1}\in\Omega'$. Hence, $f(x\cdot\lambda\cdot x^{-1})\in\Omega'$, i.e. $x\cdot\lambda\cdot x^{-1}\in f^{-1}(\Omega')$. Therefore, $f^{-1}(\Omega')$ is normal.

- We have $Kerf\subseteq f^{-1}(\Omega')$. Indeed, for $x\in Kerf$, we have $f(x)=\epsilon'(u')$ with $u'\in\Gamma'_0$ and by the condition (iii) of Definition 2.1. follows $\epsilon'(u')\in\Omega'$. Then $f(x)\in\Omega'$, i.e. $x\in f^{-1}(\Omega')$. $\triangle$

Corollary 2.1. *Let $f:\Gamma\longrightarrow\Gamma'$ be a Γ_0- groupoid morphism. Then the following assertions hold:*

(i) If $(\Omega';\Omega'_0)$ is a subgroupoid of $(\Gamma';\Gamma_0)$, then $(f^{-1}(\Omega');f_0^{-1}(\Omega'_0))$ is a subgroupoid of $(\Gamma;\Gamma_0)$.

(ii) If Ω' is a normal subgroupoid of Γ', then $f^{-1}(\Omega')$ is a normal subgroupoid of Γ such that $Kerf\subseteq f^{-1}(\Omega)$.

Proof. We apply the Proposition 2.1.$\triangle$

Remark 2.1. If $(f,f_0):(\Gamma;\Gamma_0)\longrightarrow(\Gamma';\Gamma'_0)$ is a groupoid, then not always, $Imf=\{f(x)\mid x\in\Gamma\}$ is a subgroupoid of Γ'. For example, let

$$\Gamma=\{(0,0);(0,1);(1,0);(1,1)\}=B\times B$$

the coarse groupoid associated to set $B=\{0,1\}$ and let the map $f:\Gamma\longrightarrow Z$ defined by $f(0,0)=0; f(0,1)=1; f(1,0)=-1; f(1,1)=0$. W e denote by $f_0:B\longrightarrow\{0\}$ the map defined by $f_0(0)=0$ and $f_0(1)=0$. We can prove easily the conditions of Definition 1.3. are satisfied for the pair (f,f_0) of the coarse groupoid Γ over B into the group additiv $\mathbf{Z}$ of entiers numbers over $\{0\}$, having $Imf=\{0,-1,1\}$ which is not a subgroup of $\mathbf{Z}$. Hence Imf is not a subgroupoid. $\triangle$

Definition 2.3. A *strong morphism of groupoids* or *groupoid strong morphism* is a groupoid morphism $(f,f_0):(\Gamma;\Gamma')\longrightarrow(\Gamma';\Gamma'_0)$ such that the following condition holds:

$$\text{(4)}\qquad \text{for every}\quad (f(x),f(y))\in\Gamma'_{(2)}\quad\text{we have}\quad (x,y)\in\Gamma_{(2)}.\triangle$$

Remark 2.2. The concept of strong morphism has considered by A. Ramsay (cf. [18]) in the case of Brandt groupoids, called it *true morphism* of groupoids. $\triangle$

Remark 2.3. If (f,f_0) is a strong morphism of groupoids, then

$$f_u:\Gamma_u\longrightarrow\Gamma'_{f_0(u)}; f^v:\Gamma^v\longrightarrow(\Gamma')^{f_0(v)}\text{ and } f_u^v:\Gamma_u^v\longrightarrow(\Gamma')_{f_0(u)}^{f_0(v)}$$

are also strong morphisms of groupoids. $\triangle$

Theorem 2.1. *(i) If $(f,f_0):(\Gamma;\Gamma_0)\longrightarrow(\Gamma';\Gamma'_0)$ is a groupoid morphism such that the map f_0 is injective, then (f,f_0) is a groupoid strong morphism.*

(ii) Every Γ_0- morphism of groupoids $f:\Gamma\longrightarrow\Gamma'$ is a groupoid strong morphism.

Proof. (i) We suppose that $(f(x),f(y))\in\Gamma'_{(2)}$, with $x,y\in\Gamma$. Then

$$\beta'(f(x))=\alpha'(f(y))\Longrightarrow(\beta'\circ f)(x)=(\alpha'\circ f)(y)\Longrightarrow(f_0\circ\beta)(x)=(f_0\circ\alpha)(y)\Longrightarrow$$

$$\Longrightarrow f_0(\beta(x))=f_0(\alpha(y))\Longrightarrow\beta(x)=\alpha(y)(\text{since } f_0 \text{ is injective})\Longrightarrow(x,y)\in\Gamma_{(2)}.$$

Hence (f,f_0) is a groupoid strong morphism.

(ii) This is a consequence of (ii), since $f_0=Id_{\Gamma_0}$. $\triangle$

Example 2.2. (i) The morphism $\alpha\times\beta:\Gamma\longrightarrow\Gamma_0\times\Gamma_0$, given in Definition 1.2., is a groupoid strong morphism.

(ii) The canonical morphism (f_{Γ}^*, f) of induced groupoid $f^*(\Gamma)$ of Γ by $f : X \longrightarrow \Gamma_0$ is not a groupoid strong morphism. $\triangle$

Proposition 2.2. Let $(f, f_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma'_0)$ be a groupoid strong morphism. Then the following assertions hold:

(i) If $(\Omega; \Omega_0)$ is a subgroupoid of $(\Gamma; \Gamma_0)$, then $(f(\Omega); f_0(\Omega_0))$ is a subgroupoid of $(\Gamma'; \Gamma'_0)$. In particular, Imf is a subgroupoid of Γ' over Imf_0.

(ii) If f is surjective and Ω is a normal subgroupoid of Γ, then $f(\Omega)$ is a normal subgroupoid of Γ'.

Proof. (i) We have $(\alpha'(f(\Omega)) \subseteq f_0(\Omega_0)$. Indeed, for any $u' \in \alpha'(f(\Omega))$ exists $y' \in f(\Omega)$ such that $u' = \alpha'(y')$. For $y' \in f(\Omega)$ exists $y \in \Omega$ such that $f(y) = y'$. Then $u' = \alpha'(y') = \alpha'(f(y)) = f_0(\alpha(y))$; hence $u' \in f_0(\Omega)$, since $\alpha(y) \in \Omega_0$. Similarly, $\beta'(f(\Omega)) \subseteq f_0(\Omega_0)$.

- We have $\epsilon'(u') \subseteq f(\Omega)$, for all $u' \in f_0(\Omega_0)$.Indeed, for $u' \in f_0(\Omega_0)$ exists $u \in \Omega_0$ such that $u' = f_0(u) \implies \epsilon'(u') = \epsilon'(f_0(u)) = f(\epsilon(u)) \in f(\Omega)$, since $\epsilon(u) \in \Omega$.

- Let $x', y' \in f(\Omega)$ such that $x' \cdot y'$ is defined.We prove that $x' \cdot y' \in f(\Omega)$. Indeed, $x' = f(x), y' = f(y)$ with $x, y \in \Omega$. Since, $x' \cdot y'$ is defined it implies that $(f(x), f(y)) \in \Gamma'_{(2)}$, and we have $(x, y) \in \Gamma_{(2)}$, since f is a groupoid strong morphism. Hence $x \cdot y$ is defined. We have $x \cdot y \in \Omega$, since Ω is subgroupoid of Γ,and therefore $x' \cdot y' = f(x) \cdot f(y) = f(x \cdot y) \in f(\Omega)$.

- For any $x' \in f(\Omega)$ we have $(x')^{-1} \in f(\Omega)$. Indeed, $x' = f(x)$, with $x \in \Omega \implies (x')^{-1} = (f(x))^{-1} = f(x^{-1}) \in f(\Omega)$, since $x^{-1} \in \Omega$.

Therefore $(f(\Omega); f_0(\Omega_0))$ is a subgroupoid of $(\Gamma'; \Gamma'_0)$.

(ii) By (i) $(f(\Omega); \Gamma'_0)$ is a subgroupoid of $(\Gamma'; \Gamma'_0)$, since f_0 is surjective.

Let $\lambda' \in f(\Omega)$ and $x' \in \Gamma'$ such that $\beta'(x') = \alpha'(\lambda) = \beta'(\lambda)$. We prove that $x' \cdot \lambda' \cdot (x')^{-1} \in f(\Omega)$.

Indeed, $\lambda' = f(\lambda)$ with $\lambda \in \Omega$ and $x' = f(x)$ with $x \in \Gamma$, since f is surjective. From $(f(x), f(\lambda)), (f(\lambda), (f(x))^{-1}) \in \Gamma'_{(2)}$, it follows that $(x, \lambda), (\lambda, x^{-1}) \in \Gamma_{(2)}$, since f is a groupoid strong morphism. It follows that $x \cdot \lambda \cdot x^{-1}$ is defined and $x \cdot \lambda \cdot x^{-1} \in \Omega$, since Ω is normal in Γ. Hence, $f(x \cdot \lambda \cdot x^{-1}) \in f(\Omega)$ and

$$f(x) \cdot f(\lambda) \cdot f(x^{-1}) = f(x) \cdot f(\lambda) \cdot (f(x))^{-1} = x' \cdot \lambda' \cdot (x')^{-1} \in f(\Omega).$$

Thus, $f(\Omega)$ is a normal subgroupoid of Γ'. $\triangle$

Corollary 2.2. *Let $f : \Gamma \longrightarrow \Gamma'$ be a Γ_0- morphism of groupoids. Then the following assertions hold:*

(i) If $(\Omega; \Omega_0)$ is a subgroupoid of $(\Gamma; \Gamma_0)$, then $(f(\Omega); f_0(\Omega_0))$ is a subgroupoid of $(\Gamma'; \Gamma_0)$. In particular, Imf is a subgroupoid of Γ' over Imf_0.

(ii) If f is surjective and Ω is a normal subgroupoid of Γ, then $f(\Omega)$ is a normal subgroupoid of Γ'.

Proof. We apply Theorem 2.1.(ii) and Proposition 2.2.$\triangle$

If $(\Gamma; \Gamma_0)$ is a groupoid, we denote by $\mathcal{S}(\Gamma; \Gamma_0)$ (resp., $\mathcal{N}(\Gamma)$) the set of the subgroupoids (resp., the normal subgroupoids) of $(\Gamma; \Gamma_0)$.

If $(f, f_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma'_0)$ is a groupoid morphism, we denote by $\widetilde{\mathcal{S}}(\Gamma; \Gamma_0)$ (resp., $\widetilde{\mathcal{N}}(\Gamma)$) the set of the subgroupoids (resp., the normal subgroupoids) of $(\Gamma; \Gamma_0)$, which contains the kernel of f, i.e.:

$\mathcal{S}(\Gamma; \Gamma_0) = \{\Omega \mid \Omega$ *is a subgroupoid of* $(\Gamma; \Gamma_0)$ *such that* $Kerf \subseteq \Omega\}$

$\mathcal{N}(\Gamma) = \{\Omega \mid \Omega$ *is a normal subgroupoid of* Γ *such that* $Kerf \subseteq \Omega\}$.

In view of Example 2.1.(a),(b),(c) we have that $\mathcal{S}(\Gamma;\Gamma_0)$, $\mathcal{N}(\Gamma)$), $\widetilde{\mathcal{S}}(\Gamma;\Gamma_0)$ and $\widetilde{\mathcal{N}}(\Gamma)$) are nonempty sets.

Theorem 2.2. (**the correspondence theorem for subgroupoids**) *For any surjective strong morphism of groupoids* $(f,f_0):(\Gamma;\Gamma_0)\longrightarrow(\Gamma';\Gamma'_0)$, *there exists a bijection from the set* $\mathcal{S}(\Gamma';\Gamma'_0)$ *of the subgroupoids of* $(\Gamma';\Gamma'_0)$ *to the set* $\mathcal{S}(\Gamma;\Gamma_0)$ *of the subgroupoids of* $(\Gamma;\Gamma_0)$.

Proof. We take the maps

$$\varphi:\widetilde{\mathcal{S}}(\Gamma;\Gamma_0)\longrightarrow\mathcal{S}(\Gamma';\Gamma'_0)$$

and

$$\psi:\mathcal{S}(\Gamma';\Gamma'_0)\longrightarrow\widetilde{\mathcal{S}}(\Gamma;\Gamma_0),$$

given by:

$$\varphi(\Omega)=f(\Omega),\quad(\forall)\Omega\in\widetilde{\mathcal{S}}(\Gamma);\tag{5}$$

$$\psi(\Omega')=f^{-1}(\Omega'),\quad(\forall)\Omega'\in\mathcal{S}(\Gamma').\tag{6}$$

By Proposition 2.2.(i), it follows that $f(\Omega)$ is a subgroupoid of Γ', for all $\Omega\in\widetilde{\mathcal{S}}(\Gamma)$. Hence, φ is well-defined. Also, by Proposition 2.1.(i),it follows that $f^{-1}(\Omega')$ is a subgroupoid of Γ, for all $\Omega'\in\mathcal{S}(\Gamma')$. Hence, ψ is well-defined.

The maps φ and ψ given by (5) and (6) have the following properties:

$$\psi\circ\varphi=Id_{\widetilde{\mathcal{S}}(\Gamma)}\quad\text{and}\quad\varphi\circ\psi=Id_{\mathcal{S}(\Gamma')}.\tag{7}$$

The equalities (7) are equivalently with:

$$f^{-1}(f(\Omega))=\Omega,\quad(\forall)\Omega\in\widetilde{\mathcal{S}}(\Gamma)\quad\text{and}\quad f(f^{-1}(\Omega'))=\Omega',\quad(\forall)\Omega'\in\mathcal{S}(\Gamma').$$

- (a) If $x\in\Omega$, then $f(x)\in f(\Omega)$ and we have $x\in f^{-1}(f(\Omega))$. Hence, $\Omega\subseteq f^{-1}(f(\Omega))$.

- (b) If $x\in f^{-1}(f(\Omega))$, then $f(x)\in f(\Omega)$ and exists $y\in\Omega$ such that $f(x)=f(y)$. We have $f(x)\cdot(f(y))^{-1}=\epsilon'(f(y))$. Therefore, $f(x\cdot y^{-1})=\epsilon'(f(y))$ and we obtain that $x\cdot y^{-1}\in Kerf$. Thus, $x\cdot y^{-1}=z$, with $z\in Kerf\subseteq\Omega$. Hence, $x=z\cdot y$ with $y,z\in\Omega$ and we have $x\in\Omega$. Therefore, $f^{-1}(f(\Omega))\subseteq\Omega$.

From (a) and (b), it follows the first equality of ($7'$).

- (c) If $x'\in f(f^{-1}(\Omega'))$, then $x'=f(x)\in f(\Omega)$ with $x\in f^{-1}(\Omega')$ and follows $f(x)\in\Omega'$. Hence $x'\in\Omega'$. Therefore, $f(f^{-1}(\Omega'))\subseteq\Omega'$.

- (d) If $x'\in\Omega'$, exists $x\in\Gamma$ such that $x'=f(x)$, since f is surjective. Then $x\in f^{-1}(\Omega')$, since $f(x)\in\Omega'$.Therefore, $x'\in f(f^{-1}(\Omega'))$. Hence, $\Omega'\subseteq f(f^{-1}(\Omega'))$.

From (c) and (d), it follows the second equality of ($7'$).

From (7), it follows that ψ is invertible. Hence, ψ is a bijection.$\triangle$

Corollary 2.3. (**the correspondence theorem for subgroupoids via a Γ_0-morphism**) *For any surjective* Γ_0*-morphism of groupoids* $f:\Gamma\longrightarrow\Gamma'$, *there exists a bijection from the set* $\mathcal{S}(\Gamma';\Gamma_0)$ *of the subgroupoids of* $(\Gamma';\Gamma_0)$ *to set* $\mathcal{S}(\Gamma;\Gamma_0)$ *of the subgroupoids of* $(\Gamma;\Gamma_0)$.

Proof. It is a consequence of Theorems 2.1.(ii) and 2.2. $\triangle$

Applying the Propositions 2.1.(ii) and 2.2.(ii) we can prove similarly the following theorem.

Theorem 2.3. (**the correspondence theorem for normal subgroupoids**) *For any surjective strong morphism of groupoids* $(f, f_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma'_0)$, *there exists a bijection from the set* $\mathcal{N}(\Gamma')$ *of the normal subgroupoids of* $(\Gamma'; \Gamma'_0)$ *to the set* $\widetilde{\mathcal{N}}(\Gamma)$ *of the normal subgroupoids of* $(\Gamma; \Gamma_0)$ *which contains* $Kerf$.$\triangle$

Corollary 2.4. (**the correspondence theorem for normal subgroupoids via a** Γ_0**-morphism**) *For any surjective* Γ_0*-morphism of groupoids* $f : \Gamma \longrightarrow \Gamma'$, *there exists a bijection from the set* $\mathcal{N}(\Gamma')$ *of the normal subgroupoids of* $(\Gamma'; \Gamma_0)$ *to the set* $\widetilde{\mathcal{N}}(\Gamma)$ *of the normal subgroupoids of* $(\Gamma; \Gamma_0)$ *which contains* $Kerf$.

Proof. It is a consequence of Theorems 2.1.(ii) and 2.3. $\triangle$

Remark 2.3. (i) The Theorems 2.2 and 2.3. generalise the correspondence theorems for subgroups and normal subgroups by a surjective morphism of groups.

(ii) The Theorems 2.2 and 2.3. are not true for arbitrary surjective morphisms of groupoids.

(iii) If $(f, f_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma'_0)$ is a groupoid strong morphism, then $(\widetilde{f}, \widetilde{f_0}) : (\Gamma; \Gamma_0) \longrightarrow (Imf; Imf_0)$ is a surjective strong morphism of groupoids, where $\widetilde{f}, \widetilde{f_0}$ are given by $\widetilde{f}(x) = f(x), \quad (\forall) x \in \Gamma \quad$ and $\quad \widetilde{f_0}(u) = f_0(u), \quad (\forall) u \in \Gamma_0$.$\triangle$

Theorem 2.4. *For any strong morphism of groupoids* $(f, f_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma'_0)$, *there exists a bijection from the set* $\mathcal{S}(Imf; Imf_0)$ *of the subgroupoids of* $(Imf; Imf_0)$ *to the set* $\mathcal{S}(\Gamma; \Gamma_0)$ *of the subgroupoids of* $(\Gamma; \Gamma_0)$.

Proof. We apply the Theorem 2.2. of the strong morphism of groupoids $(\widetilde{f}, \widetilde{f_0}) : (\Gamma; \Gamma_0) \longrightarrow (Imf; Imf_0)$ associated to (f, f_0). $\triangle$

Similarly, we can prove the following theorem.

Theorem 2.5. *For any strong morphism of groupoids* $(f, f_0) : (\Gamma; \Gamma_0) \longrightarrow (\Gamma'; \Gamma'_0)$, *there exists a bijection from the set* $\mathcal{N}(\Gamma')$ *of the normal subgroupoids of* $(\Gamma'; \Gamma'_0)$ *to the set* $\widetilde{\mathcal{N}}(\Gamma)$ *of the normal subgroupoids of* $(\Gamma; \Gamma_0)$ *which contains* $Kerf$. $\triangle$

Corollary 2.5. *Let* $f : \Gamma \longrightarrow \Gamma'$ *a* Γ_0*- morphism of groupoids. Then the following assertions hold:*

(i) There exists a bijection from the set $\mathcal{S}(Imf; Imf_0)$ *of the subgroupoids of* $(Imf; Imf_0)$ *to the set* $\mathcal{S}(\Gamma; \Gamma_0)$ *of the subgroupoids of* $(\Gamma; \Gamma_0)$.

(ii) There exists a bijection from the set $\mathcal{N}(Imf)$ *of the normal subgroupoids of* $(Imf; Imf_0)$ *to the set* $\widetilde{\mathcal{N}}(\Gamma)$ *of the normal subgroupoids of* $(\Gamma; \Gamma_0)$ *which contains* $Kerf$.

Proof. This is a consequence of Theorems 2.1.(ii), 2.4 and 2.5. $\triangle$

Remark 2.4. We conclude that the strong morphisms of groupoids have the same properties as the morphisms of groups. $\triangle$

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