Invariant Operators on Real and Complex Manifolds

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Abstract

The development of decomposition theory of the curvature tensors under the action of some groups was initiated in [31], [30]. Since these results and their ideas are very useful in the studies of some problems in geometry and topology of manifolds, many mathematicians have worked using this algebraic treatment of curvature tensors.

Interesting applications of the theory of decompositions are used in the study of submanifolds in conformal differential geometry, classification of almost complex manifolds [6], almost product manifolds [27]. In the Kähler geometry, decompositions of the K-curvature tensors were given in [19], [26], [32], for quaternionic Kähler manifold [35] and the case of contact geometry is studied in [18], [21]. In [20], [23] is investigated the trace decomposition problem. All these splittings are in principle consequences of the general theorems on group representations [44].

This paper develops the original ideas of C.Udrişte regarding the decomposition problem of geometrical objects, introducing some (r, r)-tensor algebras $Inv(r)$, $CInv(r), HInv(r)$, $CHInv(r)$ of operators invariant under certain groups ([13], [15], [36], [37]). These invariant tensor algebras, having the elements interpreted like endomorphisms on the space $T^1_{r-1}(M)$, enable us to get a trace decomposition of this space, the results of [20] being special cases of our theory.

The focus is on the infinite subset of projections which do provide good insight in some problems of differentiable manifolds. Let us mention that the Weyl projective curvature tensor, Weyl conformal curvature tensor, H-projective curvature tensor, Bochner curvature tensor and the Thomas projective connection, Thomas conformal connection, H-projective connection and complex conformal connection are produced by this type of operators. Also, using some invariant operators which are projections one gets the splitting of the space of tensors of type $(1,3)$ into three components invariant under some special groups into infinitely many ways. We should remark that in particular cases, one finds the Strichartz decomposition , Singer-Thorpe-Nomizu decomposition, respectively Sitaramayya-Mori decomposition of the space of curvature tensors.

The extension of the invariant operators to $\mathcal{T}^1_{r-1}(M)$ and to the $\mathcal{F}(M)$-module $\mathcal{A}^1_{r-1}(M)$, a new space which is required by our theory, leads us to the problem of decompositions of tensors and connections. Finally, one gets invariants for some transformations of geometrical object fields, extending the Thomas-Weyl theory.

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Along the line developed in [43], [45], we are studied properties of pairs of connections (Γ, Π), where Γ is an affine connection and Π is the Thomas projective connection, respectively Thomas conformal connection or Γ is the Levi-Civita connection and Π is the H-projective related J-connection, respectively complex conformal connection, we investigate some pairs of geometrical object fields associated by certain invariant operators.

The study of geodesically related Riemannian manifolds [43], [41] was extended to semi-symmetric spaces geodesically related in [5] and to subgeodesic correspondence in [28].

We continue the research in this direction studying in the last section some geodesic and subgeodesic mappings [7], [8], [11], [14]. Certain properties of semi-symmetric connection on Weyl generalized manifolds are also presented [9], [12].

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1 Pairs of geometrical object fields via absolute invariant operators

The complete decomposition in Riemannian projective geometry was studied in [1], where is solved the problem under the action of $SO(n)$. The splitting of the space of curvature tensors under the action of $GL(V)$ was obtained in [33]. It is possible to processe and develop the decomposition theory introducing a special algebra of invariant operators, built with the Kronecker tensor.

1.1 Absolut invariant tensors algebra

Let V be a real n-dimensional vector space, where $n > 2$. Let $T^r_r(V)$ be the vector space of all tensors of type (r,r) and δ^i_j be the symbol of Kronecker. In $T^r_r(V)$ we consider the vector subspace $Inv(r) = \left\{\sum_{\sigma\in S_r} x_\sigma \delta^{i_{\sigma(1)}}_{j_1} ... \delta^{i_{\sigma(r)}}_{j_r} | x_\sigma \in \mathbf{R},\ \sigma \in S_r\right\}$, where S_r is the group of permutations. Any element $P \in Inv(r)$, which is an absolute invariant tensor (i.e., $\forall A \in GL(V),\ A \circ P = P$), is interpreted like an endomorphism on $T^1_{r-1}(V)$, producing a trace decomposition of this space. So, $Inv(r)$ becomes an algebra of absolute invariant operators, the product PQ of two elements P, Q being given by the rule $P^{i_1 \dots i_{r-1}\ i_r}_{j_1\ \ \ \ \ \ \ \ j_2 \dots j_r} Q^{j_2 \dots j_r\ j_{r+1}}_{i_r\ \ \ \ \ \ \ \ i_{r+1} \dots i_{2r-1}}$.

It is interesting to study the pair (T, PT), $\forall P \in Inv(r)$, $T \in T^1_{r-1}(V)$, where
$(PT)(\omega, X_1, ..., X_{r-1}) = \sum_{\sigma\in S_{r-1}} [x_\sigma T(\omega, X_{\sigma(1)}, ..., X_{\sigma(r-1)}) +$
$+ \sum_{k=1}^{r-1} x^k_\sigma (C^1_k T)(X_{\sigma(1)}, ..., \hat{X}_{\sigma(k)}, ..., X_{\sigma(r-1)}) \omega(X_{\sigma(k)})]$,
$\forall \omega \in V'$, $\forall X_k, X_{\sigma(k)} \in V, \sigma \in S_r, C^1_k$ being the contraction map, $\forall k \in \{1, ..., r-1\}$.

Important geometrical meanings have the cases $r = 3,\ 4$, which appear in the study of connections, torsion tensors, curvature tensors, Weyl projective curvature tensors etc.

The operator $P = \{P^{bc\ \ r}_{a\ \ \ st}\} \in Inv(3)$ having the simplified expression $P = x_1 I_1 + ... + x_6 I_6$, where $\{I_1, ..., I_6\}$ is the basis, acts like un endomorphism on $T^1_2(V)$. Endowing $Inv(3)$ with a structure of Lie over R, it is possible to clasify its subalgebras

Theorem 1.1.1 *$\mathcal{D}_1 = Span\{I_1, I_2, I_6\}$ and $\mathcal{D}_2 = Span\{I_1, I_2, I_5\}$ are 3-dimensi-onal Lie subalgebras and $\mathcal{B}_1 = Span\{I_1, I_2\}, \mathcal{B}_2 = Span\{I_1, I_3\}, \mathcal{B}_3 = Span\{I_1, I_5\}$ are 2-dimensional Lie subalgebras of the Lie algebra $Inv(3)$.*

Moreover, $\mathcal{D}_1$ and $\mathcal{D}_2$ are solvable subalgebras, having the abelian ideals $\mathcal{R}_1 = \{P = \alpha I_2 - n\alpha I_6 \mid \alpha \in \boldsymbol{R}\}$, respectively $\mathcal{R}_2 = \{P = \alpha I_2 - n\alpha I_5 \mid \alpha \in \boldsymbol{R}\}$.

Any operator $\mathcal{P} = \{\mathcal{P}_a^{bcd}{}_{stp}^{r}\} \in Inv(4)$ has the simplified expression $\mathcal{P} = \sum_{i=1}^{24} y_i \mathcal{I}_i$, where $\{\mathcal{I}_1, ..., \mathcal{I}_{24}\}$ is the basis and is interpreted like an endomorphism on $T_3^1(V)$. The multiplication table $\mathcal{PQ}$, $\mathcal{P}_a^{bcd}{}_{stp}^{r} \mathcal{Q}_r^{stp}{}_{jkl}^{i}$, of two elements of $Inv(4)$ was determined using a Borland C++ Programme.

The image $P_a^{bc}{}_{st}^{r} T_{bc}^a$, where $P = \{P_a^{bc}{}_{st}^{r}\} \in Inv(3)$ and $T = \{T_{bc}^a\} \in T_2^1(V)$, represents a traceless decomposition of T. Let $\mathcal{P} = \{\mathcal{P}_i^{jkl}{}_{stp}^{r}\} \in Inv(4)$ and $T = \{T_{jkl}^i\} \in T_3^1(V)$. The image $\mathcal{P}_i^{jkl}{}_{stp}^{r} T_{jkl}^i$ is a traceless decomposition of T. The results obtained by D. Krupka, J. Mikesh concerning the traceless decomposition of tensors correspond to some particular cases of our theory.

Theorem 1.1.2 *There are infinitely many endomorphisms in $Inv(3)$, respectively $Inv(4)$, having traceless images.*

1.2 Generalization of Strichartz decomposition

Some absolut invariant operators which are projections do provide insight in some problems of differential geometry. Let us mention that projections from $\mathcal{I}nv(4)$ give the Weyl projective curvature tensor and affine transformations of elements from $\mathcal{I}nv(3)$ produce the Thomas projective connection. Moreover, properties of the projections enable us to get splittings of the space of tensors of type $(1, 3)$. So, for geometrical reasons, we study the subset of projections of $\mathcal{I}nv(r)$.

Theorem 1.2.1 *There are infinitely many projective projections $P \in Inv(3)$, respectively $\mathcal{P} \in Inv(4)$.*

These infinite families of solutions and their geometric representations can be determined using "MathCad Plus". We should remark that, for $r = 3$, in particular if $x_1 = x_3 = 0$, introducing the parameters $x_5 = \lambda, x_6 = \beta$, then x_2 and x_4, solutions of the equation $z^2 + \frac{1}{2}(\lambda + \mu - 1) + \lambda\mu = 0$, belong to one sheet hyperboloids, independent of the dimension of the vector space V.

R.S. Strichartz [33] found a decomposition of $\mathcal{K}(V)$, the space of tensors of type $(1, 3)$, verifying $R_{stl}^r + R_{slt}^r = 0$ and the first Bianchi identity, using properties of the representation theory.

The projective projections from $Span\{\mathcal{I}_1, \mathcal{I}_7, ..., \mathcal{I}_{12}\} \subset Inv(4)$ produce infinitely many splittings of the space $T_3^1(V)$ into three subspaces, invariant under to the group $GL(V)$.

Theorem 1.2.2 *There are infinitely many nonvanishing projective projections $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in Span\{\mathcal{I}_1, \mathcal{I}_7, ..., \mathcal{I}_{12}\}$, such that $\mathcal{P} + \mathcal{Q} + \mathcal{R} = \mathcal{I}_1, \mathcal{PQ} = \mathcal{QP} = \mathcal{RQ} = \mathcal{QR} = 0$ and $T_3^1(V) = Im\mathcal{P} \oplus Im\mathcal{Q} \oplus Im\mathcal{R}$ holds, where*

$$\mathcal{P} = \mathcal{I}_1 + \sum_{i=7}^{12} y_i \mathcal{I}_i,$$

$$\mathcal{Q} = \frac{y_8 - y_7}{2}(\mathcal{I}_7 - \mathcal{I}_9) + \frac{y_{10} - y_8}{2}(\mathcal{I}_8 - \mathcal{I}_{10}) + \frac{y_{12} - y_{11}}{2}(\mathcal{I}_{12} - \mathcal{I}_{11}),$$

$$\mathcal{R} = -\frac{y_8 + y_7}{2}(\mathcal{I}_7 + \mathcal{I}_9) - \frac{y_{10} + y_8}{2}(\mathcal{I}_8 + \mathcal{I}_{10}) - \frac{y_{12} + y_{11}}{2}(\mathcal{I}_{12} + \mathcal{I}_{11}),$$

where $1+ny_7+y_8+y_{12}=0, ny_9+y_{10}+y_{11}=0$.
Proposition 1.2.1 *Any tensor $R\in T_3^1(V)$ splitts into infinitely many ways $R=R'+R"+R'''$, such that $R'_{ij}=0, R"_{ij}$ is a symmetric tensor, R'''_{ij} is a skew symmetric tensor.*
Proposition 1.2.2 *The endomorphisms $\mathcal{P}\in Span\{\mathcal{I}_1,\mathcal{I}_7,...,\mathcal{I}_{12}\}$ of the algebra $\mathcal{I}nv(4)$ applying $\mathcal{K}(V)$ into the same subspace are*

$$\mathcal{P}=y_1\mathcal{I}_1+y_7\mathcal{I}_7-y_7\mathcal{I}_8-y_{10}\mathcal{I}_9+y_{10}\mathcal{I}_{10}-(y_7+y_{10})\mathcal{I}_{11}+(y_7+y_{10})\mathcal{I}_{12}.$$

Into this set of infinite family of endomorphisms from $Span\{\mathcal{I}_1,\mathcal{I}_7,...,\mathcal{I}_{12}\}$, its finite subset of projections has particular geometrical meanings.
Remark 1.2.1 In the particular case

$$\mathcal{P}=\mathcal{I}_1-\frac{n}{n^2-1}\mathcal{I}_7+\frac{n}{n^2-1}\mathcal{I}_8-\frac{1}{n^2-1}\mathcal{I}_9+\frac{1}{n^2-1}\mathcal{I}_{10}+\frac{1}{n+1}\mathcal{I}_{11}-\frac{1}{n+1}\mathcal{I}_{12},$$

$$\mathcal{Q}=\frac{1}{2(n+1)}(\mathcal{I}_7-\mathcal{I}_8-\mathcal{I}_9+\mathcal{I}_{10}-2\mathcal{I}_{11}+2\mathcal{I}_{12}), \mathcal{R}=\frac{1}{2(n-1)}(\mathcal{I}_7-\mathcal{I}_8+\mathcal{I}_9-\mathcal{I}_{10})$$

we find again the Strichartz decomposition [33] $\mathcal{K}(V)=Im\mathcal{P}\oplus Im\mathcal{Q}\oplus Im\mathcal{R}$ of the curvature tensors into three irreducible components under the action of the group $GL(V)$. This decomposition is not irreducible under $O(n)$ or $SO(n)$.

1.3 δ-decompositions of geometrical object fields

Let M be a differentiable n–dimensional manifold, $\mathcal{T}_r^r(M)$ the bundle of (r,r)–tensor fields of M. Then

$$\mathcal{I}nv(r)=\left\{P^{i_1...i_r}_{j_1...j_r}=\textstyle\sum_{\sigma\in S_r} f_\sigma\delta^{i_{\sigma(1)}}_{j_1}...\delta^{i_{\sigma(r)}}_{j_r}\mid f_\sigma\in\mathcal{F}(M),\ \sigma\in S_r\right\}$$

is the $\mathcal{F}(M)$-module of absolut invariant tensor fields.

We define the $\mathcal{F}(M)$-module denoted $\mathcal{A}^1_{r-1}(M)$, generated by the union of parallel afine spaces of geometrical object fields of type $(1,r-1)$ ([24]), whose difference or skew symmetric part with respect to a pair of indices is a tensor field of type $(1,r-1)$. Obviously, for $r=3$, the affine space $\mathcal{C}$, of affine connections on M, and the space $\mathcal{T}^1_2(M)$ are examples of such parallel spaces. Each element P of $\mathcal{I}nv(r)$ acts like an endomorphism on $\mathcal{T}^1_{r-1}(M)$ and induces an affine transformation on $\mathcal{A}^1_{r-1}(M)$, producing trace decomposition of this space.

Along the line developed by T.Y. Thomas, G. Vrănceanu which studied pairs of connections (Γ,Π), where Γ is an affine connection and Π is the projective Thomas connection, we generalize the theory studying the graph $(\Gamma,P\Gamma)$ of affine transformations P acting on the space $\mathcal{A}^1_2$.
Remark 1.3.1 If the projective projections act on the affine symmetric connections, then in certain cases, for particular values of the coeficients $x_1,...,x_6$, we get the pair of connections (Γ,Π), where $\Pi^r_{st}=\Gamma^r_{st}-\frac{1}{n+1}(\delta^r_s\Gamma_t+\delta^r_t\Gamma_s)$, $\Gamma_t=\Gamma^a_{at}$ is the projective Thomas connection.
Theorem 1.3.1 *There are infinitely many nonvanishing triplets $\mathcal{P},\mathcal{Q},\mathcal{R}$, affine transformations on $\mathcal{A}^1_3(M)$, of projective projections from $Span_{\mathcal{F}(M)}\{\mathcal{I}_1,\mathcal{I}_7,\dots,\mathcal{I}_{12}\}$, such that $\mathcal{P}+\mathcal{Q}+\mathcal{R}=\mathcal{I}_1$, $\mathcal{P}\mathcal{Q}=\mathcal{Q}\mathcal{P}=\mathcal{R}\mathcal{Q}=\mathcal{Q}\mathcal{R}=0$. Moreover, $\mathcal{A}^1_3(M)=Im\mathcal{P}\oplus Im\mathcal{Q}\oplus Im\mathcal{R}$.*

The relation between the geometrical object fields from $\mathcal{A}^1_2(M)$, respectively $\mathcal{A}^1_3(M)$ is determined by the next closed diagrame which reflects an invariance of gauge type.

Theorem 1.3.2 *Let* $P = x_1I_1 + x_2I_2 + x_6I_6$, $\mathcal{P} = y_1\mathcal{I}_1 + y_7\mathcal{I}_7 + y_8\mathcal{I}_8 + y_{21}\mathcal{I}_{21} + y_{22}\mathcal{I}_{22} + y_{24}\mathcal{I}_{24}$, $\mathcal{T} = z_1\mathcal{I}_1 + z_7\mathcal{I}_7 + z_8\mathcal{I}_8 + z_{21}\mathcal{I}_{21} + z_{22}\mathcal{I}_{22} + z_{24}\mathcal{I}_{24}$, $\mathcal{R} = \mathcal{I}_1 - \mathcal{I}_2$ *be affine transformations on* $\mathcal{A}_2^1(M)$, *respectively* $\mathcal{A}_3^1(M)$. *There are infinitely many projective projections* $\mathcal{P}, \mathcal{T}, P$ *such that next diagrame is closed*

$$(*)\qquad \begin{array}{ccc} \Gamma & \xrightarrow{P} & \Pi \\ \downarrow & & \downarrow \\ A & & \bar{A} \\ \mathcal{R}\downarrow & & \downarrow\mathcal{R} \\ R & & \pi \\ \mathcal{T}\searrow & & \swarrow\mathcal{P}\,, \\ & W & \end{array}$$

where (Γ, Π) *is a pair from* $\mathcal{A}_2^1(M)$, Γ *being an affine symmetric connection and* $\Pi = P\Gamma$ *the exotic Thomas connection,*

$(A, \bar{A})$ *is a pair from* $\mathcal{A}_3^1(M)$, *associated to* (Γ, Π) *by the rule* $A^r_{stl} = \dfrac{\partial \Gamma^r_{st}}{\partial x^l} + \Gamma^r_{mt}\Gamma^m_{sl}$,

(R, π) *is a pair from* $\mathcal{A}_3^1(M)$, R *being the curvature tensor field associated to* Γ *and* π *the exotic curvature tensor field associated to* π *and*

W *is the exotic projective Weyl curvature tensor field,* $W = (\mathcal{P} + \mathcal{Q})\,\pi = (\mathcal{T} + \mathcal{S})\,R$.

Corollary 1.3.1 *In the hypothesis of the previous theorem, if moreover the Ricci tensor associated to the affine connection* Γ *is symmetric, then the diagrame* $(*)$ *is closed for the next cases*

1) $P = I_1$, $\mathcal{T} = \mathcal{P} \in \{0, \mathcal{I}_1, y_7\mathcal{I}_7 + (1 - ny_7)\mathcal{I}_8,\ \mathcal{I}_1 - y_7\mathcal{I}_7 + (ny_7 - 1)\mathcal{I}_8\}$;

2) $P = I_1 - \dfrac{1}{n}I_2$, $\mathcal{T} = \mathcal{P} \in \{0, \mathcal{I}_1, y_7\mathcal{I}_7 + (1 - ny_7)\mathcal{I}_8,\ \mathcal{I}_1 - y_7\mathcal{I}_7 + (ny_7 - 1)\mathcal{I}_8\}$;

3) $P = I_1 + x_2I_2 - (1 + nx_2)I_6$, $x_2 \in \mathcal{F}(M) \setminus \left\{-\dfrac{1}{n}\right\}$,
$\mathcal{T} = \mathcal{P} \in \left\{0, \mathcal{I}_1 - \dfrac{1}{n-1}\mathcal{I}_7 + \dfrac{1}{n-1}\mathcal{I}_8\right\}$;

4) $P = 0$, $\mathcal{T} = 0$, $\mathcal{P} \in \{0, \mathcal{I}_1, y_7\mathcal{I}_7 + (1 - ny_7)\mathcal{I}_8,\ \mathcal{I}_1 - y_7\mathcal{I}_7 + (ny_7 - 1)\mathcal{I}_8\}$;

5) $P = \dfrac{1}{n}I_2$, $\mathcal{T} = 0$, $\mathcal{P} \in \{0, \mathcal{I}_1, y_7\mathcal{I}_7 + (1 - ny_7)\mathcal{I}_8,\ \mathcal{I}_1 - y_7\mathcal{I}_7 + (ny_7 - 1)\mathcal{I}_8\}$;

6) $P = x_2I_2 + (1 - nx_2)I_6$, $x_2 \in \mathcal{F}(M) \setminus \left\{\dfrac{1}{n}\right\}$, $\mathcal{T} = 0$, $\mathcal{P} \in \left\{0, \mathcal{I}_1 - \dfrac{1}{n-1}\mathcal{I}_7 + \right.$ $\left. + \dfrac{1}{n-1}\mathcal{I}_8\right\}$, *where* $y_7 \in \mathcal{F}(M)$.

Taking into account the case 3), we deduce that the proiective Weyl curvature tensor field W is invariant to infinitely many transformations of exotic connections

$$\Gamma \xrightarrow{P} \Pi, P = I_1 + x_2I_2 - (1 + nx_2)I_6,\ x_2 \in \mathcal{F}(M) \setminus \left\{-\frac{1}{n}\right\}.$$

The study of the closed diagrame and the geometrical interpretation of the results, when operators $P, \mathcal{P}, \mathcal{T}$ are generated by all elements of the bases of $\mathcal{I}nv(3)$ and $\mathcal{I}nv(4)$, remains an open problem.

2 Pairs of geometrical object fields via conformal invariant operators

Our algebra of conformal invariant operators $CInv(r)$ illustrates a general approach of the decomposition problem, in Riemannian geometry, in particular cases corresponding to the Singer-Thorpe-Nomizu splitting of the space of the curvature tensors and also extends the Thomas-Weyl theory.

2.1 Algebra $CInv(r)$ of conformal invariant operators

Let V be a real n-dimensional vector space, $n > 2$, endowed with the inner product $g = (g_{ij})$, where $g^{-1} = (g^{ij})$ and $\delta = (\delta^i_j)$ the Kronecker symbol. In the vector space $T^r_r(V)$ of tensors of type (r, r) we define the vector subspace

$$CInv(r) = \{P, P^{i_1 \dots i_r}_{j_1 \dots j_r} = \sum_{\sigma \in S_r} x_\sigma \delta^{i_{\sigma(1)}}_{j_1} \dots \delta^{i_{\sigma(r)}}_{j_r} + \sum_{\sigma(1)<\sigma(2)\ ,\ \tau(1)<\tau(2)} x_{\sigma\tau} g_{j_{\tau(1)} j_{\tau(2)}} g^{i_{\sigma(1)} i_{\sigma(2)}} \delta^{i_{\sigma(3)}}_{j_{\tau(3)}} \dots \delta^{i_{\sigma(r)}}_{j_{\tau(r)}} \mid x_\sigma, x_{\sigma\tau} \in R, \sigma, \tau \in S_r\},$$

where S_r is the permutation group. Any element $P \in CInv(r)$ is a conformal invariant tensor (i.e. $\forall A \in CO(V), A \circ P = P$). We endow $CInv(r)$ with a structure of real Lie algebra, as in the subsection 1.1, containing the subalgebra $Inv(r)$.

The algebra $CInv(3)$ arroses during the study of connections, torsion tensors etc. The operators $P = \{P^{bc\ r}_{a\ \ st}\} \in Inv(3)$ are endomorphisms on the space $T^1_2(V)$, having the simplified expression $P = \sum_{i=1}^{6} x_i I_i + \sum_{j=1}^{9} x_{j+6} G_j$, where $\{I_1, ..., I_6, G_1, ..., G_9\}$ is the basis.

For $r = 4$, in the algebra $CInv(4)$ containing the operators $\mathcal{P} = \{\mathcal{P}^{bcd\ r}_{a\ \ \ stl}\}$, interpreted like endomorphisms on the space $T^1_3(V)$, having the simplified expression $\mathcal{P} = \sum_{i=1}^{24} y_i \mathcal{I}_i + \sum_{j=1}^{72} z_j \mathcal{G}_j$, $\{\mathcal{I}_1, \dots, \mathcal{I}_{24}, \mathcal{G}_1, \dots, \mathcal{G}_{72}\}$ being the basis, we study for geometrical reasons the subalgebra

$$\mathcal{D} = \{\mathcal{P} = y_1 \mathcal{I}_1 + y_7 \mathcal{I}_7 + y_8 \mathcal{I}_8 + z_1 \mathcal{G}_1 + z_2 \mathcal{G}_2 + z_3 \mathcal{G}_3 + z_4 \mathcal{G}_4\}.$$

For particular values of the coeficients, an operator $\mathcal{P} \in \mathcal{D}$ produces the conformal Weyl curvature tensor field.

Theorem 2.1.1 *Endowing $\mathcal{D}$ with a structure of Lie algebra, all its subalgebras are Lie algebras.*

Remark 2.1.1 There are infinitely many endomorphisms in $CInv(3)$ and $CInv(4)$, having traceless images.

2.2 Generalization of Singer-Thorpe-Nomizu decomposition

For geometrical reasons it is worthwhile to investigate the infinite set of conformal projections on tensors of type $(1, r-1)$. Indeed, the conformal Weyl curvature tensor field is produced by a projection of the subalgebra $\mathcal{D} \subset \mathcal{C}Inv(4)$ and the conformal Thomas connection is built by one affine transformation of a projection from $\mathcal{C}Inv(3)$.

In [31] I.M. Singer and J.A. Thorpe studied the descomposition of the space $\mathcal{R}(V)$ of tensors R of type $(1, 3)$, verifying $R^r_{stl} + R^r_{slt} = 0$ and $R_{pstl} + R_{sptl} = 0$. This is also investigated by K. Nomizu [30], for generalized tensor fields.

Moreover, the space of tensors of type $(1, 3)$ splitts in infinitely many ways into three subspaces, images of some projections from $CInv(4)$. So, using the method of conformal projections, we extend Singer-Thorpe-Nomizu decomposition of curvature tensors.

Theorem 2.2.1 *There are infinitely many nonvanishing conformal projections $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in \mathcal{D}$ such that $\mathcal{P}+\mathcal{Q}+\mathcal{R}=\mathcal{I}_1, \mathcal{P}\mathcal{Q}=\mathcal{Q}\mathcal{P}=\mathcal{R}\mathcal{Q}=\mathcal{Q}\mathcal{R}=0$. Moreover, the splitting $T^1_3(V)=Im\mathcal{P}\oplus Im\mathcal{Q}\oplus Im\mathcal{R}$ into three invariant subspaces with respect to the group $CO(V)$ holds, where*

$$\mathcal{P}=\mathcal{I}_1+y_7\mathcal{I}_7+y_8\mathcal{I}_8+\sum_{i=1}^{4} z_i\mathcal{G}_i,\ \mathcal{Q}=-\frac{y_7+z_2+nz_3}{n}\mathcal{G}_3-\frac{y_8+z_1+nz_4}{n}\mathcal{G}_4,$$

$$\mathcal{R}=-y_7\mathcal{I}_7-y_8\mathcal{I}_8-z_1\mathcal{G}_1-z_2\mathcal{G}_2+\frac{y_7+z_2}{n}\mathcal{G}_3+\frac{y_8+z_1}{n}\mathcal{G}_4,$$

$ny_7+y_8+z_1+1=0, z_2+nz_3+z_4=0.$

Proposition 2.2.1. *Any tensor R of type $(1,3)$ splitts into infinitely many possibilities $R=R'+R"+R'''$ such that $R'_{ij}=0$, $R"_{ij}=R_{ij}-\frac{1}{n}Kg_{ij}$, $R'''_{ij}=\frac{1}{n}Kg_{ij}$, where $K=R_{ij}g^{ij}$.*

Remark 2.2.1. In the particular case

$$\mathcal{P}=\mathcal{I}_1-\frac{1}{n-2}\mathcal{I}_7+\frac{1}{n-2}\mathcal{I}_8+\frac{1}{n-2}\mathcal{G}_1-\frac{1}{n-2}\mathcal{G}_2+\frac{1}{(n-1)(n-2)}\mathcal{G}_3-\frac{1}{(n-1)(n-2)}\mathcal{G}_4,$$

$$\mathcal{R}=\frac{1}{n(n-1)}\mathcal{G}_3-\frac{1}{n(n-1)}\mathcal{G}_4,$$

$$\mathcal{Q}=\frac{1}{n-2}\mathcal{I}_7-\frac{1}{n-2}\mathcal{I}_8-\frac{1}{n-2}\mathcal{G}_1+\frac{1}{n-2}\mathcal{G}_2-\frac{2}{n(n-2)}\mathcal{G}_3+\frac{2}{n(n-2)}\mathcal{G}_4,$$

$T^1_3(V)=Im\mathcal{R}\oplus Im\mathcal{P}\oplus Im\mathcal{Q}$ implies the irreducible orthogonal Singer - Thorpe - Nomizu decomposition under the group $O(n)$ of curvature type tensors

$\mathcal{L}(V)=\mathcal{L}_1(V)\oplus\mathcal{L}_W(V)\oplus\mathcal{L}_2(V)$, where

$\mathcal{L}_1(V)=\{L\in\mathcal{L}(V)$ with constant sectional curvature $\}$,

$\mathcal{L}_1^{\perp}(V)=\{L\in\mathcal{L}(V)$ with vanishing scalar curvature $\}$,

$\mathcal{L}_W(V)=\{L\in\mathcal{L}(V)$ with vanishing Ricci tensor $\}$,

$\mathcal{L}_2(V)=$ the orthogonal complement of $\mathcal{L}_W(V)$ in $\mathcal{L}_1^{\perp}(V)$,

$\mathcal{L}(V)=\{L\in\mathcal{R}(V)$ verifying the first Bianchi identity $\}$.

Our general aprouch is motivated by this particular case.

2.3 δ - g - decompositions of geometrical object fields

Let (M,g) be a Riemannian n - dimensional manifold, $\mathcal{T}^1_{r-1}(M)$ the $\mathcal{F}(M)$-module of tensor fields of type $(1,r-1)$ and $\mathcal{A}^1_{r-1}(M)$ the affine $\mathcal{F}(M)$-module over $\mathcal{T}^1_{r-1}(M)$, built in the subsection 1.3.

The conformal operators P are extended to $\mathcal{T}^1_{r-1}(M)$, imposing the condition $x_\sigma, x_{\sigma\tau}\in\mathcal{F}(M)$, $\sigma,\tau\in S_r$ and generate a $\mathcal{F}(M)$-module of invariant endomorphisms under conformal transformations, denoted $CInv(r)$. We denote also with P the induced transformation on the affine $\mathcal{F}(M)$-modulule $\mathcal{A}^1_{r-1}(M)$, which produces a conformal decomposition of this space.

Let $\overset{\circ}{\Gamma}$ be the Levi-Civita connection associated to g and P a conformal projection. We find the pair of geometrical object fields $(\overset{\circ}{\Gamma}, P\overset{\circ}{\Gamma})$ of $\mathcal{A}^1_2(M)$, where $P\overset{\circ}{\Gamma}$ is called the exotic conformal connection.

For $x_1 = 1, x_2 = x_6 = -x_7 = -\frac{1}{n}$, we obtain the pair of connections $(\overset{\circ}{\Gamma}, P\overset{\circ}{\Gamma})$, where $P\overset{\circ}{\Gamma}$ is the conformal Thomas connection. This particular case motivates the general method of conformal projections.

Taking into account the study made in the vectorial case, one gets

Theorem 2.3.1 *There are infinitely many nonvanishing triplets $\mathcal{P}, \mathcal{Q}, \mathcal{R}$, conformal projections on $\mathcal{A}_3^1(M)$, affine transformations of operators from $Span_{\mathcal{F}(M)}\{\mathcal{I}_1, \mathcal{I}_7, \mathcal{I}_8, \mathcal{G}_1, ..., \mathcal{G}_4\}$ such that $\mathcal{P}+\mathcal{Q}+\mathcal{R} = \mathcal{I}_1, \mathcal{PQ} = \mathcal{QP} = \mathcal{RQ} = \mathcal{QR} = 0$, which produce the splitting $\mathcal{A}_3^1(M) = Im\mathcal{P} \oplus Im\mathcal{Q} \oplus Im\mathcal{R}$.*

Theorem 2.3.2 *Let (M,g) be a Riemann space.*

We consider $P = x_1 I_1 + x_2 I_2 + x_6 I_6 + x_7 G_1$ a conformal projection on $\mathcal{A}_2^1(M)$, $\mathcal{T} = y_1\mathcal{I}_1 + y_7\mathcal{I}_7 + y_8\mathcal{I}_8 + \sum_{i=1}^{4} z_i\mathcal{G}_i, \mathcal{T}' = y_1'\mathcal{I}_1 + y_7'\mathcal{I}_7 + y_8'\mathcal{I}_8 + \sum_{i=1}^{4} z_i'\mathcal{G}_i$ conformal projections on $\mathcal{A}_3^1(M)$ and $\mathcal{R} = \mathcal{I}_1 - \mathcal{I}_2$ affine transformation on $\mathcal{A}_3^1(M)$. Let

$(\overset{\circ}{A}, A)$ be a pair from $\mathcal{A}_3^1(M)$ associated by the rule $A^r_{stl} = \dfrac{\partial \Gamma^r_{sl}}{\partial x^t} + \Gamma^r_{mt}\Gamma^m_{st}$ to the pair $(\overset{\circ}{\Gamma}, \Gamma)$ from $\mathcal{A}_2^1(M)$, where $\overset{\circ}{\Gamma}$ is the Levi-Civita connection corresponding to g, and $P\overset{\circ}{\Gamma} = \Gamma$,

$(\overset{\circ}{R}, R)$ be a pair from $\mathcal{A}_3^1(M)$ containing the curvature tensor field associated to $\overset{\circ}{\Gamma}$ and the exotic curvature tensor field associated to $P\overset{\circ}{\Gamma} = \Gamma$,

C be the exotic conformal Weyl curvature tensor field.

The diagrame

$$(*)\qquad \begin{array}{ccc} \overset{\circ}{\Gamma} & \overset{P}{\longrightarrow} & \Gamma \\ \downarrow & & \downarrow \\ \overset{\circ}{A} & & A \\ \mathcal{R}\downarrow & & \downarrow\mathcal{R} \\ \overset{\circ}{R} & & R \\ \mathcal{T}'\searrow & & \swarrow\mathcal{T} \\ & C & \end{array}$$

is closed for the next cases

a) $P = I_1 - \dfrac{1}{n}I_2$, $\mathcal{T} = \mathcal{T}'$ arbitrary conformal projections on $\mathcal{A}_3^1(M)$;

b) $P = \dfrac{1}{n}I_2$, $\mathcal{T}' = 0, \mathcal{T}$ arbitrary conformal projections on $\mathcal{A}_3^1(M)$;

c) $P = I_1 + x_2 I_2 - (1+nx_2)I_6, x_2 \neq -\dfrac{1}{n}, \mathcal{T} = \mathcal{T}' = 0$ or $\mathcal{T} = \mathcal{T}' = \mathcal{I}_1 - \dfrac{1}{n-1}\mathcal{I}_7 + \dfrac{1}{n-1}\mathcal{I}_8$;

d) $P = x_2 I_2 + (1-nx_2)I_6, x_2 \neq \dfrac{1}{n}, \mathcal{T} = \mathcal{T}' = 0$ or $\mathcal{T}' = 0$ and $\mathcal{T} = \mathcal{I}_1 - \dfrac{1}{n-1}\mathcal{I}_7 + \dfrac{1}{n-1}\mathcal{I}_8$;

e) $P = I_1 - \dfrac{1}{n}I_2 + x_6 I_6 - x_6 G_1, \mathcal{T} = \mathcal{T}' = \mathcal{I}_1 - \dfrac{1}{n-2}\mathcal{I}_7 + \dfrac{1}{n-2}\mathcal{I}_8 +$
$$+\frac{1}{n-2}\mathcal{G}_1 - \frac{1}{n-2}\mathcal{G}_2 + \frac{1}{(n-1)(n-2)}\mathcal{G}_3 - \frac{1}{(n-1)(n-2)}\mathcal{G}_4.$$

Moreover, the diagrame reflects an invariance of gauge type of the exotic conformal Thomas connection and the exotic conformal Weyl curvature tensor field with respect to the orthogonal group.

Proposition 2.3.1 *The conformal Weyl curvature tensor field is invariant to infinitely many transformations of exotic conformal connections* $\overset{\circ}{\Gamma} \xrightarrow{P} \Gamma, P = I_1 - \frac{1}{n} I_2 + x_6 I_6 - x_6 G_1, x_6 \in \mathcal{F}(M)$.

For $x_6 = -\frac{1}{n}$ one finds $P \overset{\circ}{\Gamma} = \Gamma$, the conformal Thomas connection. This extension of the known properties of the conformal Weyl curvature tensor illustrates strikingly the generality of our method.

The study of the invariants and the geometrical interpretation of the results for the general diagram $(*)$, built with operators generated by all elements of the bases from $\mathcal{C}Inv(3)$ and $\mathcal{C}Inv(4)$ remain an open problem.

3 Pairs of geometrical object fields via H- projective invariant operators

The decomposition problem in holomorphically projective geometry was studied in [29]. We continue the research in this direction introducing the algebra of H-projective invariant operators.

3.1 Algebra $HInv(r)$ of H-projective invariant operators

Let V be a real n-dimensional vector space, $n = 2m$ and $T_r^r(V)$ the vector space of tensors of type (r, r) on V. Using the Kronecker symbol $\delta = (\delta_j^i)$ and the complex structure $J = (J_j{}^i)$ of V we define the vector subspace

$$HInv(r) = \left\{ \sum_{\sigma \in S_r} x_\sigma \delta_{j_1}^{i_{\sigma(1)}} \dots \delta_{j_r}^{i_{\sigma(r)}} + \right.$$

$$\left. + \sum_{\tau(1)<\tau(2)} x_{\sigma\,\tau} J_{j_{\tau(1)}}{}^{i_{\sigma(1)}} J_{j_{\tau(2)}}{}^{i_{\sigma(2)}} \delta_{j_{\tau(3)}}^{i_{\sigma(3)}} \dots \delta_{j\tau(r)}^{i_{\sigma(r)}} \mid x_\sigma, x_{\sigma\tau} \in R, \sigma, \tau \in S_r \right\},$$

where S_r is the group of permutations (H is the abbreviation from "holomorphic").

Every element $P \in HInv(r)$ is an invariant tensor with respect to the subgroup $G = \{A \in GL(V) | AJ = JA\}$ (i.e. $\forall A \in G, A \circ P = P$). We interpret every tensor of $HInv(r)$ like an endomorfism on $T_{r-1}^1(V)$, so $HInv(r)$ is a real algebra the algebra of absolut invariant tensors $Inv(r)$ being a subalgebra.

Every $P = \{P_a^{bc}{}_{st}^{r}\} \in HInv(3)$ is an endomorphism on $T_2^1(V)$, having the simplified expression $P = \sum_{i=1}^6 x_i I_i + \sum_{j=1}^{18} x_{j+6} H_j$, where $\{I_1, \dots, I_6, H_1, \dots, H_{18}\}$ is the basis in $HInv(3)$.

For $r = 4$, every element of $HInv(4)$ is written in the basis $\{\mathcal{I}_1, \dots, \mathcal{I}_{24}, \mathcal{H}_1, \dots, \mathcal{H}_{144}\}$ like $\mathcal{P} = \sum_{i=1}^{24} y_i \mathcal{I}_i + \sum_{j=1}^{144} v_j \mathcal{H}_j$. If the Ricci tensor is symmetric, then an operator $\mathcal{P} \in Span\{\mathcal{I}_1, \mathcal{I}_7, \mathcal{I}_8, \mathcal{H}_1, \dots, \mathcal{H}_8\}$ produces the H - proiective curvature tensor field. These operators appear during the study of some close diagrame, which reflects an invariance of gauge type of the exotic H-proiective J-connections and the exotic H-proiective curvature tensor fields.

For geometrical reasons we study in detail a subalgebra in $HInv(4)$, namely $\mathcal{E} = Span\{\mathcal{I}_1, \mathcal{I}_7, \mathcal{I}_8, \mathcal{H}_1, \mathcal{H}_3, \mathcal{H}_5, \mathcal{H}_6, \mathcal{H}_7, \mathcal{H}_8\}$. Indeed, the K - curvature tensor fields are generated by certain operators from the subalgebra $\mathcal{E}$. If $\mathcal{P} \in Span\{\mathcal{I}_1, \mathcal{I}_7, \mathcal{I}_8, \mathcal{H}_1, \ldots, \mathcal{H}_8\}$ is a H - proiective projection, then $\mathcal{P} \in \mathcal{E}$.

Endowing $\mathcal{E}$ with a structure of Lie algebra, all the subalgebras are Lie subalgebras.

It is interesting to analyse the infinite family of H-proiective projections, in particular of E and $\mathcal{E}$, where $E = Span\{I_1, I_2, I_6, H_1, H_2\}$, which aroses in the study of some geometrical object fields. Indeed, the affine transformation of a projection from $\mathcal{H}Inv(3)$ produces a symmetric H-projective J - connection. Moreover, the space of tensors of K - curvature type is connected with the subset of projections of the subalgebra $\mathcal{E}$.

3.2 δ-J-decompositions of geometrical object fields

Let (M, J) be an almost complex n-dimensional manifold ($n = 2m$) and $\mathcal{T}^1_{r-1}(M)$ the $\mathcal{F}(M)$-module of tensor fields of type $(1, r-1)$. Any H-proiective operator P on $T^1_{r-1}(M)$ is extended to $\mathcal{T}^1_{r-1}(M)$, imposing the condition $x_\sigma, x_{\sigma\tau} \in \mathcal{F}(M), \sigma, \tau \in S_r$, and generates a $\mathcal{F}(M)$-module H-proiective invariant tensor fields, denoted $\mathcal{H}Inv(r)$.

Let $\mathcal{A}^1_{r-1}(M)$ be the affine $\mathcal{F}(M)$-module over $\mathcal{T}^1_{r-1}(M)$, built in the subsection 1.3. We denote also with P the affine transformation induced on the affine $\mathcal{F}(M)$-module $\mathcal{A}^1_{r-1}(M)$.

If $\Gamma \in \mathcal{C}$, the space of affine connections, one gets the pair of geometrical object fields $(\Gamma, P\Gamma)$ from $\mathcal{A}^1_2(M)$. In the set of J-connections we determine properties of J-connections compatible with a Hermitian metric and in particular we study the case of the Kähler manifolds.

Theorem 3.2.1 *Let (M, g, J) be a Kähler manifold. The H-proiective projections $P \in E$ which produce pairs of geometrical object fields $(\overset{\circ}{\Gamma}, \Gamma = P\overset{\circ}{\Gamma})$, where $\overset{\circ}{\Gamma}$ is the Levi-Civita connection and Γ a J-connection are*

$P = I_1, P = I_1 + x_2 I_2 + x_6 I_6 + (1 + n x_2 + x_6) H_1 - x_2 H_2$,

$P = 0, P = x_2 I_2 + x_6 I_6 + (1 - n x_2 - x_6) H_1 - x_2 H_2, \ \ x_2, x_6 \in \mathcal{F}(M)$.

Remark 3.2.1 In the particular case $x_2 = x_6 = -x_7 = -x_8 = -\frac{1}{n+2}, \ x_1 = 1$ one gets the pair of connections $(\overset{\circ}{\Gamma}, P\overset{\circ}{\Gamma} = \Gamma)$, where Γ is a J-connection H-projective related with the Levi-Civita connection and $P = I_1 - \frac{1}{n+2}(I_2 + I_6 - H_7 - H_8)$ is the unique H-projective projection which applies $\overset{\circ}{\Gamma}$ to the symmetric J-connection $\Gamma = P\overset{\circ}{\Gamma}$.

Let (M, g, J) be a Kähler manifold and $\mathcal{L}_K(M)$ the $\mathcal{F}(M)$-module of K-curvature tensor fields i.e. the set of tensor fields R^i_{jkl} of type $(1,3)$ verifying

1) $R^i_{jkl} + R^i_{jlk} = 0$, 2) $R_{ijkl} + R_{jikl} = 0$, where $R_{ijkl} = g_{si} R^s_{jkl}$,

3) $\sum^c_{j,k,l} R^i_{jkl} = 0$ (Bianchi I), 4) $R^r_{stl} J_r{}^p = J_s{}^r R^p_{rtl}$ (Kähler identity).

Theorem 3.2.2 *There are infinitely many endomorphisms*
$\mathcal{P} \in Span_{\mathcal{F}(M)}\{\mathcal{I}_1, \mathcal{I}_7, \mathcal{I}_8, \mathcal{H}_1, \ldots, \mathcal{H}_8\}$ in $\mathcal{H}Inv(4)$, such that $\mathcal{P}$ apply $\mathcal{L}_K(M)$ into $\mathcal{L}_K(M)$.

Corollary 3.2.1 *The H-projective projections $\mathcal{P} \in \mathcal{E}$ defined on $\mathcal{L}_K(M)$ are*
$\mathcal{P} = \mathcal{I}_1 + y_7 \mathcal{I}_7 + y_8 \mathcal{I}_8 + v_1 \mathcal{H}_1 + v_3 \mathcal{H}_3 + v_3 \mathcal{H}_5 + v_1 \mathcal{H}_6 - y_8 \mathcal{H}_7 - y_7 \mathcal{H}_8$, *where*

a) $\frac{1}{2} + ny_7 + y_8 - v_1 = 0, v_3$ *arbitrary; b)* $y_7 = y_8 = v_1 = v_3 = 0$
and the supplementary projections
$\mathcal{Q} = y_7\mathcal{I}_7 + y_8\mathcal{I}_8 + v_1\mathcal{H}_1 + v_3\mathcal{H}_3 + v_3\mathcal{H}_5 + v_1\mathcal{H}_6 - y_8\mathcal{H}_7 - y_7\mathcal{H}_8$, *where*
a) $ny_7 + y_8 - v_1 = \frac{1}{2}, v_3$ *arbitrary; b)* $y_7 = y_8 = v_1 = v_3 = 0$. Moreover, $\mathcal{P}R = y_1 R$ and these operators are symmetric with respect to the induced inner product on $\mathcal{L}_K(M)$. Also, for every projection, one gets $\mathcal{L}_K(M) = Im\mathcal{P}$, $Im\mathcal{Q} = 0$.

Remark 3.2.2 If

$$\mathcal{P} = \mathcal{I}_1 - \frac{n}{n^2-4}\mathcal{I}_7 + \frac{n}{n^2-4}\mathcal{I}_8 - \frac{n}{n^2-4}\mathcal{H}_1 - \frac{1}{n+2}\mathcal{H}_2 + \frac{n}{n^2-4}\mathcal{H}_3 + \\ + \frac{1}{n+2}\mathcal{H}_4 + \frac{2}{n^2-4}\mathcal{H}_5 - \frac{2}{n^2-4}\mathcal{H}_6 - \frac{2}{n^2-4}\mathcal{H}_7 + \frac{2}{n^2-4}\mathcal{H}_8$$

and $R \in \mathcal{L}'_K(M)$, then $\mathcal{P}R$ is the H-proiective curvature tensor field and $\mathcal{P}R \in \mathcal{L}'_K(M)$, the set of (1,3)-tensor fields verifying 1), 3), 4).

Hence $\mathcal{L}'_K(M) = Im\mathcal{P} \oplus Ker\mathcal{P}$, where $Im\mathcal{P} = \{L \in \mathcal{L}'_K(M)$ with vanishing Ricci tensor field $\}$.

We determine some pairs of geometrical object fields from $\mathcal{A}^1_2(M)$, respectively $\mathcal{A}^1_3(M)$ associated by affine transformations of certain H-projective operators from $\mathcal{H}Inv(3), \mathcal{H}Inv(4)$, generalizing properties of J-connections H-projective related with the Levi-Civita connection and of the H-projective curvature tensor field.

Theorem 3.2.3 *Let* (M, g, J) *be a Kähler manifold and*

$P = x_1I_1 + x_2I_2 + x_6I_6 + x_7H_1 - x_2H_2, \mathcal{P} = y_1\mathcal{I}_1 + y_7\mathcal{I}_7 + y_8\mathcal{I}_8 + v_1\mathcal{H}_1 +$
$+ v_2\mathcal{H}_2 + v_3\mathcal{H}_3 + v_4\mathcal{H}_4 + v_5\mathcal{H}_5 + v_6\mathcal{H}_6 + v_7\mathcal{H}_7 + v_8\mathcal{H}_8$,
$\mathcal{T} = z_1\mathcal{I}_1 + z_7\mathcal{I}_7 + z_8\mathcal{I}_8 + w_1\mathcal{H}_1 + w_2\mathcal{H}_2 + w_3\mathcal{H}_3 + w_4\mathcal{H}_4 + w_5\mathcal{H}_5 + w_6\mathcal{H}_6 +$
$+ w_7\mathcal{H}_7 + w_8\mathcal{H}_8, \mathcal{R} = \mathcal{I}_1 - \mathcal{I}_2$

affine transformations on $\mathcal{A}^1_2(M)$, *respectively* $\mathcal{A}^1_3(M)$.

There are infinitely many H*-projective projections* P *and* H*-projective transformations* $\mathcal{P}, \mathcal{T}$ *such that next diagrame*

$$(*) \qquad \begin{array}{ccc} \overset{\circ}{\Gamma} & \xrightarrow{P} & \Gamma \\ \downarrow & & \downarrow \\ \overset{\circ}{A} & & A \\ \mathcal{R}\downarrow & & \downarrow\mathcal{R} \\ \overset{\circ}{R} & & R \\ & \mathcal{T}\searrow \quad \swarrow\mathcal{P} & \\ & H & \end{array}$$

is closed where

$(\overset{\circ}{\Gamma}, \Gamma = P\,\overset{\circ}{\Gamma})$ *is a pair from* $\mathcal{A}^1_2(M)$, $\overset{\circ}{\Gamma}$ *is the Levi-Civita connection and* Γ *the exotic* J*-connection* H*-projective related with* $\overset{\circ}{\Gamma}$,

$(\overset{\circ}{A}, A)$ *is a pair from* $\mathcal{A}^1_3(M)$, *associated to* $(\overset{\circ}{\Gamma}, \Gamma)$, *by the rule* $A^r_{stl} = \dfrac{\partial \Gamma^r_{sl}}{\partial x^t} + \Gamma^r_{mt}\Gamma^m_{sl}$,

$(\overset{\circ}{R}, R)$ *is a pair from* $\mathcal{A}^1_3(M)$, $\overset{\circ}{R}$ *being the curvature tensor field associated to* $\overset{\circ}{\Gamma}$, R *the exotic curvature tensor field associated to* Γ,

H *the exotic* H*-projective curvature tensor field.*

In the particular case

$$\mathcal{P} = \mathcal{I}_1 - \frac{n}{n^2-4}\mathcal{I}_7 + \frac{n}{n^2-4}\mathcal{I}_8 - \frac{n}{n^2-4}\mathcal{H}_1 - \frac{1}{n+2}\mathcal{H}_2 + \frac{n}{n^2-4}\mathcal{H}_3 +$$

$$+\frac{1}{n+2}\mathcal{H}_4+\frac{2}{n^2-4}\mathcal{H}_5-\frac{2}{n^2-4}\mathcal{H}_6-\frac{2}{n^2-4}\mathcal{H}_7+\frac{2}{n^2-4}\mathcal{H}_8.$$

one gets $\mathcal{P}R$ the H-projective curvature tensor field. The H-projective projection $P=I_1-\frac{1}{n+2}(I_2+I_6-H_1-H_2)$ produces the pair of connections $(\overset{\circ}{\Gamma}, P\overset{\circ}{\Gamma}=\Gamma)$, where Γ is the J-connection H-projective related with $\overset{\circ}{\Gamma}$. If $\mathcal{T}=\mathcal{P}$ we find again that the H-projective curvature tensor field is invariant under the H-projective transformations of J-connections.

It remains an open problem the geometric interpretation and the study of the generale diagrame $(*)$ built with operators generated by all elements of the bases of $\mathcal{H}Inv(3)$ and $\mathcal{H}Inv(4)$.

4 Pairs of geometrical object fields via conformal holomorphic invariant operators

The complete decomposition of K-curvature tensors [32], [26], [19] characterizes spaces of constant holomorphic sectional curvature. Introducing the algebra $CHInv(r)$ of $\delta-J-g$ operators, we extend the Sitaramayya-Mori decomposition and also we determine general invariants under some transformations of geometrical object fields.

4.1 Algebra $CHInv(r)$ of conformal holomorphic invariant operators

Let V be a real n-dimensional vector space, $n=2m$, endowed with the complexe structure $J=(J_j{}^i)$ and the Hermitian inner product $g=(g_{ij})$, $g^{-1}=(g^{ij})$. Let $\delta=(\delta^i_j)$ be the Kronecker symbol, $J_{ij}=J_i{}^k g_{kj}$ and $T^r_r(V)$ the vector space of tensors of type (r,r) on V.

The triplet (δ,J,g) determines the next vector subspace

$$CHInv(r)=\Bigg\{P^{i_1\ldots i_r}_{j_1\ldots j_r}=\sum_{\sigma\in S_r}x_\sigma\delta^{i_{\sigma(1)}}_{j_1}\ldots\delta^{i_{\sigma(r)}}_{j_r}+$$
$$+\sum_{\sigma,\tau\in S_r,\,\tau(1)<\tau(2),\,\sigma(1)<\sigma(2)}x_{\sigma\tau}g_{j_{\tau(1)}j_{\tau(2)}}g^{i_{\sigma(1)}i_{\sigma(2)}}\delta^{i_{\sigma(3)}}_{j_{\tau(3)}}\ldots\delta^{i_{\sigma(r)}}_{j_{\tau(r)}}+$$
$$+\sum_{\sigma,\tau\in S_r,\,\tau(1)<\tau(2)}x_{\sigma\tau}J_{j_{\tau(1)}}{}^{i_{\sigma(1)}}J_{j_{\tau(2)}}{}^{i_{\sigma(2)}}\delta^{i_{\sigma(3)}}_{j_{\tau(3)}}\ldots\delta^{i_{\sigma(r)}}_{j_{\tau(r)}}+$$
$$+\sum_{\sigma,\tau\in S_r,\,\sigma(1)<\sigma(2),\,\tau(1)<\tau(2)}x_{\sigma\tau}J_{j_{\tau(1)}j_{\tau(2)}}J^{i_{\sigma(1)}i_{\sigma(2)}}\delta^{i_{\sigma(3)}}_{j_{\tau(3)}}\ldots\delta^{i_{\sigma(r)}}_{j_{\tau(r)}}+$$
$$+\sum_{\sigma,\tau\in S_r,\,\tau(1)<\tau(2),\,\sigma(2)<\sigma(3)}x_{\sigma\tau}J_{j_{\tau(1)}j_{\tau(2)}}J_{j_{\tau(3)}}{}^{i_{\sigma(1)}}g^{i_{\sigma(2)}i_{\sigma(3)}}\delta^{i_{\sigma(4)}}_{j_{\tau(4)}}\ldots\delta^{i_{\sigma(r)}}_{j_{\tau(r)}}+$$
$$+\sum_{\sigma,\tau\in S_r,\,\sigma(1)<\sigma(2),\,\tau(2)<\tau(3)}J^{i_{\sigma(1)}i_{\sigma(2)}}J_{j_{\tau(1)}}{}^{i_{\sigma(3)}}g_{j_{\tau(2)}j_{\tau(3)}}\delta^{i_{\sigma(4)}}_{j_{\tau(4)}}\ldots\delta^{i_{\sigma(r)}}_{j_{\tau(r)}}\Bigg|$$
$$x_\sigma,x_{\sigma\tau}\in\mathbf{R},\ \sigma,\tau\in S_r\Big\},$$

where $J_{ij} = J_i{}^k g_{kj}$, $J^{ij} = J_k{}^i g^{kj}$, S_r is the group of permutations and CH is the abbreviation of "conformal holomorphic". The subspace $CHInv(r)$ is endowed with a structure of Lie algebra, considering any P as an endomorphism on $T^1_{r-1}(V)$.

Any element $P \in CHInv(r)$ is invariant with respect to the group $\bar{G}$, where $\bar{G} = CO(V) \cap Sp(m, V) \cap G$, $G = \{A \in GL(V) | AJ = JA\}$; $\forall A \in \bar{G}$, $A \circ P = P$.

We should remark that $CHInv(r)$ contains the subalgebras $Inv(r)$, $CInv(r)$ and $HInv(r)$. Hence the δ-operators, the δ-g-operators and the δ-J-operators belong to a much more general algebra of δ-J-g-operators.

For $r = 3$, the algebra $CHInv(3)$ is generated by the endomorphisms $P = \sum_{i=1}^{6} x_i I_i + \sum_{j=1}^{9} x_{j+6} G_j + \sum_{k=1}^{18} x_{15+k} H_k + \sum_{l=1}^{9} x_{33+l} C_l + \sum_{s=1}^{9} x_{42+s} B_s + \sum_{r=1}^{9} x_{51+r} K_r$.

For geometrical reasons one studies the subalgebra $F = Span\{I_1, I_2, I_6, G_1, H_1, H_2, C_1\}$, which aroses during the developement of the theory of the conformal complex connections on Kähler manifolds [45]. Introducing a structure of Lie algebra on F, all its subspaces are Lie subalgebras.

In the case $r = 4$, any element $\mathcal{P} = \{\mathcal{P}_{a\ \ stl}^{bcd\ r}\}$ of $CHInv(4)$, has the simplified expression $\mathcal{P} = \sum_{i=1}^{24} y_i \mathcal{I}_i + \sum_{j=1}^{72} z_j \mathcal{G}_j + \sum_{k=1}^{144} v_r \mathcal{H}_r + \sum_{l=1}^{72} u_l \mathcal{C}_l + \sum_{s=1}^{72} w_s \mathcal{B}_s + \sum_{p=1}^{72} x_p \mathcal{K}_p$, where $\{\mathcal{I}_1, ..., \mathcal{I}_{24}, \mathcal{G}_1, ..., \mathcal{G}_{72}, \mathcal{H}_1, ..., \mathcal{H}_{144}, \mathcal{C}_1, ..., \mathcal{C}_{72}, \mathcal{B}_1, ..., \mathcal{B}_{72}, \mathcal{K}_1, ..., \mathcal{K}_{72}\}$ is the basis in $CHInv(4)$.

The subset $\mathcal{F} = Span\{\mathcal{I}_1, \mathcal{I}_7, \mathcal{I}_8, \mathcal{G}_1, \ldots, \mathcal{G}_4, \mathcal{B}_1, \ldots, \mathcal{B}_6, \mathcal{H}_5, \mathcal{H}_6, \mathcal{H}_{11}\}$, which contains the algebra $\mathcal{F}' = Span\{\mathcal{I}_1, \mathcal{I}_7, \mathcal{I}_8, \mathcal{G}_1, \ldots, \mathcal{G}_4, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3\}$, appears in the generalization of the theory of the Bochner curvature tensors on almost Hermitian manifolds. In the infinite set of conformal holomorphic projections one projection from $\mathcal{C}Hnv(4)$ produces the Bochner curvature tensor field and the affine transformation of a projection from $\mathcal{C}Hnv(3)$ a complex conformal connection in the case of a Kähler manifold.

Remark 4.1.1 In particular, if $\mathcal{P} \in \mathcal{F}'$ is a conformal holomorphic projection, for $w_1 = w_2 = w_3 = 0$, one gets a conformal projection and for $z_1 = z_2 = z_3 = z_4 = 0$ one finds a projective projection.

Theorem 4.1.1 *There are infinitely many conformal holomorphic nonvanishing projections $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in \mathcal{F}'$ such that $\mathcal{P} + \mathcal{Q} + \mathcal{R} = \mathcal{I}_1$, $\mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} = \mathcal{R}\mathcal{Q} = \mathcal{Q}\mathcal{R} = 0$ and $T^1_3(V) = Im\mathcal{P} \oplus Im\mathcal{Q} \oplus Im\mathcal{R}$.*

4.2 δ-J-g-decompositions of geometrical object fields

Let (M, g, J) be an almost Hermitian manifold, $\mathcal{T}^1_{r-1}(M)$ the $\mathcal{F}(M)$-module of tensor fields of type $(1, r-1)$ and $\mathcal{A}^1_{r-1}(M)$ the affine $\mathcal{F}(M)$-module over $\mathcal{T}^1_{r-1}(M)$, built in the subsection 1.3.

The conformal holomorphic operators P on $T^1_{r-1}(M)$ are extended to $\mathcal{T}^1_{r-1}(M)$ imposing x_σ, $x_{\tau\sigma} \in \mathcal{F}(M)$, $\sigma, \tau \in S_r$ and generate a $\mathcal{F}(M)$-module of invariant endomorphisms denoted $\mathcal{CHI}nv(r)$. The affine transformation of P induced on the $\mathcal{F}(M)$–module $\mathcal{A}^1_{r-1}(M)$, denoted also with P, produces a conformal holomorphic decomposition of the space $\mathcal{A}^1_{r-1}(M)$.

Theorem 4.2.1 *Let (M, g, J) be a Kähler manifold. The conformal holomorphic projections of $F = Span_{\mathcal{F}(M)}\{I_1, I_2, I_6, G_1, H_1, H_2, C_1\}$ which produce pairs of geometrical object fields $(\overset{\circ}{\Gamma}, \Gamma = P\,\overset{\circ}{\Gamma})$, where $\overset{\circ}{\Gamma}$ is the Levi–Civita connection and $\Gamma = P\,\overset{\circ}{\Gamma}$ is a J-connection, called the exotic complex conformal connection, are*

$P = I_1,\ P = I_1 + x_2 I_2 + x_6 I_6 + x_7(G_1 + C_1) + (1 + nx_2 + x_6 + 2x_7)H_1 - x_2 H_2,$
$P = 0, P = x_2 I_2 + x_6 I_6 + x_7(G_1 + C_1) + (1 - nx_2 - x_6 - 2x_7)H_1 - x_2 H_2.$

Remark 4.2.1 In the particular case $x_2 = x_6 = x_{34} = -x_7 = -x_{16} = -x_{17} = -\dfrac{1}{n+2}, x_1 = 1$ one finds $P = I_1 - \dfrac{1}{n+2}(I_2 + I_6 - G_1 - H_1 - H_2 + C_1)$ and the pair of connections $(\overset{\circ}{\Gamma}, P\,\overset{\circ}{\Gamma})$, where $P\,\overset{\circ}{\Gamma}$ is a complex conformal connection on a Kähler manifold (M, g, J) [45].

For $r = 4$, let (M, g, J) be an almost Hermitian manifold and let $\mathcal{F}$ be the $\mathcal{F}(M)$-module generated by the operators
$\mathcal{P} = y_1\mathcal{I}_1 + y_7\mathcal{I}_7 + y_8\mathcal{I}_8 + \sum_{i=1}^{4} z_i\mathcal{G}_i + \sum_{j=1}^{6} w_j\mathcal{B}_j + v_5\mathcal{H}_5 + v_6\mathcal{H}_6 + v_{11}\mathcal{H}_{11}.$

Let $R \in \mathcal{L}_K(M)$, $\mathcal{L}_K(M)$ being the space of K–curvature tensor fields on M. We define $\bar{R} = \mathcal{P}R$, the exotic Bochner curvature tensor field and we study the pair of geometrical object fields $(R, \mathcal{P}R)$.

Theorem 4.2.2 *The endomorphisms $\mathcal{P}$ of $\mathcal{F}$ applying the space $\mathcal{L}_K(M)$ into $\mathcal{L}_K(M)$ are*

$\mathcal{P} = y_1\mathcal{I}_1 + y_7(\mathcal{I}_7 - \mathcal{I}_8 - \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{B}_4 - \mathcal{B}_5 + \mathcal{H}_5 - \mathcal{H}_6 + 2\mathcal{H}_{11} + 2\mathcal{B}_6)+$
$+z_3(\mathcal{G}_3 - \mathcal{G}_4 + \mathcal{B}_1 - \mathcal{B}_2 + 2\mathcal{B}_3),$

where $y_1, y_7, z_3 \in \mathcal{F}(M)$.

Remark 4.2.2 Our general approuch is motivated by the particular case $y_7 = -\dfrac{1}{n+4}$, $z_3 = \dfrac{1}{(n+1)(n+4)}$ which produces the pair of tensor fields $(R, \mathcal{P}R)$, R being the curvature tensor field and $\mathcal{P}R = \bar{R}$ the Bochner curvature tensor field defined for the class of almost Hermitian manifolds.

Theorem 4.2.3 *Let (M, g, J) be an almost Hermitian manifold and*

$P = \sum_{i=1}^{6} x_i I_i + \sum_{j=1}^{9} x_{j+6} G_j + \sum_{k=1}^{18} x_{15+k} H_k + \sum_{l=1}^{9} x_{33+l} C_l +$
$+ \sum_{s=1}^{9} x_{42+s} B_s + \sum_{r=1}^{9} x_{51+r} K_r,$

$\mathcal{P} = y_1\mathcal{I}_1 + y_7\mathcal{I}_7 + y_8\mathcal{I}_8 + \sum_{i=1}^{4} z_i\mathcal{G}_i + \sum_{j=1}^{6} w_j\mathcal{B}_j + v_5\mathcal{H}_5 + v_6\mathcal{H}_6 + v_{11}\mathcal{H}_{11},$

$\mathcal{P}' = y'_1\mathcal{I}_1 + y'_7\mathcal{I}_7 + y'_8\mathcal{I}_8 + \sum_{i=1}^{4} z'_i\mathcal{G}_i + \sum_{j=1}^{6} w'_j\mathcal{B}_j + v'_5\mathcal{H}_5 + v'_6\mathcal{H}_6 + v'_{11}\mathcal{H}_{11},$

$\mathcal{T} = \mathcal{I}_1 - \mathcal{I}_2$ *affine transformations on $\mathcal{A}_2^1(M)$ and $\mathcal{A}_3^1(M)$.*

There are infinitely many transformations $P, \mathcal{P}$ and $\mathcal{P}'$ such that

$$\begin{array}{ccccc}
\overset{\circ}{\Gamma} & \xrightarrow{P} & \Gamma \\
\downarrow & & \downarrow \\
\overset{\circ}{A} & & A \\
\mathcal{T}\downarrow & & \downarrow\mathcal{T} \\
\overset{\circ}{R} & & R \\
\mathcal{P}\longmapsto & B & \longleftarrow\!\mid \mathcal{P}'
\end{array}$$

is a closed diagrame, where

$(\overset{\circ}{\Gamma}, \Gamma)$ *is a pair from $\mathcal{A}_2^1(M)$, $\overset{\circ}{\Gamma}$ being the Levi-Civita connection and $\Gamma = P\,\overset{\circ}{\Gamma}$,*

$(A, \overset{\circ}{A})$ *is a pair from $\mathcal{A}_3^1(M)$ associated to $(\Gamma, \overset{\circ}{\Gamma})$, by the rule $A^r_{stl} = \dfrac{\partial \Gamma^r_{sl}}{\partial x^t} + \Gamma^r_{mt}\Gamma^m_{sl}$,*

$(R, \overset{\circ}{R})$ *the exotic curvature tensor fields associated to $(\Gamma, \overset{\circ}{\Gamma})$, $R = \mathcal{T}A, \overset{\circ}{R} = \mathcal{T}\ \overset{\circ}{A}$ and*

B the exotic Bochner curvature tensor $B = \mathcal{P}'R = \mathcal{P}\ \overset{\circ}{R}$.

This is a general method to determine the transformations of exotic connections under which the Bochner curvature tensor field is invariant.

4.3 Extension of the Sitaramayya - Mori decomposition

In [26] H.Mori studied the decomposition of the space of K-curvature tensors. The same orthogonal decomposition, for a real Hermitian n-dimensional vector space, V, $n = 2m$, is determined by M.Sitaramayya [32] : $\mathcal{L}_K(V) = \mathcal{L}^1_K(V) \oplus \mathcal{L}^W_K(V) \oplus \mathcal{L}^2_K(V)$, where

$\mathcal{L}^1_K(V) = \{L \in \mathcal{L}_K(V) \mid L$ with constant holomorphic sectional curvature $\}$,
$\mathcal{L}^W_K(V) = \{L \in \mathcal{L}_K(V) \mid L$ with vanishing Ricci tensor $\}$,
$\mathcal{L}^W_K(V) \oplus \mathcal{L}^2_K(V) = \{L \in \mathcal{L}_K(V) \mid Tr(L) = 0\}$,
$\mathcal{L}^1_K(V) \oplus \mathcal{L}^W_K(V) = \{L \in \mathcal{L}_K(V) \mid K(L) = \lambda I, \lambda \in R\}$.

The theory of conformal holomorphic operators gives a new method to get the Sitaramayya-Mori decomposition. Studing the nonvanishing conformal holomorphic operators from $\mathcal{F}$, one finds the orthogonal decomposition.

Theorem 4.3.1 *The conformal holomorphic projections* $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in \mathcal{F}$*, defined on* $\mathcal{L}_K(M)$ *such that* $\mathcal{P} + \mathcal{Q} + \mathcal{R} = \mathcal{I}_1$, $\mathcal{PQ} = \mathcal{QP} = 0$, $\mathcal{RQ} = \mathcal{QR} = 0$ *and* $\mathcal{L}_K(M) = Im\mathcal{P} \oplus Im\mathcal{Q} \oplus Im\mathcal{R}$ *are:*

$$\mathcal{P} = \frac{1}{n+2}(\mathcal{H}_3 - \mathcal{H}_4 + \mathcal{B}_1 - \mathcal{B}_2 + 2\mathcal{B}_3),$$

$$\begin{aligned}\mathcal{Q} &= \mathcal{I}_1 - \frac{1}{n+4}\{[\mathcal{I}_7 - \mathcal{I}_8 - \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{B}_4 - \mathcal{B}_5 + \mathcal{H}_5 - \mathcal{H}_6 + 2\mathcal{H}_{11}] - \\ &- \frac{1}{n+2}[\mathcal{H}_3 - \mathcal{H}_4 + \mathcal{B}_1 + 2\mathcal{B}_3]\},\end{aligned}$$

$$\begin{aligned}\mathcal{R} &= \frac{1}{n+4}\{[\mathcal{I}_7 - \mathcal{I}_8 + \mathcal{H}_1 - \mathcal{H}_2 + \mathcal{B}_4 - \mathcal{B}_5 + \mathcal{H}_5 - \mathcal{H}_6 + 2\mathcal{H}_{11}] - \\ &- \frac{n}{2}[\mathcal{H}_3 - \mathcal{H}_4 + \mathcal{B}_1 - \mathcal{B}_2 + 2\mathcal{B}_3]\}.\end{aligned}$$

Using general affine transformations of conformal holomorphic projections on $\mathcal{A}^1_3(M)$ it is possible to characterize all splittings into three components of this affine space.

Theorem 4.3.2 *There are infinitely many nonvanishing triplets* $\mathcal{P}, \mathcal{Q}, \mathcal{R}$*, conformal holomorphic projections on* $\mathcal{A}^1_3(M)$*, affine transformations of operators from* $Span_{\mathcal{F}(M)}\{\mathcal{I}_1, \mathcal{I}_7, \mathcal{I}_8, \mathcal{G}_1, ..., \mathcal{G}_4, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3\}$*, such that* $\mathcal{P} + \mathcal{Q} + \mathcal{R} = \mathcal{I}_1, \mathcal{PQ} = \mathcal{QP} = 0, \mathcal{RQ} = \mathcal{QR} = 0$*. Moreover* $\mathcal{A}^1_3(M) = Im\mathcal{P} \oplus Im\mathcal{Q} \oplus Im\mathcal{R}$,

$$\mathcal{P} = \mathcal{I}_1 + y_7\mathcal{I}_7 + y_8\mathcal{I}_8 + \sum_{i=1}^{4} z_i\mathcal{G}_i + \sum_{i=1}^{3} w_i\mathcal{B}_i,$$

$$\mathcal{Q} = -y_7\mathcal{I}_7 - y_8\mathcal{I}_8 - z_1\mathcal{G}_1 - z_2\mathcal{G}_2 + \frac{y_7 + z_2}{n}\mathcal{G}_3 + \frac{y_8 + z_1}{n}\mathcal{G}_4,$$

$$\mathcal{R} = -\left(z_3 + \frac{y_7 + z_2}{n}\right)\mathcal{G}_3 - \left(z_4 + \frac{y_8 + z_1}{n}\right)\mathcal{G}_4 - \sum_{i=1}^{3} w_i\mathcal{B}_i,$$

where $1 + ny_7 + y_8 + z_1 = 0$, $z_2 + nz_3 + z_4 + w_1 + w_3 = 0$, w_2 *being arbitrary.*

The $\mathcal{F}(M)$-module of invariant (r, r)-tensor fields $\mathcal{I}nv(r), \mathcal{C}Inv(r), \mathcal{H}Inv(r), \mathcal{CH}Inv(r)$ were built interpreting every element as an operator on $\mathcal{T}^1_{r-1}(M)$, which has affine induced transformations on the affine $\mathcal{F}(M)$ - modulule $\mathcal{A}^1_{r-1}(M)$ of geometrical object fields of type $(1, r-1)$. It is possible to built another algebras considering any operator as an endomorphism on $\mathcal{T}^0_r(M)$ (the multiplication PQ

being given by the rule $P^{i_1\dots i_r}_{j_1\dots j_r} Q^{j_1\dots j_r}_{k_1\dots k_r}$), having an affine transformation on an affine $\mathcal{F}(M)$-module $\mathcal{A}^0_r(M)$ of geometrical object fields of type $(0,r)$, defined similar with $\mathcal{A}^1_{r-1}(M)$.

We should remark that it is worthwhile to study also the cases $r = 2$ or $r = 6$. Indeed $R\cdot R, Q(g,R)$, the tensor fields which characterize the pseudo-symmetry or the semi-symmetry property are produced by affine transformations on $\mathcal{A}^0_6(M)$ of some absolut invariant operators of $\mathcal{I}nv(6)$. Also, the Obata operators Ω and Ω', which determine the pairs of linear connections compatible with certain G-structures are built by some affine transformations on $\mathcal{A}^0_2(M)$ of conformal invariant operators of $\mathcal{C}Inv(2)$.

Particular projective projections, defined using the representation theory, were studied in [22], illustrating that there is a connection between the subgroups of the permutation group S_r and the corresponding subalgebras of operators, built using our theory.

5 Geometry of pairs of connections

5.1 Geodesically and subgeodesically related manifolds

Spaces with constant curvature and Einstein spaces geodesically related were studied in [43]. Properties of recurrent and birecurrent spaces geodesically related were obtained in [41], [28]. In [5] appeared another direction of study on Riemannian manifolds. Pseudo-symmetric spaces, which verify

$(*) R\cdot R$ and $Q(g,R)$ are linear dependent at every point of M,

represent a natural generalization of the semi-symmetric manifolds ($R\cdot R = 0$) [34] and arrose during the study of the totally umbilical submanifolds of semi-symmetric spaces, as well as during the consideration of geodesic mappings. The condition $(*)$ is equivalent to

$R\cdot R = LQ(g,R)$, L being defined on the set $U = \{x\in M \mid R\neq R(1) \text{ at } x\}$,

$R(1) = \dfrac{1}{n(n-1)}(\mathcal{G}_3 - \mathcal{G}_4)R$,

$(R\cdot R)_{hijklm} = (\delta^{adefbc}_{hijklm} - \delta^{daefbc}_{hijklm} - \delta^{efadbc}_{hijklm} - \delta^{efdabc}_{hijklm})(C^1_4(R\otimes R))_{abcdef}$,

C^1_4 being the contraction of the (1,7)-tensor $R\otimes R$,

$(Q(g,R))_{hijklm} = (\delta^{adefcb}_{hijklm} - \delta^{bdcfea}_{hijklm} - \delta^{caefdb}_{hijklm} - \delta^{caefbd}_{hijklm} -$

$-\delta^{cdafeb}_{hijklm} - \delta^{cdafbe}_{hijklm} - \delta^{cdeafb}_{hijklm} - \delta^{cdeabf}_{hijklm})(g\otimes R)_{abcdef}$.

Replacing the Riemann tensor with the conharmonic tensor,

$Z = [\mathcal{I}_1 - \dfrac{1}{n-2}(\mathcal{I}_7 - \mathcal{I}_8 - \mathcal{G}_1 + \mathcal{G}_2)]R$, one gets the tensor $R\cdot Z$. If $R\cdot Z = 0$, then (M,g) is a conharmonic semi-symmetric space (or Z-semi-symmetric).

For A, D symmetric tensors of type $(0,2)$, we define

$(R\cdot A)_{hijk} = (-\delta^{abcd}_{hijk} + \delta^{abdc}_{hijk})(C^1_1(A\otimes R))_{abcd}$,

$Q(g,D) = (\delta^{abcd}_{hijk} - \delta^{acbd}_{hijk} - \delta^{cadb}_{hijk} - \delta_{hijk}$

6 Geometry of pairs of connections

6.1 Geodesically and subgeodesically related manifolds

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$R \cdot R = LQ(g, R)$, L being defined on the set $U = \{x \in M \mid R \neq R(1) \text{ at } x\}$,

$$R(1) = \frac{1}{n(n-1)}(\mathcal{G}_3 - \mathcal{G}_4)R,$$

$$(R \cdot R)_{hijklm} = (\delta^{adefbc}_{hijklm} - \delta^{daefbc}_{hijklm} - \delta^{efadbc}_{hijklm} - \delta^{efdabc}_{hijklm})(C^1_4(R \otimes R))_{abcdef},$$

C^1_4 being the contraction of the (1,7)-tensor $R \otimes R$,

$$(Q(g, R))_{hijklm} = (\delta^{adefcb}_{hijklm} - \delta^{bdcfea}_{hijklm} - \delta^{caefdb}_{hijklm} - \delta^{caefbd}_{hijklm} - $$
$$-\delta^{cdafeb}_{hijklm} - \delta^{cdafbe}_{hijklm} - \delta^{cdeafb}_{hijklm} - \delta^{cdeabf}_{hijklm})(g \otimes R)_{abcdef}.$$

Replacing the Riemann tensor with the conharmonic tensor,

$Z = [\mathcal{I}_1 - \frac{1}{n-2}(\mathcal{I}_7 - \mathcal{I}_8 - \mathcal{G}_1 + \mathcal{G}_2)]R$, one gets the tensor $R \cdot Z$. If $R \cdot Z = 0$, then (M, g) is a conharmonic semi-symmetric space (or Z-semi-symmetric).

For A, D symmetric tensors of type $(0, 2)$, we define

$$(R \cdot A)_{hijk} = (-\delta^{abcd}_{hijk} + \delta^{abdc}_{hijk})(C^1_1(A \otimes R))_{abcd},$$

$$Q(g, D) = (\delta^{abcd}_{hijk} - \delta^{acbd}_{hijk} - \delta^{cadb}_{hijk} - \delta^{cabd}_{hijk})(g \otimes D)_{abcd},$$

where C^1_1 is the contraction and "," is the covariant derivative with respect to g.

Two Riemann spaces $V_n = (M, g)$ and $\overline{V_n} = (M, \overline{g})$ are ξ^i-subgeodesically related, the tensor of correspondence being $-g_{ij}$, if the Yano formulae are satisfied

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \psi_k + \delta^i_k \psi_j - g_{kj}\xi^i,$$

where ψ_i şi ξ^i are the components of a 1-form, respectively vector field. This is equivalent with the existence of a difeomorphism f, called subgeodesic map, between these two spaces, which applies the ξ^i-subgeodesics into the ξ^i-subgeodesics.

V_n and $\overline{V_n}$ are trivial subgeodesically related if $\Psi_i - \xi_i = 0$, $\forall i \in \{1, ..., n\}$.

If $\xi^i = 0$ then V_n and $\overline{V_n}$ are geodesically related and the Yano formulae become the Weyl formulae. If $\Psi_{ij} = f g_{ij}$, where $f \in \mathcal{F}(M)$, then V_n and $\overline{V_n}$ are special geodesically related. The correspondence is trivial if $\Psi_i = 0$, $\forall i \in \{1, ..., n\}$.

The Weyl projective curvature tensor field $W = [\mathcal{I}_1 - \frac{1}{n-1}(\mathcal{I}_7 - \mathcal{I}_8)]R$ is invariant under geodesic maps.

We study Riemannian manifolds geodesically related with conharmonic semi-symmetric spaces and also pseudo-symmetric manifolds subgeodesically related.

Theorem 5.1.1 *Let $V_n = (M, g)$ and $\overline{V_n} = (M, \overline{g})$, $n \geq 3$, be nontrivial geodesically related Riemannian manifolds. If $\overline{V_n}$ is Z-semi-symmetric, then V_n and $\overline{V_n}$ are spaces with constant curvature or the geodesic correspondence is speciale.*

Theorem 5.1.2 *Let $V_n = (M, g)$ and $\overline{V_n} = (M, \overline{g})$, $n \geq 3$, be nontrivial geodesically related Riemannian manifolds. If $\overline{V_n}$ is a space with irreducible curvature tensor field and $\overline{V_n}$ is $\overline{Z}$-semi-symmetric, then V_n and $\overline{V_n}$ are spaces with constant curvature.*

Theorem 5.1.3 *Let $V_n = (M, g)$ and $\overline{V_n} = (M, \overline{g})$, $n \geq 3$, nontrivial ξ^i-subgeodesically related Riemannian manifolds, the tensor of correspondence being $-g$. If $\overline{V_n}$ is a space with irreducible curvature tensor field and $\overline{V_n}$ is $\overline{Z}$-semi-symmetric, then $\overline{V_n} = (M, \overline{g})$ and $\widetilde{V_n} = (M, \widetilde{g} = e^{2\xi} g)$ are spaces with constant curvature.*

Theorem 5.1.4 *Let $V_n = (M, g)$ and $\overline{V_n} = (M, \overline{g})$, $n \geq 3$, be ξ^i-subgeodesica-lly related Riemannian manifolds, the tensor of correspondence being $-g$, such that $\xi_{hk} = 0$, where $\xi_{hk} = \xi_{h,k} - \xi_h \xi_k + \frac{1}{2}\xi_i \xi^i g_{hk}$. If $\overline{V_n} = (M, \overline{g})$ a conharmonic semi-symmetric space and with irreducible curvature tensor filed, then $V_n = (M, g)$ is Einstein space.*

Theorem 5.1.5 *Let $V_n = (M, g)$ and $\overline{V_n} = (M, \overline{g})$, $n \geq 3$, be two Riemann spaces ξ^i-subgeodesically related, the tensor of correspondence being $-g$, such that $B = \frac{1}{n} Tr(B) g$, where B is the (0,2)-tensor field having the components $B_{rs} = \xi_{r,s} - \xi_r \xi_s$.*

If V_n is a pseudo-symmetric space, then $\overline{V_n}$ is a pseudo-symmetric space.

Theorem 5.1.6 *Let $V_n = (M, g)$ and $\overline{V_n} = (M, \overline{g})$, $n \geq 3$, be ξ^i-subgeodesically related Riemannian spaces, the tensor of correspondence being $-g$.*

Then $\overline{R} \cdot g = Q(g, F)$, where F is the symmetric tensor of type $(0,2)$, having the components $F_{ij} = \psi_{i;j} - \xi_{i;j} - (\psi_i - \xi_i)(\psi_j - \xi_j)$, where ";" represents the covariant derivatives with respect to the metric $\overline{g}$.

Proposition 5.1.1 *Let $V_n = (M, g)$ and $\overline{V_n} = (M, \overline{g})$, $n \geq 3$, be ξ^i-subgeodesically related Riemannian spaces, the tensor of correspondence being $-g$.*

If $\overline{V_n}$ is a pseudo-symmetric space, the map $\overline{L}$ which satisfies $(*)$ on $\overline{U}$ is constant and $F_{ij} = f g_{ij} + h \overline{g}_{ij}$, where $f, h \in \mathcal{F}(M)$, h being nonzero, then the relation

$$(\overline{L} + h)\left[g - \frac{1}{n}(\overline{g}^{ij} g_{ij})\overline{g}\right] = 0$$

holds on the set $\overline{U}$.

6.2 Semi-symmetric connections on Weyl generalized manifolds

Let (M, g) be a Riemannian space, n-dimensional, $f \in \mathcal{F}(M)$ be a nowhere vanishing function, $\hat{g} = \{e^\lambda g | \lambda \in \mathcal{F}(M)\}$ be the conformal structure generated by g and and the Weyl f-structure $W^f : \hat{g} \mapsto \Lambda^1(M)$, where $W^f(e^\lambda g) = W^f(g) - f d\lambda$, $\forall \lambda \in \mathcal{F}(M)$, $\Lambda^1(M)$ being the space of the 1-forms on M. Weyl generalized manifolds $(M,\ \hat{g},\ W^f)$ are a natural extension of Weyl spaces $(M, \hat{g}, W)$, obtained for $f = 1$ ([2]).

A linear connection ∇ on M is compatible with the Weyl f-structure W^f and associated to the 1-form ω if

$$(\nabla_{fX} g)(Y, Z) + W^f(g)(X) g(Y, Z) + \omega(Y) g(X, Z) + \omega(Z) g(X, Y) = 0.$$

∇ is called σ-semi-symmetric connection if $T(X, Y) = \sigma(Y)X - \sigma(X)Y$, where σ is a 1-form and T is the torsion tensor.

We study semi-symmetric connections on Weyl generalized manifolds.

Theorem 5.2.1 *Let $(M,\ \hat{g},\ W^f)$ be a Weyl generalized manifold. There exists a unique σ-semi-symmetric connection compatible with the Weyl f-stucture and associated to the 1-form ω, where $g, f, W^f(g), \omega$ are given, determined by*

$$\nabla_{fX} Y = \overset{\circ}{\nabla}_{fX} Y + \frac{1}{2} W^f(g)(X) Y + \left(\frac{1}{2} W^f(g) + f\sigma\right)(Y) X - g(X, Y)$$

$\left(\frac{1}{2}W^f(g) + f\sigma - \omega\right)^{\#}$,

$\overset{\circ}{\nabla}$ *being the Levi-Civita connection associated to* g.

Theorem 5.2.2 *If* $n > 2$, *then the tensor field*

$$D^i_{jkl} = R^{*\,i}_{jkl} - \frac{1}{n-2}\left\{\Omega^{mi}_{jk}\left(R^*_{ml} - \frac{K}{2(n-1)}g_{ml}\right) - \Omega^{mi}_{jl}\left(R^*_{mk} - \frac{K}{2(n-1)}g_{mk}\right)\right\},$$

where K *is the scalar curvature,* $R^{*\,i}_{jkl} = R^i_{jkl} - \frac{1}{n}\delta^i_j B_{hk}$, $B_{hk} = R^s_{shk}$, $R^*_{jk} = R^{*\,i}_{jki}$ *and* $\Omega = \frac{1}{2}(I\otimes I - g\otimes\widetilde{g})$, *is an invariant under the transformation of a* σ*-semi-symmetric connection, compatible with Weyl* f*-structure* W^f *into the* $\overline{\sigma}$*-semi-symmetric connection* $\overline{\nabla}$, *compatible with the Weyl* f*-structure* $\overline{W}^f$.

We should remark that this invariant is produced by an affine transformation on $\mathcal{A}^1_3(M)$ of an operator of $CInv(3)$.

The invariant tensor D characterizes the flatness of the Weyl generalized spaces.

Theorem 5.2.3 *Let* $(M,\hat{g},W^f)$*be a Weyl generalized manifold,* ∇ *semi-symmetric connection, compatible with the Weyl* f*-structure* W^f. *Then*

1. There exist local the 1-forms p *and* q *such that the space endowed with the semi-symmetric connection given by*

(5.2.1) $$\overline{\nabla}_{fX}Y = \nabla_{fX}Y + q(Y)X + p(X)Y - g(X,Y)q^{\#}$$

is flat;

2. The tensor D *is vanishing*

are equivalent.

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References

[1] N. Bokan, *The decomposition theory and its applications,* In: The math. heritage of C.F. Gauss, G.M. Rassias, ed., 66-99, Singapore (1991).

[2] N. Bokan, P. Gilkey, U. Simon, *Geometry of differential operators on Weyl manifolds,* Phil. Trans. R. Soc. Lond. A (1996).

[3] B.-y. Chen, K. Yano, *Manifolds with vanishing Weyl or Bochner curvature,* J. Math. Soc. Japan, 27 (1975), 106-112.

[4] V. Cruceanu, *Sur l'ensemble des connections sur un espace fibre*, Bull. Math. de la Soc. Sci. Math. de la Roumanie, 13, 16, 1 (1969), 315-322.

[5] J. Deprez, R. Deszcz, L. Verstraelen, *Examples of pseudo-symmetric conformally flat warped products*, Chinese J. Math., 17, 1 (1989), 51-65.

[6] A. Gray, L. Helvella, *The sixteen classes of almost Hermitian manifolds and their linear invariants,* Ann. Mat. Pura Appl., 123 (1980), 325-58.

[7] I.E. Hirică, *On geodesically and subgeodesically related pseudo-Riemannian spaces,* Rendiconti del Sem. Mat. Messina, 5 (1999).

[8] I.E. Hirică, *Asupra spaţiilor Riemann în corespondenţă geodezică*, Studii şi Cercetări Matematice, 1-2, 49 (1997), 73-76.

[9] I.E. Hirică, *On semi-symmetric connections on Weyl generalized manifolds*, Analele Univ. Bucureşti, (1997), 9-16,

[10] I.E. Hirică, *Jacobi fields on manifolds endowed with f-structures*, Rev. Roum. Math. Pures et Appl., vol. 43, nr. 9-10 (1998), 813-817.

[11] I.E. Hirică, *On geodesically related pseudo-Riemannian spaces*, An. Univ. "Al.Cuza"-Iaşi, Secţ. Matematică, Tom XLIII, fasc. 2, (1997), 323-328.

[12] I.E. Hirică, *Notes on semi-symmetric connections on spaces endowed with Weyl f-structures*, General Mathematics, vol. 5 (1997), 191-198.

[13] I.E. Hirică, *Absolut invariant operators on differentiable manifold*, Proc. Second Conf. of Balkan Society of Geometers and Workshop "Diff.Geometry, Global Analysis and Lie Algebras ", Thessaloniki, Greece, June 24-27, (1998).

[14] I.E. Hirică, *Notes on subgeodesic and geodesic mappings of Riemannian manifolds,* Proc. Balkan Society of Geometers and Workshop "Diff.Geometry, Global Analysis and Lie Algebras ", Thessaloniki, Greece, 1996 (Gr. Tsagas Editor), Geometry Balkan Press (1999), 205-212.

[15] I.E. Hirică, C. Udrişte, *Decompositions of tensor fields and connections*, Hadronic Journal Supplement, U.S.A., 1999 - Proceedings of International Workshop on Fundamental Open Problems in Mathematics, Physics and other Sciences, Chinese Academy of Sciences, Beijing, August 28, 1997.

[16] I.E. Hirică, L. Nicolescu, M.S. Stupariu, *Champs m-speciaux dans l'algebre de deformation*, An. Univ. Oradea, Fasc. Matem., Tom V, (1995-1996), 119-134.

[17] I.E. Hirică, L. Nicolescu, M.S. Stupariu, *On conformally semi-symmetric connections*, An. Univ. Oradea, Fasc. Matem., Tom V, (1995-1996), 135-149.

[18] D. Janssens, L. Vanhecke, *Almost contact structures and curvature tensors*, Kodai Math. J., 4 (1981), 1-27.

[19] D.L. Johnson, *Sectional curvature and curvature normal forms,* Michigan Math., 27 (1980), 275-294.

[20] D. Krupka, *The trace decomposition problem*, Beiträge zür Algebra und Geometrie, 36 (1995), 2, 303-315.

[21] P.Matzeu, *Decomposition of curvature tensors on almost contact manifolds,* Rend. Sem. Mat. Univ. Politec. Torino, 47 (1989), 2, 179-210.

[22] R. Merris, *Multilinear Algebra*, Algebra, Logic and Applications Series, volume 8, Gordon and Breach Science Publishers, (1997).

[23] J. Mikesh, *On general trace decomposition problem*, Diff. Geom. and Its Appl. (Brno, 1995), 45-50, Masaryk, Univ. Brno, (1996).

[24] R. Miron, M. Anastasiei, *Vector bundles and Lagrange spaces with applications to relativity,* Geometry Balkan Press, Bucureşti (1997).

[25] R. Miron, *Connexions compatible aux structures presque symplectique,* C.R. Acad. Sci. Paris, t. 256 (1967), 685-687.

[26] H. Mori, *On the decomposition of generalized K-curvature tensor fields*, Tôhoku Math. J. 25(1973), 225-235.

[27] A.M. Naveira, *A classification of Riemannian almost-product manifolds*, Rendiconti di Matematica 3, VII (1983), 577-592.

[28] L. Nicolescu, *Les espaces de Riemann en representation subgeodesique*, Tensor, N.S., 32 (1978), 183-187.

[29] S. Nikcevic, *On the decomposition of curvature tensor fields on Hermitian manifolds,* Colloq. Math. Soc. Janos Bolyai 56. Diff. Geom., Eger (Hungary), (1989), 555-568.

[30] K. Nomizu, *On the decomposition of generalized curvature tensor fields*, Differential Geometry, in honor of K.Yano, Kinokuniya, Tokyo, (1972), 335-345.

[31] I.M. Singer, J.A. Thorpe, *The curvature of 4-dimensional Einstein spaces*, Global Analysis (Paper in honor of K. Kodaira), University of Tokyo Press, Tokyo, (1969), 355-365.

[32] M. Sitaramayya, *Curvature tensors in Kähler manifolds*, Trans. Amer. Math. Soc., 183 (1973), 341-383.

[33] R.S. Strichartz, *Linear algebra of curvature tensors and their covariant derivatives*, Can. J. Math. 15 (1988), 1105-1143.

[34] Z.I. Szabo, *Structure theorems on Riemannian spaces satisfying* $R(X,Y)\cdot R=0$*, I. The local version*, J. Diff. Geom., 17 (1982), 531-582.

[35] F. Tricceri, L. Vanhecke, *Decomposition of a space of curvature tensors on a quaternionic Kähler manifold and spectrum theory*, Simon Stevin 53 (1979), 163-173.

[36] C. Udrişte, I.E. Hirică, *Family of projective projections on tensors and connections*, Balkan Journal of Geometry and Its Applications, 2, 2 (1997), 139-156.

[37] C. Udrişte, I.E. Hirică, *Conformal operators on Riemannian manifolds*, Rendiconti del Sem. Mat. di Messina, 5 (1998).

[38] C. Udrişte, *On the Riemannian curvature tensor*, Bull. Math. de la Soc. Sci. Math. de la Roumanie, 16 (64), 4 (1972), 471-476.

[39] C. Udrişte, *Coquaternion structures and almost coquaternion structures,* Mathematica Balkanica, 4 (1974), 635-642.

[40] L. Vanhecke, *Curvature tensors*, J. Korean Math. Soc. 14 (1977), 143-151.

[41] P. Venzi, *Geodatische Abbildungen mit* $\lambda_{ij} = \Delta g_{ij}$, Tensor, N.S., 3 (1980), 230-241.

[42] L. Verstraelen, *Comments on pseudosymmetry in the sense of Ryszard Deszcz,* Geometry and Topology of Submanifolds, 6 (1994), 199-209, World Sci., Singapore.

[43] G.Vrănceanu, *Leçons de Geometrie differentielle*, I,II,III,IV, Ed. Acad. Române, Bucureşti.

[44] H. Weyl, *The Classical Groups, their invariants and representations*, Princeton, (1946).

[45] K. Yano, *On complex conformal connections*, Kodai Math. Sem. Rep. 26 (1975), 137-151.

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