Extrema Constrained by C^k Curves

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Abstract

In this paper we generalize the results obtained in Ref. [17].

§1 raises the following problem: what conection there exists between the local extrema of the function $f : D \subset R^p \to R$ and the local extrema of the functions $f \circ \alpha$, $\alpha \in \Gamma$, where Γ is a given family of parametrized curves ?

§2 proves the existence of a C^k curve containing a given sequence of points.

§3 solves the problem which was presented in §1 in the case of C^k curves.

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1 Introduction

Let us consider the extremum problem

$$\min f(x), \text{ subject to } x \in M,$$

where M is a subset of R^p with a given structure. If M is an open set of R^p which coincides to the domain of f, then the extremum problem is called *unconstrained*; in any other case, the extremum problem is called *constrained*. Such problems, in which M is a $C^k, k \geq 2$, finite–dimensional differentiable were developed recently in Refs. [1]–[4], [6]–[9].

The extremum conditions (necessary and sufficient) depend on the fashion of defining the subset M. If M is a differentiable manifold, then they depend also on the geometrical structure of M. Frequently, M is considered as the union of a family of its subsets (a plane as the union of straight lines, an integral manifold of a Pfaff system as the union of some integral curves, and so on, Refs. [10]–[16]). Then the following two problems arise:

1) Let D be an open subset in R^p and $\{A_i\}_{i\in I}$ be a family of subsets of D having a common point $x_* \in A_i, \forall i \in I$. Suppose x_* is a local minimum point for each restriction $f\mid_{A_i}$ of the function $f : D \to R$ to the subset $A_i, i \in I$. Is x_* a local minimum point of f ?

2) Let $f : D \subset R^p \to R$ and $\alpha_i : I_i \subset R \to D, i \in J$ a family of parametrized curves. What connection we have between of functions $f \circ \alpha_i$, the extrema of the restrictions $f\mid_{\alpha_i(I_i)}$, and the extrema of the function f ?

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The problem 1) and the problem 2), for the C^1 or C^2 curves, were solved in Ref. [17]. In this paper we shall solve the problem 2) for the general case of C^k curves and even for the analytic curves. For that reason we shall recall some notions about the curves.

Definition 1.1. Let $I \subset R$ be an interval. A function $\alpha : I \to R^p$ of class $C^k,\ k \geq 1$, is called *parametrized curve of class* C^k and is denoted by α. We shall say that:

1) α passes (just once) through the point $x_* \in R^p$ if there exists (only one) $t_0 \in IntI$ such that $\alpha(t_0) = x_*$;

2) α is a *simple parametrized curve* if α is injective;

3) α is *regular at the point* $x_* = \alpha(t_0)$ if $\alpha'(t_0) \neq 0$;

4) α *has a tangent at the point* $x_* = \alpha(t_0)$ if there exists $m \in \overline{1,k}$, such that $\alpha^{(m)}(t_0) \neq 0$.

Definition 1.2. Two parametrized curves $\alpha : I \to R^p, \beta : J \to R^p$ of class C^k are called *equivalent* if there exists a diffeomorphism $h : I \to J$ of class C^k such that $\alpha = \beta \circ h$. We shall write $\alpha \sim \beta$.

Definition 1.3. The set $\tilde{\alpha}$ of C^k parametrized curves equivalent to $\alpha : I \to R^p$ is called *curve of class* C^k. The curve $\tilde{\alpha}$ has qualities 1) to 4) in the Definition 1.1, if a representative α has these properties.

From now we shall refer to a function $f : D \to R$, where D is an open subset in R^p.

Definition 1.4. Let $f : D \to R$, let $x_* \in D$, and $\alpha : I \to D$ be a parametrized curve passing through x_*. We shall say that

1) x_* is a *minimum point for f constrained by* α if for any $t_0 \in I$, with $\alpha(t_0) = x_*$, the point t_0 is a local minimum point for $f \circ \alpha$, i.e., there exists a neighborhood $I_{t_0} \subset I$ of t_0 such that

$$f(x_*) = f(\alpha(t_0)) \leq f(\alpha(t)), \quad \forall t \in I_{t_0}.$$

2) x_* is a *minimum point for f constrained* by $\tilde{\alpha}$ if there exists a neighborhood V of x_* such that

$$f(x_*) \leq f(x), \quad \forall x \in V \cap \alpha(I).$$

Remark. If x_* is a minimum point of f constrained by the curve $\tilde{\alpha}$, then x_* is a minimum point of f constrained by the parametrized curve α. The converse is not true even so α is a simple parametrized curve. However, in case that $\alpha : I \to D$ is a simple and regular parametrzied curve and I is a compact set, both notions coincide.

Definition 1.5. Let Γ_{x_*} be a family of parametrized curves (curves) passing through the point x_*. We shall say that x_* is a *minimum point of f constrained by the family* Γ_{x_*} if x_* is a minimum point of f constrained by every curve of the family Γ_{x_*}.

2 C^k curves by given sequences of points

The aim of this paragraph is to show that certain conditions assume the existence of C^k curves which contain a given sequence of points. To this we recall shortly the prolongation theorem of Whitney (Ref. [5]).

Let $K \subset R^p$ be a compact set , $k = (k_1, \ldots, k_p)$ be a multiindex and $|k| = k_1 + \ldots + k_p$. A family of continuous functions $F = (f^k)_{|K| \leq m}, f^k : K \to R$, is called *jet of order m*. Denote $F(x) = f^0(x)$, $x \in K$ and $D^k F = (f^{k+l})_{|l| \leq m-|k|}$, $|k| \leq m$. Naturally, for any function $g \in C^m(K)$ one can define the jet

$$J^m(g) = \left(\frac{\partial^k g}{\partial x^k}\right)_{|k|\le m}.$$

For any $x \in R^p$ and a fixed $a \in K$, we introduce the Taylor polynomial function

$$T_a^m F(x) = \sum_{|k|\le m} \frac{(x-a)^k}{k!} f^k(a).$$

Denote

$$\tilde{T}_a^m F = J^m(T_a^m F), \quad R_a^m F = F - \tilde{T}_a^m F.$$

Prolongation Theorem 2.1. *Let $F = (f^k)_{|k|\le m}$. There exists a function $f \in C^m(R^p)$ with $g^m(f) = F$ if and only if*

$$(R_x^m F)^k(y) = \mathcal{O}(|x-y|^{m-|k|},$$

when $|x-y| \to 0$, for any $x, y \in K$ and any $|k| \le m$.

In the sequel we shall apply the preceding theorem in the case $p = 1$ and $K = \{0\} \cup \{t_n | n \in N\}$; where $t_n \in R$ and $t_n \to 0$.

Corollary 2.1. *Let us consider $k \in N^*$. Given the real sequences $t_n \to 0$, $x_n^{(0)} \to 0$, $x_n^{(i)} \to a^{(i)}$, $i = \overline{1,k}$, there exists $f \in C^k(R)$ with $f^{(i)}(t_n) = x_n^{(i)}$, $\forall i \in \overline{0,k}$, if and only if*

$$(*) \qquad \frac{x_m^{(p)} - \displaystyle\sum_{i=p}^{k-1} \frac{(t_m - t_n)^{i-p}}{(i-p)!} x_n^{(i)}}{(t_m - t_n)^{k-p}} \to \frac{a^{(k)}}{(k-p)!}$$

for $m, n \to \infty$ and for any $p \in \overline{0, k-1}$.

Lemma 2.1. (Ref. [17]) *Let $(x_n), (y_n)$ be two sequences of real numbers such that*

1) $x_n \ne 0$, $x_n \ne x_{n+1}$, $\forall n \in N$;

2) there exists $\lim\limits_{n\to\infty} \dfrac{y_n}{x_n} = r$;

3) there exists $\lambda > 0$ with $\left|\dfrac{x_n}{x_{n+1}} - 1\right| \ge \lambda$, $\forall n \in N$.

Then the sequence $\dfrac{y_{n+1} - y_n}{x_{n+1} - x_n}$ is convergent towards r.

Lemma 2.2. *If (x_n) is a sequence of strictly positive real numbers and $x_{n+1} \le \dfrac{1}{2^k} x_n$, $\forall n \in N$, where $k \in N^*$, then there exists $\mu > 0$ such that $\left|\dfrac{x_n - x_m}{(x_n^{1/k} - x_m^{1/k})^k}\right| \le \mu$, $\forall m, n \in N$, $m \ne n$.*

Proof. For $m = n + p$, $p \ge 1$, it follows $x_n^{1/k} \le \dfrac{1}{2^p} x_m^{1/k}$. Denote $t_n = x_n^{1/k}$. We have

$$\frac{x_n - x_m}{(x_n^{1/k} - x_m^{1/k})^k} = \frac{t_n^k - t_m^k}{(t_n - t_m)^k} = \frac{t_n^{k-1} + \ldots + t_m^{k-1}}{(t_n - t_m)^{k-1}} \le$$

$$\leq \frac{t_n^{k-1}\left(1+\frac{1}{2^p}+\ldots+\left(\frac{1}{2^p}\right)^{k-1}\right)}{t_n^{k-1}\left(1-\frac{1}{2^p}\right)^{k-1}} = \frac{1-\left(\frac{1}{2^p}\right)^k}{\left(1-\frac{1}{2^p}\right)^k} < \frac{1}{\left(1-\frac{1}{2}\right)^k}.$$

Lemma 2.3. *Let $(x_n), (y_n)$ two sequences of real numbers such that (x_n) is strictly monotone,*

$$y_n \to 0,\ \frac{y_n}{x_n} \to 0,\ |x_{n+1}| \leq \frac{1}{2^k}|x_n|,\ \forall n \in N$$

where $k \in \boldsymbol{N}^*$. Then $\frac{y_m - y_n}{x_m - x_n} \to 0$ for $m, n \to \infty$.

Proof. Suppose, for example, $x_n > 0$, $\forall n \in N$. Let $b_n = \frac{y_{n+1} - y_n}{x_{n+1} - x_n}$. From Lemma 2.1 it follows $b_n \to 0$. Let $m > n$ and $\mu_{mn} = \max\{|b_n|, \ldots, |b_{m-1}|\}$. Then, for any $i \in \overline{n, m-1}$ we have $\left|\frac{y_{i+1} - y_i}{x_{i+1} - x_i}\right| \leq \mu_{mn}$, namely

$$-\mu_{mn} \leq \frac{y_{i+1} - y_i}{x_i - x_{i+1}} \leq \mu_{mn}$$

and therefore

$$-\mu_{mn}\sum_{i=n}^{m-1}(x_i - x_{i+1}) \leq \sum_{i=n}^{m-1}(y_{i+1} - y_i) \leq \mu_{mn}\sum_{i=n}^{m-1}(x_i - x_{i+1}).$$

It results

$$\left|\frac{y_m - y_n}{x_m - x_n}\right| \leq \mu_{mn}.$$

Now the conclusion is obvious.

Lemma 2.4. *Let $(x_n), (y_n)$ be two sequences of real numbers such that (x_n) is strictly monotone, $y_n \to 0$, $\frac{y_n}{x_n} \to 0$, $|x_{n+1}| \leq \frac{1}{2^k}|x_n|$, $\forall n \in N$, where $k \in N^*$. Then, there exist two functions $f, g \in C^k(R)$ and a sequence (t_n) of real numbers such that $t_n \to 0$, $f(t_n) = x_n$, $g(t_n) = y_n$, $f^{(i)}(0) = 0$, $\forall i \in \overline{0, k-1}$, $g^{(i)}(0) = 0$, $\forall i \in \overline{0, k}$ and $f^{(k)}(0) \neq 0$.*

Moreover, the function f and the sequence (t_n) do not depend on the sequence (y_n).

Proof. Let $t_n = x_n^{1/k}$. Then, the function $f(x) = x^{1/k}$ and the sequence (t_n) satisfy the required properties. In order to get the function g we should apply the Corollary 2.1 to $t_n = x_n^{1/k}$, $x_n^{(0)} = y_n$, $x_n^{(i)} = 0$ and $a^{(i)} = 0$, $\forall i \in \overline{1, k}$. Obviously, the condition $(*)$ from the statement (Corollary 2.1) is fulfilled, for any $p \geq 1$. For $p = 0$ this condition becomes $\frac{y_m - y_n}{(t_m - t_n)^k} \to 0$, for $m, n \to \infty$. Taking into account that

$$\frac{y_m - y_n}{(t_m - t_n)^k} = \frac{y_m - y_n}{x_m - x_n} \cdot \frac{x_m - x_n}{(t_m - t_n)^k}$$

and using the Lemmas 2.2 and 2.3, it results $\frac{y_m - y_n}{(t_m - t_n)^k} \to 0$, for $m, n \to \infty$.

Thus we obtain a function $g \in C^k(R)$ which satisfies the required properties.

Theorem 2.2. *Let (x_n) be a sequence of distinct points of R^p, convergent to the point $a \in R^p$. Then, for any $k \in N^*$, there exist a subsequence (x_{n_m}) and a parametrized curve $\alpha : R \to R^P$ of class C^k, which contains the set of points $\{x_{n_m}, a\}$ such that α has a tangent at the point a. Moreover, if $a = \alpha(t_0)$, then $\alpha^{(i)}(t_0) = 0$, $\forall i \in \overline{1, k-1}$, $\alpha^{(k)}(t_0) \neq 0$ and there exists a sequence (t_m) with $t_m \to t_0$ and $\alpha(t_m) = x_{n_m}$.*

Proof. By a translation, we can suppose $a = (0, \ldots, 0) \in R^p$. Since $u_n = \dfrac{x_n}{||x_n||}$ is bounded, considering it likely a subsequence, we can asssume $u_n \to u \in R^p$. By a rotation, we can suppose $u = (1, 0, \ldots, 0)$. Consequently, if $x_n = (x_u^1, \ldots, x_n^p)$, it follows

$$\frac{x_n^1}{|x_n^1|\sqrt{1 + \left(\frac{x_n^2}{x_n^1}\right) + \ldots + \left(\frac{x_n^p}{x_n^1}\right)}} \to 1.$$

Hence $x_n^1 > 0$ for sufficiently large n and $\dfrac{x_n^i}{x_n^1} \to 0$, $\forall i \in \overline{2, p}$. Obviously, there exists a subsequence (x_{n_m}) such that $x_{n_{m+1}}^1 > 0$ and $x_{n_{m+1}}^1 \leq \frac{1}{2^k} x_{n_m}^1$, $\forall m \in N$.

Apllying Lemma 2.4 to the pair of sequences $(x_{n_m}^1)$ and $(x_{n_m}^i)$, $i \in \overline{2, p}$, we obtain the functions $\varphi_i : R \to R$, $i \in \overline{1, p}$ of class C^k and a sequence (t_m) of real numbers such that $t_m \to 0$, $\varphi_i(t_m) = x_{n_m}^i$, $i \in \overline{1, p}$, $\varphi_i^{(j)}(0) = 0$, $i \in \overline{1, p}$, $j \in \overline{0, k-1}$, $\varphi_i^k(0) = 0$, $i \in \overline{2, p}$ and $\varphi_1^{(k)}(0) \neq 0$. Then, the parametrized curve $\alpha(t) = (\varphi_1(t), \ldots, \varphi_p(t))$, $t \in R$, has the required properties.

3 Minimum constrained by C^k curves

Let D be an open subset in R^p and $x_* \in D$. For any $k \in N^*$ we denote by $\Gamma_{x_*}^k$ the family of all C^k parametrized curves passing just once through the point x_*, each having a tangent at x_*. Let

$$A_{x_*}^k = \left\{\alpha \in \Gamma_{x_*}^k | \alpha(t_0) = x_*, \quad \alpha^{(i)}(t_0) = 0,\ i \in \overline{1, k-1},\ \alpha^{(k)}(t_0) \neq 0\right\}$$

and

$$B_{x_*}^k = \left\{\alpha \in \Gamma_{x_*}^k | \alpha(t_0) = x_*,\ \exists m \in \overline{1, k-1} \text{ such that } \alpha^{(m)}(t_0) \neq 0\right\}, \text{ for } k \geq 1.$$

Theorem 3.1. *Let $f : D \to R$. Then, x_* is a local minimum point of f, if and only if, there exists $k \in N^*$ such that x_0 is a minimum point constrained by the family $A_{x_*}^k$.*

Proof. We can suppose $f(x_*) = 0$. If x_* would not be a local minimum point of f, then there exists a sequence of distinct points (x_n) of R^p, with $x_n \to x_*$ and $f(x_n) < 0$, $\forall n \in N$. Taking into account the Theorem 2.2 we find a curve $\alpha \in A_{x_*}^k$ such that x_* is not a minimum point of f constrained by α. Contradiction.

It is interesing to remark that Theorem 3.1 does not impose any condition upon the function f. Then, more surprising is the fact that Theorem 3.1 fails for the family $B_{x_*}^k$ or for the family of all analytic curves passing through the point x_*, even if f is of class C^∞.

Examples. 1) Let $f : R^3 \to R$,

$$f(x,y) = (y^k - x^{k+1})(y^k - 2^k x^{k+1}),$$

where $k \in N,\ k \geq 1$. It is obvious that f is of class C^∞ and that the critical point $x_* = (0,0)$ is not a local minimum point of f. Let us show that $x_* = (0,0)$ is a minimum point of f constrained by the family $B^k_{x_*}$.

Let $D^- = \{(x,y) \in R^2 | f(x,y) < 0\}$. For any $(x,y) \in D^-$ it results

$$(*) \qquad \left|\frac{y}{x}\right| < 2|x|^{1/k}$$

and

$$(**) \qquad \frac{|y|}{|x|^{(m+1)/m}} > |x|^{-1/m(m+1)},$$

with $|x| < 1$ and $m \in \overline{1, k-1}$.

Let $\alpha \in B^k_{x_*}$. We can suppose $x_* = \alpha(0)$. If, by reductio ad absurdum, the point x_* would not be a minimum point of f constrained by the parametrized curve α, then would exist a sequence (t_n) with $t_n \to 0$ and $\alpha(t_n) = (x_n, y_n) \in D^-$, $\forall n \in N$. Consequently, the numbers x_n and y_n satisfy the above conditions $(*)$ and $(**)$. Obviously, if $\alpha(t) = (x(t), y(t))$, then $x(t) = t^m(a + tf(t))$ and $y(t) = t^m(b + tg(t))$ where $a^2 + b^2 > 0$, $m \in \overline{1, k-1}$ and f, g are continuous functions. We assume that $a \neq 0$. Hence, $\left|\frac{y_n}{x_n}\right| = \frac{|b + t_n g(t_n)|}{|a + t_n f(t_n)|}$. Using the relation $(*)$, we get that $\left|\frac{y_n}{x_n}\right| \to 0$ and therefore $b = 0$. Hence,

$$\frac{|y_n|}{|x_n|^{(m+1)/m}} = \frac{g(t_n)}{|a + t_n f(t_n)|^{(m+1)/m}} \to |g(0)|.$$

On the other hand, using the relation $(**)$, it results that $\frac{|y_n|}{|x_n|^{(m+1)/m}} \to \infty$. Contradiction. Now, we assume that $a = 0$. Then,

$$\left|\frac{x_n}{y_n}\right| = \frac{|t_n||f(t_n)|}{|b + t_n g(t_n)|} \to 0,$$

which is a contradiction to the relation $(*)$.

2) Let $g : R \to R$, $g(x) = e^{-1/x^2}$, for any $x > 0$ and $g(x) = 0$ for any $x \leq 0$. Let $f : R^2 \to R$, $f(x,y) = y(y - g(x))$ which is of class C^∞. Also, it is obvious that the critical point $x_* = (0,0)$. Let us show that $x_* = (0,0)$ is a minimum point of f constrained by the family $\Gamma^\omega_{x_*}$, where $\Gamma^\omega_{x_*}$ is the family of all analytic parametrized curves passing through the point x_*.

Let $D^- = \{(x,y) | f(x,y) < 0\}$. It follows that for any $(x,y) \in D^-$ we have $x > 0$ and

$$(*) \qquad 0 < ye^{1/x^2} < 1$$

Let $\alpha \in \Gamma^\omega_{x_*}$. We can suppose $x_* = \alpha(0)$. If, by reductio ad absurdum, the point x_* would not be a minimum point of f constrained by α, then would exist a sequence

(t_n) with $t_n \to 0$ and $\alpha(t_n) = (x_n, y_n) \in D^-$, $\forall n \in N$. Hence, the numbers x_n and y_n satisfy the above condition $(*)$ and $x_n \to 0$, $y_n \to 0$. Obviously, if $\alpha(t) = (x(t), y(t))$, then $x(t) = t^p(a + \ldots)$ and $y(t) = t^q(b + \ldots)$, with $ab \neq 0$. It follows that

$$y_n e^{1/x_n^2} = [(x_n^2)^{q/2p} e^{1/x_n^2}] y_n (x_n^2)^{-q/2p} \to \infty \cdot \frac{|b|}{|a|^{q/p}} = \infty,$$

which is in contradiction to the relation $(*)$.

In the following, we shall denote by $\tilde{\Gamma}^k_{x_*}$ the family of all C^k curves passing through the point x_*, regular at x_*.

Theorem 3.2. *Let $f : D \subset R^p \to R$. If there exists $k \in N^*$ such that for any $\tilde{\alpha} \in \tilde{\Gamma}^k_{x_*}$ the point x_* is an extrema point of f constrained by $\tilde{\alpha}$, then x_* is a local extrema point of f.*

Proof. Let us suppose that x_* is not a local extrema point for f and $f(x_*) = 0$. Then, there exist two sequences (x_n) and (y_n) of distinct points of D with $x_n \to x_*$, $y_n \to x_*$, $f(x_n) < 0$ and $f(y_n) > 0$, $\forall n \in N$. By the Theorem 2.2 there exist two subsequences (x_{n_m}) and (y_{n_r}), two C^k parametrized curves α and β, and two sequences of real numbers (t_m) and (t'_r) with $t_m \to 0$, $t'_r \to 0$, $t_m > 0$ $t'_r > 0$ such that $\alpha(t_m) = x_{n_m}$ and $\beta(t'_r) = y_{n_r}$, $\forall m, r \in N$. Then, it is easy to show that there exists a parametrized curve $\gamma : R \to R^p$ of class C^k such that $\gamma(t) = \alpha(t)$, $\forall t \leq 1$, $\gamma(t) = \beta(1/t)$, $\forall t \geq 3$, $\gamma(2) = x_*$ and $\gamma'(2) \neq 0$. It follows that $\tilde{\gamma} \in \tilde{\Gamma}^k_{x_*}$ and $\tilde{\gamma}$ contains the points x_{n_m} and y_{n_r}, $\forall m, r \in N$. Hence, the point x_* is not a local extrema point of f.

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