Theory of Semisymmetric Conformally Flat and Biharmonic Submanifolds

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Abstract

We consider immersions of sets of manifolds with some defining relation specified, expressed in terms of an intrinsic or extrinsic curvature condition, and analyse the consequences and relationships among various curvature functions this implies.

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1 Introduction

A first contribution concerns immersions of semisymmetric and Ricci - semisymmetric manifolds as hypersurfaces in Euclidean spaces. The set of all manifolds which are Ricci-semisymmetric and satisfy $R \cdot S = 0$ contains the set of manifolds which are semisymmetric and satisfy $R \cdot R = 0$ as a proper subset. However, considering only those manifolds (M^n, g) which can be immersed as a hypersurface in a Euclidean space $\mathbf{E}^{n+1}$, one might ask whether this can lead to nonsemisymmetric Ricci-semisymmetric hypersurfaces. This is commonly known as the Problem of P.J. Ryan, and has been an open question since 1972. We discuss examples of Ricci-semisymmetric hypersurfaces M^n of $\mathbf{E}^{n+1}$ $(n \geq 5)$ which are not semisymmetric; this provides an answer to the Problem of P.J. Ryan.

Next, we consider conformally flat hypersurfaces of $\mathbf{E}^4$. In contrast to hypersurfaces of Euclidean spaces in all other dimensions, in this particular case there do exist conformally flat hypersurfaces which are not quasi-umbilical. Such hypersurfaces allow introducing a special type of coordinates such that the coordinate lines coincide with the curvature lines; they are called Guichard coordinates. In this context we discuss 2 theorems on 3-dimensional conformally flat hypersurfaces: a conformally flat hypersurface M^3 of $\mathbf{E}^4$ with constant mean curvature H, and having 3 different principal curvatures, must be minimal; for a conformally flat hypersurface M^3 of $\mathbf{E}^4$ with constant Gauss-Kronecker curvature τ, and having 3 different principal curvatures, the value of τ must be equal to zero.

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Finally, we turn our attention to certain generalisations of minimal submanifolds. In the context of B.-Y. Chen's theory of submanifolds of finite type, the set of minimal submanifolds is contained in larger classes of submanifolds, e.g. the classes of submanifolds of finite type, coordinate finite type, and restricted type, amongst others. In this framework, we consider the submanifolds with harmonic mean curvature, or equivalently the biharmonic submanifolds ($\Delta^2\vec{x} = \vec{0}$), and their generalisations defined by the requirement that the mean curvature vector is an eigenvector of the Laplacian $\Delta\vec{H} = \lambda\vec{H}$. We discuss the following 2 theorems: every hypersurface M^3 of $\mathbf{E}^4$ with harmonic mean curvature vector field is minimal; a hypersurface M^3 of $\mathbf{E}^4$ satisfying $\Delta\vec{H} = \lambda\vec{H}$ must necessarilly have constant mean curvature.

2 The problem of P.J. Ryan

A semi-Riemannian manifold (M^n, g), $n = \dim\ M \geq 3$, is called semisymmetric if

$$R \cdot R = 0 \tag{1}$$

holds on M. It is well known that the class of semisymmetric manifolds includes the set of locally symmetric manifolds ($\nabla R = 0$) as a proper subset.

A semi-Riemannian manifold (M^n, g), $n \geq 3$, is said to be Ricci-semisymmetric, if the following condition is satisfied

$$R \cdot S = 0\,. \tag{2}$$

Again, the class of Ricci-semisymmetric manifolds includes the set of Ricci-symmetric manifolds ($\nabla S = 0$) as a proper subset. It is clear that every semisymmetric manifold is Ricci-semisymmetric. The converse statement is however not true.

Although the conditions (1) and (2) do not coincide for manifolds in general, there has been a long standing question:

Question 2.1. Are the conditions $R\cdot R = 0$ and $R\cdot S = 0$ equivalent for hypersurfaces of Euclidean spaces ?

This question has been first raised by P.J. Ryan in 1972 (cfr. Problem P 808 of [18] and references therein), and has been an open problem ever since. Question 2.1 is commonly refered to as the Problem of P.J. Ryan.

Whereas the conditions $R\cdot R = 0$ and $R\cdot S = 0$ are equivalent on any 3-dimensional manifold, for $n > 3$ we have the following results. It had been proved in [20] that (1) and (2) are equivalent for hypersurfaces which have positive scalar curvature in a Euclidean space $\mathbf{E}^{n+1}$, $n > 3$. In [17] this result was generalized to hypersurfaces of a Euclidean space $\mathbf{E}^{n+1}$, $n > 3$, which have nonnegative scalar curvature and also to hypersurfaces of constant scalar curvature. [17] also proves that (1) and (2) coincide for hypersurfaces of Riemannian space forms with nonzero constant sectional curvature. Further, in [16] it was proved that (1) and (2) are equivalent for hypersurfaces of a Euclidean space $\mathbf{E}^{n+1}$, $n > 3$, under the additional global condition of completeness. In [8], it has been shown that the conditions (1) and (2) are equivalent for hypersurfaces of the Euclidean space $\mathbf{E}^5$.

In [6] a negative answer to Question 2.1 was given for hypersurfaces of a Euclidean space $\mathbf{E}^{n+1}$, $n \geq 5$. Indeed, [6] gives an example of a hypersurface M^5 of $\mathbf{E}^6$ which satisfies $R \cdot S = 0$, but which is not semisymmetric. The existence of such a hypersurface M^5 of $\mathbf{E}^6$ which is Ricci-semisymmetric, but does not fulfill $R \cdot R = 0$, is recalled in Theorem 2.1 here below. This proves that the conditions $R \cdot R = 0$ and $R \cdot S = 0$ are not equivalent for hypersurfaces of Euclidean space in general, thus solving the Problem of P.J. Ryan.

W.r.t. a local orthonormal frame $\{e_i\}_{i=1}^n$ which diagonalises the shape operator $\mathcal{A}$, with principal curvatures $\lambda_i (i = 1, \ldots n)$, the only nonzero components of the Riemann-Christoffel curvature tensor are

$$R_{ijji} = \lambda_i \lambda_j \,, \qquad i \neq j \,, \quad 1 \leq i, j \leq n \,,$$

and the Ricci tensor is diagonal $S_{ii} = \lambda_i (\sum_{i \neq j} \lambda_j)$.

The set of equations for $R \cdot R = 0$ (1) amounts to:

$$\lambda_i \lambda_j \lambda_k (\lambda_i - \lambda_j) = 0 \,, \qquad i \neq j, j \neq k, k \neq i, \qquad 1 \leq i, j \leq n \,. \tag{3}$$

Analogously, the set of equations for $R \cdot S = 0$ (2) amounts to:

$$\lambda_i \lambda_j (\lambda_i - \lambda_j) \left(\sum_{k \neq i, k \neq j} \lambda_k \right) = 0 \,, \qquad i \neq j \,, \quad 1 \leq i, j, k \leq n \,. \tag{4}$$

We remark that a solution of (3) is indeed automatically a solution of (4). Theorem 2.1 proves that there exists a 5-dimensional hypersurface of $\mathbf{E}^6$, for which the principal curvatures are a solution of (4), but do not satisfy (3).

Theorem 2.1. *There exists an isometric immersion of a 5-dimensional manifold* M^5 *into* $\mathbf{E}^6$ *with a metric*

$$\begin{aligned} ds^2 &= e^{2x^1} \big((dx^1)^2 + \cos^2 \phi(x^2, x^3)(dx^2)^2 \\ &+ \sin^2 \phi(x^2, x^3)(dx^3)^2 + \cos^2 \psi(x^4, x^5)(dx^4)^2 \\ &+ \sin^2 \psi(x^4, x^5)(dx^5)^2 \big) \,, \end{aligned} \tag{5}$$

and principal curvatures $(0, b, b, -b, -b)$*; where*

$$b(x^1) = e^{-x^1} \,, \tag{6}$$

and ϕ *and* ψ *are a solution of the equation*

$$\frac{\partial^2 \zeta}{(\partial x^i)^2} - \frac{\partial^2 \zeta}{(\partial x^j)^2} = -\sin(2\zeta) \,, \tag{7}$$

for $(i, j) = (2, 3)$*, and* $(4, 5)$*, respectively.* M^5 *satisfies* $R \cdot S = 0$*, but is not semisymmetric.*

This result was generalized in [7], where it was proven that Ricci-semisymmetric hypersurfaces M^n which are not semisymmetric exist in Euclidean spaces $\mathbf{E}^{n+1}$ for all dimensions $n \geq 5$. The existence of the immersions of M^n in $\mathbf{E}^{n+1}$ for which $R \cdot S = 0$ but $R \cdot R \neq 0$, relies on the (complete) integrability of a system of partial differential equations of Bourlet type. In particular [7] thus show that the example of Theorem

2.1 is not an isolated case, but belongs to an infinite family of which it is the simplest representative. The construction for all dimensions $n \geq 5$ of nontrivial hypersurfaces M^n of $\mathbf{E}^{n+1}$ for which $R \cdot S = 0$ but $R \cdot R \neq 0$ relies on Theorem 2.2 here below. The approach identifies links with the theory of completely integrable systems, and thus gives insight into the nonlinearity underlying the geometry.

Theorem 2.2. *There exists an isometric immersion of the n-dimensional manifold $M^n_{(1,q,r)}$, with $q \geq 3$, $r \geq 3$, and $q + r + 1 = n$, into $\mathbf{E}^{n+1}$ with the metric*

$$(8)\qquad \begin{aligned} ds^2 &= e^{2hx^1}\left((dx^1)^2 + B^2 \sum_{i=2}^{q+1} l_i{}^2(x^2, \cdots, x^{q+1})(dx^i)^2 \right. \\ &+ \left. C^2 \sum_{i=n-r+1}^{n} l_i{}^2(x^{n-r+1}, \cdots, x^n)(dx^i)^2\right), \end{aligned}$$

and principal curvatures,

$$\lambda_1 = 0,\ \lambda_i = (1-r)\beta\ e^{-hx^1}(2 \leq i \leq q+1),\ \lambda_i = -(1-q)\beta e^{-hx^1}(n-r+1 \leq i \leq n).$$

The parameters h, β, B, and C are related by the following conditions

$$(9)\qquad (1-q)(1-r)\beta^2 = h^2,$$

$$(10)\qquad \left(h^2 + (1-r)^2\beta^2\right) B^2 = 1,$$

$$(11)\qquad \left(h^2 + (1-q)^2\beta^2\right) C^2 = 1,$$

and the functions $\{l_i(x^{\alpha+1}, \cdots, x^{\alpha+m})\}_{i=\alpha+1}^{\alpha+m}$ are a solution of the completely integrable system

$$(12)\qquad \frac{\partial \gamma_{ij}}{\partial x^i} + \frac{\partial \gamma_{ji}}{\partial x^j} + \sum_{k\neq i, k\neq j} \gamma_{ki}\gamma_{kj} + l_i l_j = 0 \qquad (\alpha+1 \leq i, j \leq \alpha+m)(i \neq j),$$

$$(13)\qquad \frac{\partial \gamma_{jk}}{\partial x^i} = \gamma_{ji}\gamma_{ik} \qquad (\alpha+1 \leq i, j, k \leq \alpha+m)(i \neq j, j \neq k, k \neq i),$$

with

$$(14)\qquad \gamma_{ij} = \frac{1}{l_i}\frac{\partial l_j}{\partial x^i} \qquad (\alpha+1 \leq i, j \leq \alpha+m)(i \neq j),$$

and for $(\alpha, m) = (1, q)$, and $(\alpha, m) = (q+1, r)$, respectively. $M^n_{(1,q,r)}$ satisfies $R\cdot S = 0$, but is not semisymmetric.

It is now clear how to construct genuine Ricci-semisymmetric nonsemisymmetric hypersurfaces of all Euclidean spaces $\mathbf{E}^{n+1}$ $(n \geq 5)$ corresponding to all possible (p, q, r), thus with $p > 0, q > 1, r > 1$ and $p + q + r = n$. First, when $p > 1$, take a product immersion in $\mathbf{E}^{n+1}$ of $\mathbf{E}^{p-1}$ with a hypersurface $M^{n-p+1}_{(1,q,r)}$ of $\mathbf{E}^{n-p+2}$; if both $q \geq 3$ and $r \geq 3$, Theorem 2.2 proves the existence of this hypersurface

$M^{n-p+1}_{(1,q,r)}$ of $\mathbf{E}^{n-p+2}$. When e.g. $q = 2$, then i, j, k range over only 2 possible values, and consequently equations of the type (13) cannot occur. For the same reason (12) gives only 1 single equation. If we make the Ansatz

$$l_2(x^2, x^3) = \cos\phi(x^2, x^3), \tag{15}$$

$$l_3(x^2, x^3) = \sin\phi(x^2, x^3), \tag{16}$$

the remaining Gauss equation (12) turns into

$$\frac{\partial^2\phi}{(\partial x^2)^2} - \frac{\partial^2\phi}{(\partial x^3)^2} = -\sin(\phi). \tag{17}$$

This is the sine Gordon equation and essentially (upon adjustment of the normalisation, which is conventional) the equation which was encountered in Theorem 2.1.

3 Conformally flat hypersurfaces

Let (M^n, g) be an n-dimensional Riemannian manifold of class C^∞. (M^n, g) is called conformally flat if every point has a neighborhood which is conformal to an open set in the Euclidean space $\mathbf{E}^n$. Or, equivalently, (M^n, g) is conformally flat if there exists locally a function u such that $e^u g$ is a flat metric.

For 2-dimensional manifolds, the existence of isothermal coordinates shows that every surface is conformally flat. For manifolds of dimension $n \geq 4$, the necessary and sufficient condition for conformal flatness is given by the vanishing of the Weyl-conformal curvature tensor. The Weyl conformal curvature tensor C involves second order derivatives of the metric tensor, and is defined as

$$C(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-2}\left(\frac{\kappa}{n-1}\left(g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\right)\right.$$
$$\left. - \left(S(Y,Z)g(X,U) - g(X,Z)S(Y,U)\right) - \left(g(Y,Z)S(X,U) - S(X,Z)g(Y,U)\right)\right).$$

In dimension $n = 3$, however, the criterium for conformal flatness is that the Schouten tensor is a Codazzi tensor; this condition involves third order derivatives of the metric. The Schouten tensor T is defined (for an n-dimensional manifold) as

$$T(X,Y) = \frac{1}{(n-2)}\left(S(X,Y) - \frac{\kappa}{2(n-1)}g(X,Y)\right).$$

Equivalently, this can be formulated as the vanishing of the Bach tensor, $B = 0$, where

$$B(X,Y,Z) = (\nabla_X T)(Y,Z) - (\nabla_Y T)(X,Z).$$

In particular, for hypersurfaces M^n of a Euclidean space $\mathbf{E}^{n+1}$, we have in dimensions $n \geq 4$ a classical (and often reproved) result by Cartan-Schouten. The induced metric of a hypersurface M^n of $\mathbf{E}^{n+1}$ $(n \geq 4)$ is conformally flat if and only if at least

$n-1$ of the principal curvatures coincide at each point, if and only if it is "quasi umbilical". Whence, in dimensions $n \geq 4$, a conformally flat hypersurface can have at most 2 different principal curvatures at each point. This condition is conformally invariant as the following characterization shows more clearly: as quasi umbilic hypersurfaces, conformally flat hypersurfaces of dimension $n \geq 4$ are "channel hypersurfaces", i.e. envelopes of 1-parameter families of hyperspheres. As such, they are foliated with $(n-1)$-spheres.

In dimension $n = 3$, the result of Cartan-Schouten does not longer hold in it's full generality: still, channel hypersurfaces are conformally flat, but there are examples of conformally flat hypersurfaces with three distinct principal curvatures showing that the channel hypersurfaces form only a strict subclass of the conformally flat hypersurfaces. Indeed, [15] has given examples of conformally flat hypersurfaces with exactly 3 different principal curvatures. Recently, [12] described more examples of such conformally flat hypersurfaces. For example, "conformal product hypersurfaces" over surfaces of constant Gauss curvature — cones over constant Gauss curvature surfaces in S^3, cylinders over constant Gauss curvature surfaces in $\mathbf{E}^3$, and hypersurfaces of revolution over constant Gauss curvature surfaces in the hyperbolic half space H^3 — are conformally flat and have generically three different principal curvatures [12] [13]. In spite of this interesting phenomenon with 3 different principal curvatures, there are not so many particular results for 3-dimensional conformally flat hypersurfaces of $\mathbf{E}^4$. This is perhaps mainly due to the fact that the condition with the Schouten tensor keeps its nature of a set of coupled partial differential equations of third order.

Cartan gave the following characterization for 3-dimensional conformally flat hypersurfaces M^3 of $\mathbf{E}^4$: consider the six (complex) 2-dimensional distributions of planes in TM where the second fundamental form is a multiple of the first fundamental form (umbilical distributions). M^3 of $\mathbf{E}^4$ is conformally flat if and only if these distributions are integrable.

Another criterium, closely related to Cartan's, was given in [13]: a hypersurface in M^3 of $\mathbf{E}^4$ is conformally flat if and only if the "conformal fundamental forms"

$$\begin{aligned} \alpha_1 &= \sqrt{\lambda_3 - \lambda_1}\,\sqrt{\lambda_1 - \lambda_2}\,\omega_1 \\ \alpha_2 &= \sqrt{\lambda_1 - \lambda_2}\,\sqrt{\lambda_2 - \lambda_3}\,\omega_2 \\ \alpha_3 &= \sqrt{\lambda_2 - \lambda_3}\,\sqrt{\lambda_3 - \lambda_1}\,\omega_3 \end{aligned}$$

are integrable (closed). Here, $\omega_i : TM^3 \to \mathbf{E}$ denote the first fundamental forms with respect to a principal curvature frame and $\lambda_i : M^3 \to \mathbf{E}$ denote the principal curvatures of M^3. Note, that these forms are conformally invariant. As a consequence, each conformally flat hypersurface (with distinct principal curvatures) carries principal curvature line coordinates $(x^1, x^2, x^3) : M^3 \to \mathbf{E}^3$. Indeed, [12] proves the following structural theorem:

Theorem 3.1. *If M^3 is a conformally flat hypersurface of $\boldsymbol{E}^4$ with 3 different principal curvatures, then the curvature lines form a "Guichard net", i.e. there are principal curvature line coordinates $(x^1, x^2, x^3) : M^3 \to \boldsymbol{E}^3$ such that*

$$0 = {l_1}^2 + {l_2}^2 - {l_3}^2 \tag{18}$$

for the metric coefficients

$$g = \sum_{i=1}^{3} {l_i}^2 (dx^i)^2 .$$

This gives a necessary condition for a non-quasi-umbilical hypersurface of $\mathbf{E}^4$ to be conformally flat. Whether this condition is also sufficient, is still an open problem; at least no counterexamples are known.

Such a Guichard curvature line net can be considered as a 3-dimensional analog of isothermic curvature line coordinates on surfaces: thus, every conformally flat hypersurface M^3 of $\mathbf{E}^4$ is an "isothermic hypersurface". Therefore, concerning the converse of Theorem 3.1 which is still open and does not seem easy to decide, one can formulate the following:

Conjecture 3.1. *Every "isothermic hypersurface" M^3 of $\mathbf{E}^4$ is conformally flat.*

On the other hand, mapping the Guichard curvature line net of a conformally flat hypersurface into $\mathbf{E}^3$ via conformal coordinates yields a Guichard net in $\mathbf{E}^3$. Up to conformal transformations, the hypersurface can be uniquely reconstructed from the Guichard net in $\mathbf{E}^3$ [12], [13].

In order to gain more insight into the above conjecture — to decide whether it is true or not — it would be desirable to have more explicit results and examples of conformally flat hypersurfaces in $\mathbf{E}^4$.

We have e.g. the result of [10], classifying the conformally flat hypersurfaces of $\mathbf{E}^4$ whose mean curvature vector $\vec{H}$ is an eigenvector of their Laplacian, i.e. $\Delta\vec{H} = \lambda\vec{H}$. In view of [3], a hypersurface M^3 of $\mathbf{E}^4$ satisfying $\Delta\vec{H} = \lambda\vec{H}$, has both constant mean curvature and constant scalar curvature. From the results in [3] and [10] combined, one thus deduces that a conformally flat hypersurface M^3 of $\mathbf{E}^4$ with constant mean curvature and constant scalar curvature is locally an open part of a plane, a cylinder $S^k \times \mathbf{E}^{3-k}$ $(k = 1, 2)$, or a hypersphere; in all cases the hypersurface is isoparametric.

In [4] and [5] we look for information on conformally flat hypersurfaces of $\mathbf{E}^4$ with 3 different principal curvatures and one of the generalized curvatures being constant. In [4], we consider conformally flat hypersurfaces M^3 of $\mathbf{E}^4$ with constant mean curvature and prove the following

Theorem 3.2. *A conformally flat hypersurface M^3 of $\mathbf{E}^4$ with constant mean curvature and having 3 different principal curvatures at every point must be a minimal hypersurface.*

In [5], we consider 3-dimensional conformally flat hypersurfaces of $\mathbf{E}^4$ with constant Gauss-Kronecker curvature and prove the following

Theorem 3.3. *For a conformally flat hypersurface M^3 of $\mathbf{E}^4$ with constant Gauss-Kronecker curvature τ and 3 different principal curvatures, the value of this constant τ must be zero.*

The following example shows that conformally flat hypersurfaces M^3 with 3 different principal curvatures and having constant (zero) mean curvature and constant (zero) Gauss-Kronecker curvature do really exist.

Example 3.1. The manifold M^3 equiped with the metric g, given by

$$g = e^{2y^1}\left((dy^1)^2 + (dy^2)^2 + (dy^3)^2\right),$$

can be immersed into $\mathbf{E}^4$ with the following shape operator

$$S = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

where the principal curvatures are given by

(19) $$\lambda_1 = 0\,, \lambda_2 = e^{-y^1}\,, \lambda_3 = -e^{-y^1}\,.$$

4 Biharmonic hypersurfaces

The submanifolds of finite type [1] constitute a class of submanifolds generalizing the well known minimal submanifolds. Another line of thought, closely related however to the above one, leads to the biharmonic submanifolds, and the manifolds satisfying $\Delta\vec{H} = \lambda\vec{H}$. For this discussion, we restrict ourselves to a Euclidean ambient space.

Let therefore M^n be an n-dimensional, connected submanifold of the Euclidean space $\mathbf{E}^m$. Denote by $\vec{x}$, $\vec{H}$, and Δ respectively the position vector field of M^n, the mean curvature vector field of M^n, and the Laplace operator on M^n, with respect to the Riemannian metric g on M^n, induced from the Euclidean metric of the ambient space $\mathbf{E}^m$. Then, as is well known,

(20) $$\Delta\vec{x} = -n\vec{H}\,.$$

This shows, in particular, that M^n is a minimal submanifold of $\mathbf{E}^m$ if and only if its coordinate functions are harmonic (i.e. they are eigenfunctions of Δ with eigenvalue 0):

(21) $$\vec{H} = \vec{0} \iff \Delta\vec{x} = \vec{0}\,.$$

This condition (21) can be generalized in several directions. T. Takahashi [19] studied and classified submanifolds in Euclidean space for which

(22) $$\Delta\vec{x} = \lambda\vec{x}\,, \qquad \lambda \in \mathbf{R}\,,$$

i.e. submanifolds for which all coordinate functions are eigenfunctions of Δ with the same eigenvalue λ. Rephrased in terms of B.-Y. Chen's theory of submanifolds of finite type [1], Takahashi's condition (22) characterizes the 1-type submanifolds of $\mathbf{E}^m$.

Another direction, generalizing the condition (21), was taken by B.-Y. Chen, who initiated in 1985 the study of submanifolds of $\mathbf{E}^m$ satisfying

(23) $$\Delta\vec{H} = \vec{0}\,.$$

A submanifold M^n of $\mathbf{E}^m$ satisfying this condition (23) is said to have harmonic mean curvature vector field. In view of (20), submanifolds with harmonic mean curvature vector field are equivalently characterized by the condition

(24) $$\Delta^2\vec{x} = \vec{0}\,.$$

Therefore, submanifolds satisfying (23) are also called *biharmonic submanifolds.*

Both Takahasi's condition (22), and the condition (23), are generalized and combined into the condition

(25) $$\Delta\vec{H} = \lambda\vec{H}\,, \qquad \lambda \in \mathbf{R}\,.$$

For submanifolds satisfying (25), the mean curvature vector $\vec{H}$ is thus an eigenvector of the Laplacian.

We can summarize the inclusions between the different conditions (21), (22), (23), and (25) in the following table:

$$\begin{array}{ccc} \Delta\vec{H}=\vec{0} & \subset & \Delta\vec{H}=\lambda\vec{H} \\ \cup & & \cup \\ \vec{H}=\vec{0} & \subset & \Delta\vec{x}=\lambda\vec{x} \end{array}$$

Introducing a local orthonormal frame $\{e_i\}_{i=1}^3$, the Laplace operator Δ acting on a vector valued function $\vec{V}$, is given by

$$\Delta\vec{V}=\sum_{i=1}^{3}\left(\tilde{\nabla}_{\nabla_{e_i}e_i}\vec{V}-\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i}\vec{V}\right). \tag{26}$$

With (26), we find the following necessary and sufficient conditions for a hypersurface M^3 of $\mathbf{E}^4$ to satisfy (25):

$$\mathcal{A}(\nabla H)=-\frac{3H}{2}(\nabla H), \tag{27}$$

$$\Delta H+H\,\mathrm{tr}\mathcal{A}^2=\lambda H, \qquad \lambda\in\mathbf{R}, \tag{28}$$

where the Laplace operator Δ acting on a scalar valued function f, is given by

$$\Delta f=\sum_{i=1}^{3}\left(\nabla_{e_i}e_i f-e_ie_i f\right). \tag{29}$$

With $\lambda=0$, the case of a biharmonic hypersurface M^3 of $\mathbf{E}^4$ is recovered as a special case.

As remarked, minimal submanifolds are immediately seen to be biharmonic. Conversely, the question arises whether the class of submanifolds with harmonic mean curvature vector field is essentially larger than the class of minimal submanifolds. Otherwise stated, we consider the problem to classify the biharmonic submanifolds of $\mathbf{E}^m$, other than the minimal ones. For a survey of this and related problems, see e.g. [1]. Concerning this problem B.-Y. Chen conjectured the following

Conjecture 4.1. *The only biharmonic submanifolds of Euclidean spaces are the minimal ones.*

Concerning biharmonic submanifolds in Euclidean spaces, we have the following results, which support the above mentioned conjecture. B.-Y. Chen proved in 1985 that every biharmonic surface in $\mathbf{E}^3$ is minimal. Thereafter, I. Dimitrić generalized this result [9] and proved that a biharmonic submanifold M^n of a Euclidean space $\mathbf{E}^m$ is minimal if it is one of the following: (a) a curve, (b) a submanifold with constant mean curvature, (c) a hypersurface with at most two distinct principal curvatures, (d) a pseudo-umbilical submanifold of dimension $n\neq 4$, (e) a submanifold of finite type. Th. Hasanis and Th. Vlachos proved [11] that every biharmonic hypersurface in $\mathbf{E}^4$ is minimal.

In [2], we give an alternative proof of this last theorem by a different method. We summarize the main differences between the approach in [11] and the line of proof in [2]: [11] first classifies explicitely the H-hypersurfaces in $\mathbf{E}^4$, therefore introducing coordinates. Afterwards, the biharmonic hypersurfaces are singled out, invoking the use of a computer for lengthy calculations, concluding that there are none besides the minimal ones. Unfortunately, the possibilities for generalizing this method seem rather remote.

In our proof in [2], we proceed in an entirely coordinate independent way and with purely analytical arguments, avoiding the use of a computer. This makes the proof more concise. However, the main advantage of our approach, as we see it, is that it gives insight in the structure of the hypersurface and opens perspectives for generalization to higher (co)dimensions.

Finally, let us remark that [11] contains complementary information which, although not needed in [2], is of independent interest.

Summarizing, [2] gives a new proof of the following

Theorem 4.1. *Every hypersurface of $\mathbf{E}^4$ with harmonic mean curvature vector field, is minimal.*

The study of submanifolds of $\mathbf{E}^m$ satisfying (25) was initiated by B.-Y. Chen in 1988. A general implicit classification theorem for submanifolds of a Euclidean space which satisfy the condition $\Delta\vec{H} = \lambda\vec{H}$ for some $\lambda \in \mathbf{R}$ was obtained: a submanifold M^n of $\mathbf{E}^m$ satisfies (25) if and only if

(i) either M^n is biharmonic ($\lambda = 0$),

(ii) or M^n is of 1-type,

(iii) or M^n is of null 2-type.

However, only for surfaces in $\mathbf{E}^3$ a complete classification of the surfaces satisfying the condition (25) has been achieved:

A surface M^2 in $\mathbf{E}^3$ satisfies the condition $\Delta\vec{H} = \lambda\vec{H}$ for some $\lambda \in \mathbf{R}$ if and only if M^2 is either minimal ($\lambda = 0$), or M^2 is an open part of one of the following: a round 2-sphere $S^2(r)$, or a cylinder over a circle $S^1(r) \times \mathbf{E}^1$.

Concerning hypersurfaces of $\mathbf{E}^4$, [10] classifies the hypersurfaces satisfying (25) which are conformally flat:

A conformally flat hypersurface M^3 in $\mathbf{E}^4$ satisfies the condition $\Delta\vec{H} = \lambda\vec{H}$ for some $\lambda \in \mathbf{R}$ if and only if M^3 is either a minimal hypersurface ($\lambda = 0$), or M^3 is an open part of one of the following: a round 3-sphere $S^2(r)$, a cylinder over a circle $S^1(r) \times \mathbf{E}^2$, or a cylinder over a 2-sphere $S^2(r) \times \mathbf{E}^1$.

From this and other partial results, one may remark that all known examples of hypersurfaces of $\mathbf{E}^4$ which satisfy (25), have constant mean curvature. In [3] we prove, without any additional assumptions, that this is a property which holds in general. More precisely, we prove the following

Theorem 4.2. *A hypersurface of $\mathbf{E}^4$ satisfying $\Delta\vec{H} = \lambda\vec{H}$ must necessarilly have constant mean curvature.*

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