On Normal Sections of Veronese Submanifold

Kadri Arslan & Cihan Özgür

Abstract

In this study we consider Veronese Submanifold $\mathbf{V}^m$ which is a projective m-space $\mathbf{P}^m$ isometrically imbedded in $\mathbf{R}^{m+\frac{1}{2}m(m+1)}$ by its first standard imbedding. We also consider the normal sections of $\mathbf{V}^m$. Finally we show that $\mathbf{V}^m$ is of $AW(3)$-type.

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1 Introduction

Let M be an n-dimensional submanifold in $(m+d)-$ dimensional Euclidean space $\mathbf{R}^{n+d}$. Let ∇, $\bar{\nabla}$, and $\tilde{\nabla}$ denote the covariant derivatives in $T(M)$, $N(M)$ and $\mathbf{R}^{n+d}$ respectively. Thus $\tilde{\nabla}_X$ is just the directional derivative in the direction X in $\mathbf{R}^{n+d}$. Then for tangent vector fields X, Y and Z and normal vector field v over M we have $\tilde{\nabla}_X\ Y = \nabla_X Y + h(X,Y)$ and $\tilde{\nabla}_X\ v = -A_v X + \bar{\nabla}_X\ v$ where h is the second fundamental form and A_v is the shape operator of M [3].

For tangent vector fields X, Y, Z, W over M we define $\bar{\nabla}\ h$ and $\bar{\nabla}\bar{\nabla}\ h$ as usual by

$$\bar{\nabla}_X\ h(Y,Z) = (\bar{\nabla}_X\ h)(Y,Z) + h(\nabla_X Y, Z) + h(Y, \nabla_X Z),$$

$$\bar{\nabla}_W\ ((\bar{\nabla}_X\ h)(Y,Z)) = (\bar{\nabla}_W \bar{\nabla}_X\ h)(Y,Z) + (\bar{\nabla}_X\ h)(\nabla_W Y, Z) -$$

$$-(\bar{\nabla}_X\ h)(Y, \nabla_W Z) + (\bar{\nabla}_Y\ h)(\nabla_W X, Z).$$

2 Normal sections

Let M be a smooth n-dimensional submanifold in $(n+d)$-dimensional Euclidean space $\mathbf{R}^{n+d}$. For a point x in M and a non-zero tangent vector $X \in T_x M$, we define the $(d+1)-$dimensional affine subspace $E(x,X)$ of $\mathbf{R}^{n+d}$ by

$$E(x,X) = x + span\{X, T_x^{\perp} M\}.$$

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In a neighborhood of x the intersection $M\cap E(x,X)$ is a regular curve $\gamma:(-\varepsilon,\varepsilon)\to M$. We suppose the parameter $t\in(-\varepsilon,\varepsilon)$ is a multiple of the arc-length such that $\gamma(0)=x$ and $\gamma'(0)=X$. Each choose of $X\in T(M)$ yields a different curve which is called the *normal section of* M at x in the direction of X, where $X\in T_x(M)$ [5]. For such a normal section we can write

$$\gamma(t)=x+\lambda(t)X+N(t), \tag{1}$$

where $N(t)\in T_x^{\perp}M$ and $\lambda(t)\in\mathbf{R}$.

Definition 1. The submanifold M is said to have *pointwise k-planar normal sections* ($Pk-PNS$) if for each normal section γ, the higher order derivatives $\gamma'(0)$, $\gamma''(0)$,..., $\gamma^{(k+1)}(0)$ are linearly dependent as vectors in $\mathbf{R}^{n+d}$.

If $k=1$, then M is totally geodesic. Taking $k=2$, we note that submanifolds with pointwise 2-planar normal sections have been classified. They have parallel second fundamental form (i. e. $\bar{\nabla}\ h=0$) or hypersurfaces [6], [1]. Taking $k=3$, we note that submanifolds with pointwise 3-planar normal sections have been studied by various mathematicians (see [7],[2]).

Proposition 1. Let $\gamma(t)$ be a normal section of an $n-$dimensional submanifold M in $\mathbf{R}^{n+d}$. Then M has $Pk-PNS$ if and only if for every normal section, the normal vectors $N''(0), N'''(0)$,..., $N^{(k+1)}(0)$ are linearly dependent.

Proposition 2. Let $\gamma(t)$ be a normal section of M at point $\gamma(0)=x$ in the direction of $\gamma'(0)=X$. Then

$$N''(0)=h(X,X),$$

$$N'''(0)=(\bar{\nabla}_X\ h)(X,X),$$

$$N^{(iv)}(0)=4\lambda^{(iv)}(0)h(X,X)+3h(A_{h(X,X)}X,X)+(\bar{\nabla}_X\bar{\nabla}_X\ h)(X,X).$$

Proof. Let M be an n-dimensional submanifold of $\mathbf{R}^{n+d}$. Let X be a unit vector tangent to M at $x\in M$ and let $\gamma(s)$ be a normal section of M at p in the direction of X with its arc-length and $\gamma(0)=x$. We denote $T=\gamma'(s)$ the unit vector tangent to the normal section. Then we have

$$\gamma''(s)=\tilde{\nabla}_T\ T=\nabla_T T+h(T,T)$$

$$\gamma'''(s)=\tilde{\nabla}_T\tilde{\nabla}_T\ T=\nabla_T\nabla_T T+h(T,\nabla_T T)-A_{h(T,T)}T+\tilde{\nabla}_T\ (h(T,T))$$

$$\begin{aligned}\gamma^{(iv)}(s)=\tilde{\nabla}_T\tilde{\nabla}_T\tilde{\nabla}_T\ T&=\nabla_T\nabla_T\nabla_T T+2h(\nabla_T\nabla_T T,T)-h(A_{h(T,T)}T,T)\\&-\nabla_T(A_{h(T,T)}T)-3A_{h(\nabla_T T,T)}T+6(\bar{\nabla}_T\ h)(\nabla_T T,T)\\&-A_{(\bar{\nabla}_T h)(T,T)}T+h(\nabla_T T,\nabla_T T)+(\bar{\nabla}_T\bar{\nabla}_T\ h)(T,T),\end{aligned}$$

at $s=0$. After some calculation we obtain

$$\gamma'(0)=X\ ,\ N'(0)=0$$

$$\gamma''(0)=h(X,X)=N''(0)$$

$$\gamma'''(0) = \nabla_X \nabla_X X - A_{h(X,X)}X + (\bar{\nabla}_X\, h)(X,X) = \lambda'''(0)X + N'''(0)$$

$$\gamma^{(iv)}(0) = (terms\ in\ T_xM) + 4\lambda^{(iv)}(0)h(X,X) + 3h(A_{h(X,X)}X,X) + (\bar{\nabla}_X \bar{\nabla}_X\, h)(X,X) = \\ = \lambda^{(iv)}(0)X + N^{(iv)}(0).$$

For the normal parts we obtain the result.
Definition 2. Submanifolds are of $AW(3)$ *type* if

$$\|N''(0)\|^2\, N^{(iv)}(0) = < N''(0), N^{(iv)}(0) > N''(0). \tag{2}$$

(see [2]).

3 Veronese submanifolds

In this section we will consider the Veronese submanifold which is the first standard imbedding of the real projective space $\mathbf{P}^m$ in $\mathbf{R}^{m+\frac{1}{2}m(m+1)}$. The notation here is essentially the same as in [4].

Let $M(m+1;\mathbf{R})$ be the space of $(m+1)\times(m+1)$ matrices over $\mathbf{R}$. It is considered as an $(m+1)^2$-dimensional Euclidean space with the inner product $< A, B >= \dfrac{1}{2} traceAB^T$, where B^T is the transpose of the matrix B.

Someone consider $\mathbf{P}^m$ as the quotient space of the hypersphere

$$\mathbf{S}^m = \{\zeta \in \mathbf{R}^{m+1} : \zeta^T\zeta = 1\}$$

obtained by identifying ζ with $\zeta\lambda$, where ζ is a column vector and $\lambda \in \mathbf{R}$ such that $|\lambda| = 1$.

Define a mapping

$$\widetilde{\varphi} : \mathbf{S}^m \to H(m+1;\mathbf{R}) = \{A \in M(m+1;\mathbf{R}) : A^T = A\}$$

as follows

$$\widetilde{\varphi}(\zeta) = \zeta\zeta^T = \begin{pmatrix} |\zeta_0|^2 & \zeta_0\zeta_1 & - & \zeta_0\zeta_m \\ - & - & - & - \\ \zeta_m\zeta_0 & \zeta_m\zeta_1 & - & |\zeta_m|^2 \end{pmatrix} \tag{3}$$

for

$$\zeta = (\zeta_i) \in \mathbf{S}^m \subset \mathbf{R}^{m+1}, \quad 0 \le i \le m.$$

Then it is easy to verify that $\widetilde{\varphi}$ induces a mapping φ of $\mathbf{P}^m$ into $H(m+1;\mathbf{R})$:

$$\varphi(\pi(\zeta)) = \widetilde{\varphi}(\zeta) = \zeta\zeta^T, \tag{4}$$

where $\pi : \mathbf{S}^m \to \mathbf{P}^m$ is a Riemannian submersion [4]. We simply denote $\varphi(\pi(\zeta))$ by $\varphi(\zeta)$.

From (3) the image of $\mathbf{P}^m$ under φ is given by

$$\varphi(\mathbf{P}^m) = \{A \in H(m+1;\mathbf{R}) : A^2 = A \text{ and trace} A = 1\}.$$

Let $A=\zeta\zeta^T$ be a point in $\varphi(\mathbf{P}^m)$. Consider the curve

$$A(t)=\zeta\zeta^T;\ \zeta\in\mathbf{R}^{m+1} \tag{5}$$

in $\varphi(\mathbf{P}^m)$ with $A(0)=A$ and $A^{'}(0)=X\in T_A(\mathbf{P})^m)$. From $A^2(t)=A(t)$ one gets $XA^T+AX^T=X$. So we have

$$T_A(\mathbf{P}^m)=\{X\in H(m+1;\mathbf{R}):XA^T+AX^T=X\}. \tag{6}$$

Proposition 3 [4]. Let Y be a vector field tangent to $\mathbf{P}^m$ and $X\in T_A(\mathbf{P}^m)$. Consider a curve $A(t)$ in $\varphi(\mathbf{P}^m)$ so that $A(0)=A$ and $A'(0)=X$. Denote by $Y(t)$ the restriction of Y to $A(t)$. Then

$$A(t)Y(t)+Y(t)A(t)=Y(t).$$

Corollary 4. *Let $A(t)=\zeta\zeta^T$; $\zeta\in\boldsymbol{R}^{m+1}$ be a curve in $\varphi(\boldsymbol{P}^m)$. Then*

$$A(0)=X=\xi a^T+a\xi^T, \tag{7}$$

where

$$\zeta(0)=\xi,\zeta^{'}(0)=a. \tag{8}$$

Proof. Let $A(t)=\zeta\zeta^T$ be a curve in $\varphi(\mathbf{P}^m)$ with $A(0)=A$ and $A'(0)=X\in T_A(\mathbf{P}^m)$. Differentiating (5) we get

$$A'(t)=\zeta'\zeta^T+\zeta(\zeta')^T. \tag{9}$$

Substituting (8) into (9) we obtain (7).

A vector ν in $H(m+1;\mathbf{R})$ is normal to $\mathbf{P}^m$ at A if and only if $<X,\nu>=0$ for all X in $T_A(\mathbf{P}^m)$. Thus, ν is in $T_A^{\perp}(\mathbf{P}^m)$ if and only if $trace(X\nu)=0$ for all X in $T_A(\mathbf{P}^m)$. Therefore by (6) we obtain

$$T_A^{\perp}(\mathbf{P}^m)=\{\nu\in H(m+1;\mathbf{R}):A\nu=\nu A\}. \tag{10}$$

Proposition 5 [4]. If Y is a vector field as in the previous proposition and $X\in T_A(\mathbf{P}^m)$, then

$$\widetilde{\nabla}_X Y=Y'(0)=A(\widetilde{\nabla}_X Y)+(\widetilde{\nabla}_X Y)A+XY+YX,$$

$$\widetilde{h}(X,Y)=(XY+YX)(I-2A), \tag{11}$$

and

$$\nabla_X Y=2(XY+YX)A+A(\widetilde{\nabla}_X Y)+(\widetilde{\nabla}_X Y)A,$$

where $\widetilde{h}$ is the second fundamental form of $\mathbf{P}^m$ in $H(m+1;\mathbf{R})$ at A, ∇ and $\widetilde{\nabla}$ denote the induced connection on $\mathbf{P}^m$ and the Riemannian connection of the Euclidean space $H(m+1;\mathbf{R})$ at A respectively.

Corollary 6. *Let $A(t)=\zeta\zeta^T$ be a curve in $\varphi(\boldsymbol{P}^m)$ with $A(0)=A$ and $A'(0)=X\in T_A(\boldsymbol{P}^m)$. Then*

(12) $$\widetilde{h}(X,X) = 2aa^T - 2\xi a^T a \xi^T.$$

Proof. Differentiating $\xi^T \xi = 1$ and using (9) we obtain

(13) $$a^T \xi = 0.$$

Substituting (13) and (7) into (11) we get the result.

Theorem 7 [4]. *The isometric imbedding $\varphi : \boldsymbol{S}^m \to H(m+1;\boldsymbol{R})$, determined by (4), is the first standard imbedding of $\boldsymbol{P}^m$ into $H(m+1,\boldsymbol{R})$ and $\boldsymbol{P}^m$ lies in a hypersphere $\boldsymbol{S}(r)$ of $H(m+1;\boldsymbol{R})$ centered at $\dfrac{1}{m+1}I$ and with radius $r = \sqrt{\dfrac{m}{2}(m+1)}$.*

Definition 3. A real projective m-space $\mathbf{P}^m$ isometrically imbedded in $\mathbf{R}^{m+\frac{1}{2}m(m+1)}$ by its first standard imbedding φ determined by (4) is called the *Veronese submanifold* $\mathbf{V}^m$ [4].

4 Normal section of $\mathbf{V}^m$

Let $A \in H(m+1;\mathbf{R})$ be a symmetric matrix. Then we can decompose A into

$$\begin{aligned} A &= \xi\xi^T A \xi\xi^T + [(1-\xi\xi^T)A\xi\xi^T + \\ &+ \xi\xi^T A(1-\xi\xi^T)] + (1-\xi\xi^T)A(1-\xi\xi^T), \end{aligned}$$

where $\xi \in \mathbf{R}^{m+1}$ is the constant vector.

Interchanging A with the curve $A(t)$ in $\varphi(\mathbf{P}^m)$ we get

$$\begin{aligned} A(t) &= \xi\xi^T A(t) \xi\xi^T + [(1-\xi\xi^T)A(t)\xi\xi^T + \\ &+ \xi\xi^T A(t)(1-\xi\xi^T)] + (1-\xi\xi^T)A(t)(1-\xi\xi^T). \end{aligned}$$

We consider $A(t)$ as a normal section of $\mathbf{V}^m$ at point $A(0)$ in the direction of $A^{'}(0)$. Then by (1) we get

(14) $$\lambda(t)X = (1-\xi\xi^T)A(t)\xi\xi^T + \xi\xi^T A(t)(1-\xi\xi^T)$$

and

(15) $$N(t) = \xi\xi^T A(t)\xi\xi^T + (1-\xi\xi^T)A(t)(1-\xi\xi^T).$$

Combining (14) with (15) we have

(16) $$\lambda(t)a = (1-\xi\xi^T)A(t)\xi,$$

where $\xi = \zeta(0)$ and $a = \zeta^{'}(0)$.

Theorem 8. *Veronese submanifold $\boldsymbol{V}^m$ is of $AW(3)$ type.*

Proof. The normal section $A(t)$ of $\mathbf{V}^m$ at point $A(0) \in \mathbf{V}^m$ in the direction of

$$A^{'}(0) = X = \xi a^T + a\xi^T$$

is given by

$$A(t) = \zeta(t)\zeta^T(t),$$

where $\zeta(t) \in \mathbf{V}^m \subset \mathbf{R}^{m+1}$ and

$$\zeta^T(t)\zeta(t) = 1. \tag{17}$$

Differentiating (5) and (17) (surpressing the dependence of ζ on t to simplify the notation) we get

$$A^{'}(t) = \zeta'\zeta^T + \zeta(\zeta')^T = \lambda^{'}(t)X + N^{'}(t),$$
$$(\zeta^{'})^T\zeta + \zeta^T\zeta^{'} = 0.$$

Differentiating the previous equations and (10) respectively we obtain

$$(\zeta^{''})^T\zeta + 2(\zeta^{'})^T\zeta^{'} + \zeta^T\zeta^{''} = 0, \tag{18}$$

$$A^{''}(t) = \zeta^{''}\zeta^T + 2\zeta^{'}(\zeta^{'})^T + \zeta(\zeta^{''})^T, \tag{19}$$

and

$$\lambda^{''}(t)a = (1 - \xi\xi^T)A^{''}(t)\xi. \tag{20}$$

At the point $t = 0$, substituting (8) into (18)-(20) we have

$$\zeta'' = -\xi \parallel a \parallel^2, \tag{21}$$

$$\lambda''(0) = 0, \tag{22}$$

$$A^{''}(0) = -2 \parallel a \parallel^2 \xi\xi^T + 2aa^T = N^{''}(0). \tag{23}$$

Differentiating (19)-(21) we get

$$(\zeta^{'''})^T\zeta + 3(\zeta^{''})^T\zeta^{'} + 3(\zeta^{'})^T\zeta^{''} + \zeta^T\zeta^{'''} = 0 \tag{24}$$

$$A^{'''}(t) = \zeta^{'''}\zeta^T + 3\zeta^{''}(\zeta^{'})^T + 3\zeta^{'}(\zeta^{''})^T + \zeta(\zeta^{'''})^T, \tag{25}$$

$$\lambda^{'''}(t)a = (1 - \xi\xi^T)A^{'''}(t)\xi. \tag{26}$$

At the point $t = 0$, substituting (8), and (21) into (24)-(26) we have

$$(\zeta''')^T(0)\xi = 0, \tag{27}$$

$$\begin{aligned} A^{'''}(0) &= \zeta'''(0)\xi^T - 3 \parallel a \parallel^2 \xi a^T - 3 \parallel a \parallel^2 a\xi^T + \xi(\zeta''')^T(0) \\ &= \lambda^{'''}(0)X + N^{'''}(0), \end{aligned}$$

$$\lambda'''(0)a = \zeta'''(0) - 3a \parallel a \parallel^2 . \tag{28}$$

So by the use of (27) we also get

$$\lambda'''(0)X = \zeta'''(0)\xi^T + \xi(\zeta''')^T(0) - 3a \parallel a \parallel^2 \xi^T - 3\xi a^T \parallel a \parallel^2,$$

which implies

(29) $$N'''(0) = 0.$$

By definition $\mathbf{V}^m$ has parallel second fundamental form.

Now we want to show that $\mathbf{V}^m$ is of $AW(3)$ type. First, differentiating (24)-(27) we get

(30) $$(\zeta^{(iv)})^T\zeta + 4(\zeta^{'''})^T\zeta^{'} + 6(\zeta^{''})^T\zeta^{''} + 4(\zeta^{'})^T\zeta^{'''} + \zeta^T\zeta^{(iv)} = 0,$$

(31) $$A^{(iv)}(t) = \zeta^{(iv)}\zeta^T + 4\zeta^{'''}(\zeta^{'})^T + 6\zeta^{''}(\zeta^{''})^T + 4\zeta^{'}(\zeta^{'''})^T + \zeta(\zeta^{(iv)})^T,$$

(32) $$\lambda^{(iv)}(t)a = (1 - \xi\xi^T)A^{(iv)}(t)\xi.$$

At the point $t = 0$, substituting (8), (21) and (27) into (30)-(32) we have

$$\zeta^{(iv)}(0) = \parallel a \parallel^4 \xi,\ \lambda^{(iv)}(0) = 0,$$

(33) $$A^{(iv)}(0) = 8 \parallel a \parallel^4 \xi\xi^T - 8 \parallel a \parallel^4 = N^{(iv)}(0).$$

Comparing (33) with (23) we get $N^{(iv)}(0) = 4 \parallel a \parallel^2 N^{''}(0)$. So by Definition 2, $\mathbf{V}^m$ is of $AW(3)$ *type.*

References

[1] K. Arslan and A. West, *Non-Spherical Submanifolds with Pointwise, 2-Planar Normal Sections*, Bulletin London Math. Soc.28, 1996, 88-92.

[2] K. Arslan and A. West, *Product Submanifolds with Pointwise, 3-Planar Normal Sections*, Glasgow Math.J.37, 1995, 73-81.

[3] B.Y. Chen, *Geometry of Submanifolds*, Dekker 1973.

[4] B.Y. Chen, *Total Mean Curvature and Submanifolds of Finite Type*, World Scientific, 1984.

[5] B.Y. Chen, *Submanifolds with Planar Normal Sections*, Soochow J.Math.7, 1981, 19-24.

[6] D. Ferus, *Immersions with Parallel Second Fundamental Form.*, Math Z. 140(1974), 87-93.

[7] S.J. Li, *Isotropic Submanifolds with Pointwise 3-Planar Normal Sections*, Bollettino U. M. I. (7) 1-B (1987), 373-385.

Uludag University
Faculty of Art and Sciences
Department of Mathematics
Campus of Görükle, 16059
BURSA-TURKEY
e-mail: arslan@uu20.bim.edu.tr
e-mail: cozgur@mail.balikesir.edu.tr