On rank one matrices and invariant subspaces

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Abstract

This paper contains a characterization of complex matrices of rank 1 and characterization of a matrix -invariant subspace.

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Let M_n be the set of $n \times n$ matrices over **R**, **C** and for $x \in M_n(C)$, x^T denotes the complex transpose of x.

Definition 1. If $A \in M_n(C)$, $x \in \mathcal{C}^n$, $Ax = \lambda x$, $x \neq 0$, $\lambda \neq 0$,λ is called eigenvalue and x eigenvector of A associated with λ.

Definition 2. The set of all eigenvalues of the matrix A is called the spectrum of A and is denoted by $\sigma(A)$.

If x is an eigenvector associated with the eigenvalue λ of A , any nonzero scalar multiple of X is an eigenvector as well.

If p is a given polinomial and λ is an eigenvalue of the matrix A with x the corresponding eigenvector, then $p(\lambda)$ is an eigenvalue of the matrix $p(A)$ and x is the eigenvector of $p(A)$ associated with $p(\lambda)$.

Definition 3. The characteristic polynomial of A $\in M_n$ is defined as

$$p_A(t) = det(tI - A) \tag{1.1}$$

If A $\in M_n$, the characteristic polynomial p_A has degree n and the set of roots of this polynomial coincides with $\sigma(A)$. This result is nothing but the consequence of the expansion of det(tI-A) by minors ; each row of tI-A contributes one and only one power of t as the determinant is expanded .

Definition 4. A matrix B $\in M_n$ is said to be similar to a matrix A $\in M_n$ if there exists a nonsingular matrix S $\in M_n$ such that $B = S^{-1}AS$.

Similarity is an equivalence relation on M_n and partitions the set of n dimensional matrices into disjoint equivalence classes.

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Proposition 1. (characterization of a rank 1 matrix) *Let $A \in M_n$, $n \geq 2$, be a matrix of rank 1.Then*

i) There exist x,y vectors in $\mathcal{C}^n$; x,y $\neq$ 0 such that $A = xy^T$;

ii) A has at most one non-zero eigenvalue with algebraic multiplicity 1;

ii) This eigenvalue is $y^T x$;

iii) x is the right and y is the left eigenvector corresponding to this eigenvalue.

Proof. i) A is a matrix of rank 1 which means that any row of A can be expressed in terms of any other row of A. Let $A = [a_1, ..., an]$ where a_i, $i = 1, ..., n$ represent the rows of matrix A. Then rank A is 1 if and only if

$a_{i,1} = \alpha_{i-1}a_{1,1}$,$a_{i,2} = \alpha_{i-1}a_{1,2}$,....,$a_{i,n} = \alpha_{i-1}a_{1,n}$, for $i = 2, ..., n$.

Consider the vectors x,y defined as : $x = [1, \alpha_1, ..., \alpha_{n-1}]^T \in \mathbf{C}^n$ and $y = [a_{1,1}, ..., a_{1,n}]^T$.From the definition x, y satisfy the relation $A = xy^T$ and they are two non-zero vectors. Otherwise A would be the zero-matrix, contradiction with A having rank 1.

ii) Consider the characteristic polynomial

$$p_A(t) = t^n - (\sum_{i=1}^{n} \lambda_i)t^{n-1} + ... + det(A) \tag{1.2}$$

Since $det(A) = 0$ then $t = 0$ is a root of p_A. Thus 0 is an eigenvalue of A. Suppose that A has at least two non-zero distinct eigenvalues. They are defined by the relations:

$$Ax_1 = \lambda_1 x_1, x_1 \neq 0, x_1 \in \mathbf{C}^n \tag{1.3}$$

$$Ax_2 = \lambda_2 x_2, x_2 \neq 0, x_2 \in \mathbf{C}^n \tag{1.4}$$

and x_1, x_2 are linear independent vectors. $A = xy^T$ so the following relations takes place

$$xy^T x_1 = \lambda_1 x_1 \tag{1.5}$$

$$x = \frac{\lambda_1}{y^T x_1} x_1 \tag{1.6}$$

$$xy^T x_2 = \lambda_2 x_2 \tag{1.7}$$

$$x = \frac{\lambda_2}{y^T x_2} x_2 \tag{1.8}$$

The quantities $y^T x_1$, $y^T x_2$ are scalars so the relations (1.6),(1.8) are in contradiction with the linear independence of the vectors x_1, x_2. This eigenvalue , if exists has algebraic multiplicity 1. It is known that if an eigenvalue has multiplicity $k \geq 1$ then the rank of the matrix $A - \lambda I$ is $n - k$. But the O eigenvalue has multiplicity at least $n - 1$ therefore the possible non-zero eigenvalue would have multiplicity less than n-(n-1)=1. Which means that this non-zero eigenvalue, if exists, has multiplicity 1.

ii) Suppose there exists a non zero eigenvalue. Let v be an associated eigenvector. Then

$$Av = \lambda v \tag{1.9}$$

Then using the result from i)

$$xy^Tv = \lambda v, x, v \neq 0, \lambda \neq 0 \tag{1.10}$$

$$y^Txy^Tv = \lambda y^Tv, y^Tv \neq 0 \tag{1.11}$$

$$(y^Tx - \lambda)y^Tv = 0, y^Tv \neq 0 \tag{1.12}$$

Therefore $y^Tx = \lambda$.

iv) The right eigenvector for the non zero eigenvalue found above is x

$$Ax = (xy^T)x \tag{1.13}$$

and the left eigenvector is y :

$$y^TA = (y^Tx)y^T \tag{1.14}$$

which completes the proof.

It is also true that any matrix that can be expressed as the product between two non-zero vectors, $A = xy^T$ has rank 1, due to the fact that $a_{i,j} = x_i\bar{y}_j$, $i, j = 1, 2, ..., n$.

Definition 5. A subspace $W \subseteq \mathbf{C}^n$ is said to be $A - invariant$, for $A \in M_n$, if $Aw \in W$ for every $w \in W$.

If $A \in M_n$, each nonzero element of a one dimensional $A - invariant$ subspace of $\mathbf{C}^n$ is an eigenvector of A.

Proposition 2. *Let $A \in M_n$. If W is an $A - invariant$ subspace of $\mathbf{C}^n$ of dimension at least 1, then there is an eigenvector of A in W.*

Proof. Let $\{w_1, ...w_K\}$ be basis for the subspace W. Let $w \in W$. Then

$$w = \beta_1 w_1 + ... + \beta_K wK \tag{1.15}$$

$$Aw = \beta_1 Aw_1 + ... + \beta_K Aw_K \tag{1.16}$$

Each of the vectors $Aw_i \in W$, $i = 1, 2, ..., K$, therefore each of them can be expressed as a linear combination of the basis vectors of W. Let

$$Aw_i = \alpha_{i,1}w_1 + ... + \alpha_{i,K}w_K \tag{1.17}$$

So the expression of Aw is

$$Aw = \sum_{i=1}^{K} (\beta_i\alpha_{1,i} + ... + \beta_K\alpha_{K,i})w_i \tag{1.18}$$

If w is an eigenvector then $Aw = \lambda w$. To show that there exists an eigenvector of A in w it is equivalent to show that the matrix $\alpha = [\alpha_{i,j}]_{i,j}$ has an eigenvalue. This is true because the $det(tI - \alpha)$ has at least a complex root .

Definition 6. The subspace W is called $\mathcal{F}invariant$, for a family $\mathcal{F} \in M_n$, if W is invariant of each $A \in \mathcal{F}$.

For a commuting family a similar result takes place.

Proposition 3. *If $\mathcal{F}$ is a commuting family, then there is a vector $x \in \mathbf{C}^n$ that is an eigenvector of every matrix $A \in \mathcal{F}$.*

Proof. Let W be such an invariant subspace that has minimum dimension. Every nonzero vector in W is an eigenvector of every $A \in \mathcal{F}$. If this is not the case, then for some matrix $A \in \mathcal{F}$, not every nonzero vector in W is an eigenvector of A. Since W is $\mathcal{F}$ invariant then it is A-invariant. Then by proposition 2 there is an eigenvector of A in W. Define

(1.19) $$W_0 = \{y \in W : Ay = \lambda y\}$$

W_0 is a subspace of W. Because of the assumption about A, $W_0 \neq W$, so the dimension of W_0 is strictly smaller than that of W. W_0 is $\mathcal{F}$ invariant, which contradicts the choice of W. Therefore every nonzero vector in W is an eigenvector of every $A \in \mathcal{F}$, which completes the proof .

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