On some special vector fields

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Abstract

We introduce the notion of F-distinguished vector fields in a deformation algebra, where F is a $(1,1)$-tensor field. The aim of this paper is to study these special vector fields and, using their properties, to characterize spherical hypersurfaces, when F is the shape operator. The last section is devoted to the relation between the geometrical properties of Weyl manifolds and the algebraic properties of Weyl algebras.

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1 F-distinguished vector fields

Let M be a connected paracompact, smooth manifold of dimension $n \geq 2$. Let TM be the tangent bundle of M and $\mathcal{T}_s^r(M)$ be the $\mathcal{C}^\infty(M)$-module of tensor fields of type (r,s) on M. We denote $\mathcal{T}_0^1(M)$ (respectively $\mathcal{T}_1^0(M)$) by $\mathcal{X}(M)$ (respectively $\Lambda^1(M)$).

Let A be a $(1,2)-$tensor field on M. The $\mathcal{C}^\infty(M)-$module $\mathcal{X}(M)$ becomes a $\mathcal{C}^\infty(M)-$algebra if we consider the multiplication rule given by $X \circ Y = A(X,Y)$, $\forall X, Y \in \mathcal{X}(M)$. This algebra is denoted by $\mathcal{U}(M,A)$ and it is called the algebra associated to A. If ∇ and $\overline{\nabla}$ are two linear connections on M, then $\mathcal{U}\left(M, \overline{\nabla} - \nabla\right)$ is called the deformation algebra defined by the pair $\left(\nabla, \overline{\nabla}\right)$ [9].

Let (M,g) be a Riemannian manifold and F be a $(1,1)$-tensor field on M.

Definition 1.1 $X \in \mathcal{X}(M)$ *is called a* (∇, F)*-Killing vector field if*

$$g(\nabla_Z F(X), Y) + g(Z, \nabla_Y F(X)) = 0, \forall Y, Z \in \mathcal{X}(M) \tag{1.1}$$

holds.

One should remark that this is equivalent to the condition that $F(X)$ is a ∇-Killing vector field.

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Definition 1.2 *Let A be a $(1,2)$-tensor field on M. X is called a F-distinguished vector field in the algebra $\mathcal{U}(M,A)$ if one has*

$$g(A(Z,F(X)),Y) + g(Z,A(Y,F(X)) = 0, \forall Y,Z \in \mathcal{X}(M). \tag{1.2}$$

In the particular case when F is the identity tensor field of type $(1,1)$ one gets the known notion of distinguished vector fields on M [10].

Let $\overset{\circ}{\nabla}$ be the Levi-Civita connection, associated to g and $\nabla, \overline{\nabla}$ be linear connections on M, given by

$$\nabla = \overset{\circ}{\nabla} - \frac{1}{2}A \ , \ \overline{\nabla} = \overset{\circ}{\nabla} + \frac{1}{2}A.$$

Proposition 1.1 *Let $X \in \mathcal{U}(M,A)$. The following assertions are equivalent:*

i) X is a (∇, F)-Killing vector field and a F-distinguished vector field in the algebra $\mathcal{U}(M,A)$;

ii) X is a $(\overline{\nabla}, F)$-Killing vector field and a F-distinguished vector field in the algebra $\mathcal{U}(M,A)$;

iii) X is a (∇, F) and $(\overline{\nabla}, F)$-Killing vector field.

Proof. i)$\Leftrightarrow$ii) Let X be F-distinguished vector field in the algebra $\mathcal{U}(M,A)$. Hence $g(A(Z,F(X)),Y) + g(Z,A(Y,F(X)) = 0, \forall Y,Z \in \mathcal{X}(M)$. Since $A = \overline{\nabla} - \nabla$, then $g(\nabla_Z F(X),Y) + g(Z,\nabla_Y F(X)) = 0 \Leftrightarrow g(\overline{\nabla}_Z F(X),Y) + g(Z,\overline{\nabla}_Y F(X)) = 0$.

iii) $\Leftrightarrow$i) It is a consequence of (1.1) and (1.2).

Remark 1.1 Let A^i_{jk}, g_{ij} and X^i be the local components of A, g and X, respectively, in a local system of coordinates. The formula (1.2) becomes

$$(A^p_{js}g_{pk} + A^p_{ks}g_{jp})F^s_i X^i = 0. \tag{1.3}$$

The integral curves of F-distinguished vector fields, called F-distinguished curves, verify the following differential system of equations

$$(A^p_{js}g_{pk} + A^p_{ks}g_{jp})F^s_i \frac{dx^i}{dt} = 0. \tag{1.4}$$

Remark 1.2 Let (M,g) be a Riemannian manifold, $\overset{\circ}{\nabla}$ be the Levi-Civita connection associated to g and $\pi \in \Lambda^1(M)$. Let ∇ be the Lyra connection associated to π, hence

$$\nabla_X Y = \overset{\circ}{\nabla}_X Y + \pi(Y)X - g(X,Y)P, \forall X,Y \in \mathcal{X}(M), \tag{1.5}$$

where P is the dual vector field associated to π i.e. $g(P,Z) = \pi(Z), \forall Z \in \mathcal{X}(M)$. Then $A = \nabla - \overset{\circ}{\nabla}$ verifies

$$A^i_{jk} = \delta^i_k \pi_j - g_{jk}\pi^i, \tag{1.6}$$

where $\pi^i = g^{ik}\pi_k$. So, from (1.6) we notice that (1.3) is satisfied. Hence all the elements of the Lyra algebra $\mathcal{U}(M,A)$ are F-distinguished vector fields.

2 On spherical hypersurfaces

Let M^n be a hypersurface in the Euclidean space $\mathbf{E}^{n+1}$. Let us denote by g, b and h the first, the second and the third fundamental forms on M, respectively. We suppose that b is nondegenerated. Let $\overset{1}{\nabla}, \overset{2}{\nabla}$ and $\overset{3}{\nabla}$ be the Levi-Civita connections associated to g, b and h, respectively. Let us denote by

$$A = \overset{1}{\nabla} - \overset{2}{\nabla} \ , \ A' = \overset{2}{\nabla} - \overset{3}{\nabla} \ , \ A'' = \overset{1}{\nabla} - \overset{3}{\nabla}$$

We note that

$$b(A(X,Y),Z) = b(A'(X,Y),Z) = 2b(A''(X,Y),Z) = -\frac{1}{2}(\overset{1}{\nabla}_X b)(Y,Z). \tag{2.1}$$

We suppose that the $(1,1)$-tensor field F is the shape operator of the hypersurface M. Then $F_i^s = b^{sq} g_{qi}$.

Remark 2.1 The deformation algebras $\mathcal{U}(M,A)$, $\mathcal{U}(M,A')$ and $\mathcal{U}(M,A'')$ have the same F-distinguished vector fields.

Indeed, this is a consequence of (1.3) and (2.1).

Remark 2.2 Let M^2 be a surface in the Euclidean space $\mathbf{E}^3$, given by

$$\begin{aligned} x &= (a + b\cos x^1)\cos x^2, \\ y &= (a + b\cos x^1)\sin x^2, \\ z &= b\sin x^1, \end{aligned}$$

where $a > b > 0, a$ and b are constants, $x^2 \in \mathbf{R}$ and $x^1 \in \mathbf{R} \setminus \{(2k+1)\frac{\pi}{2}\}$, $k \in \mathbf{Z}$. One has the following nonvanishing components of A, A' and A'' :

$$A_{22}^1 = \frac{2a\sin x^1}{b}, A_{21}^2 = A_{12}^2 = \frac{2a\sin x^1}{(a + b\cos x^1)\cos x^1},$$

$$A'^1_{22} = -\frac{a\sin x^1}{b}, A'^2_{21} = A'^2_{12} = -\frac{a\sin x^1}{(a + b\cos x^1)\cos x^1},$$

$$A''^1_{22} = \frac{a\sin x^1}{b}, A''^2_{21} = A''^2_{12} = \frac{a\sin x^1}{(a + b\cos x^1)\cos x^1}.$$

We point out that $x^1 = k\pi, k \in \mathbf{Z}$, the equatorial circles, are F-distinguished curves of the algebras $\mathcal{U}(M,A)$,$\mathcal{U}(M,A')$ and $\mathcal{U}(M,A'')$. Indeed these curves verify (1.4).

Theorem 2.1 *Let $M^n \subset \mathbf{E}^{n+1}$ be a hypersurface and F be the shape operator of M. Then the following conditions are equivalent:*

i) All the elements of the algebra $\mathcal{U}(M,A)$ are F-distinguished vector fields.

ii) M is a spherical hypersurface.

Proof. i) $\Rightarrow$ ii) One has $g(A(Z,F(X)),Y) + g(Z, A(Y,F(X)) = 0$, $\forall X, Y, Z, \in \mathcal{X}(M)$.

Therefore, using (2.1), in local coordinates, we obtain

$$(2.2) \qquad (g_{sj} b^{sr} \overset{1}{\nabla}_r b_{ki} + g_{sk} b^{sr} \overset{1}{\nabla}_r b_{ji}) b^{iq} g_{ql} = 0.$$

Moreover, (2.2) implies

$$(2.3) \qquad (g_{is} b^{sr} \overset{1}{\nabla}_r b_{jk}) b^{iq} g_{ql} = 0.$$

Then (2.3) lead to $\overset{1}{\nabla}_r b_{jk} = 0$. Hence one gets i) ([8]).

ii) ⇒i) is obvious.

3 Weyl manifolds

Let g be a semi-Riemannian metric on M and let $\widehat{g} = \{e^u g | u \in \mathcal{C}^\infty(M)\}$ be the conformal class defined by g.

Let W be a Weyl structure on the conformal manifold $(M, \widehat{g})$ i.e. a mapping $W : \widehat{g} \mapsto \Lambda^1(M)$. Hence $W(e^u g) = W(g) - du, \forall u \in \mathcal{C}^\infty(M)$. The triple $(M, \widehat{g}, W)$ is called a Weyl manifold. There exists a unique torsion free connection ∇, compatible with the Weyl structure W i.e.

$$(3.1) \qquad \nabla g + W(g) \otimes g = 0,$$

given by

$$(3.2) \qquad \begin{array}{c} 2g(\nabla_X Y, Z) = X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) + \\ +W(g)(X)g(Y,Z) + W(g)(Y)g(X,Z) - W(g)(Z)g(X,Y) + \\ +g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X). \end{array}$$

∇ is called the Weyl conformal connection. Let $\overset{\circ}{\nabla}$ be the Levi-Civita connection associated to g and $A = \nabla - \overset{\circ}{\nabla}$. $\mathcal{U}(M, A)$ is called the Weyl algebra. One has

$$(3.3) \qquad 2g(A(X,Y),Z) = W(g)(X)g(Y,Z) + W(g)(Y)g(X,Z) - W(g)(Z)g(X,Y).$$

The torsion free connections ∇ and $\overset{\circ}{\nabla}$ are called projectively equivalent if their unparametrized geodesic coincide [5].

The goal of this section is to study the Weyl algebra. Our algebraic approach gives some insights of geometrical nature.

Theorem 3.1 *Let $(M, \widehat{g}, W)$ be a Weyl manifold. Let R, S and $\overset{\circ}{R}, \overset{\circ}{S}$ be the curvature tensor field and the Ricci tensor field associated to ∇ and $\overset{\circ}{\nabla}$, respectively. Let F be a $(1,1)$-tensor field . We suppose that the mapping $F_p : T_pM \mapsto T_pM$ is surjective, $\forall p \in M$. Then the following assertions are equivalent:*

i) Every element of the algebra $\mathcal{U}(M, A)$ is a F-distinguished vector field.

ii) The algebra $\mathcal{U}(M, A)$ is associative.

iii) ∇ and $\overset{\circ}{\nabla}$ are projectively equivalent.

iv) $R = \overset{\circ}{R}$, *when* S *is nondegenerated.*

v) $S = \overset{\circ}{S}$, *when* S *is nondegenerated and the 1-form* $W(g)$ *is exact.*

vi) $\nabla = \overset{\circ}{\nabla}$.

Proof. i)⇒vi). Let X be a F-distinguished vector contained in the Weyl algebra $\mathcal{U}(M, A)$. From (1.2) and

$$\begin{aligned} 2g(A(Z,F(X)),Y) = \; & W(g)(Z)g(F(X),Y) + W(g)(F(X))g(Y,Z) - \\ & -W(g)(Y)g(Z,F(X)), \\ 2g(A(F(Y),Y),Z) = \; & W(g)(Y)g(F(X),Z) + W(g)(F(X))g(Y,Z) - \\ & -W(g)(Z)g(F(X),Y) \end{aligned}$$

one gets

$$W(g)(F(X))g(Z,Y) = 0, \forall X,Y,Z \in \mathcal{X}(M). \tag{3.4}$$

Since the mapping $F_p : T_pM \mapsto T_pM$ is surjective, $\forall p \in M$, (3.3) and (3.4) imply $g(A(X,Y),Z) = 0, \forall X,Y,Z \in \mathcal{X}(M)$. Therefore $A = 0$ i.e. vi).

vi) ⇒i) If $A = 0$, then (1.2) is satisfied.

ii) ⇔ iii) ⇔ iv) ⇔ v) ⇔ iv) [6].

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