Symmetry groups and Lagrangians associated with Tzitzeica surfaces

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Abstract

The aim of this paper is to apply the symmetry group approach to the partial differential equations arising in the Tzitzeica surfaces theory. As a consequence, we find a new solution of the Tzitzeica equation which defines a ruled Tzitzeica surface. The symmetry groups for the Liouville-Tzitzeica equation and the Tzitzeica equation are determined. We prove that these are also Euler-Lagrange equations. The variational symmetry groups and respectively, the associated conservation laws are found. Recently, we have shown that the simple Tzitzeica surfaces equation is an Euler-Lagrange equation. According to these results, the Tzitzeica surfaces theory is strongly related to variational problems, and hence this is a subject of global differential geometry.

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1 Introduction

Tzitzeica – the founder of the centroaffine geometry – introduced in 1907 the so-called *S-surfaces*, with the property that $\frac{K}{d^4}$=constant, where K is the Gaussian curvature and d is the distance from the origin to the tangent plane at an arbitrary point [24]. These surfaces are called *Tzitzeica surfaces* by Gheorghiu, *affine spheres* by Blaschke, and *projectives spheres* by Wilczynski. The spheres and the quadrics are the simplest examples of Tzitzeica surfaces. Tzitzeica also considered their generalization to hypersurfaces (see for details [25] and [26]). Mayer [20], Gheorghiu [11], Dobrescu [9] and Vranceanu [30], [31] studied the properties of these hypersurfaces. Gheorghiu proved that the Tzitzeica hypersurfaces can be considered as affine spaces A_{n-1} embedded in an affine Euclidean space E_n, and he introduced a new class of affine space A_n^0. Udrişte [27] gave and studied new examples of these affine spaces.

Let us briefly explain the basic notions of Tzitzeica surfaces theory. Consider $D \subset \mathbf{R}^2$ an open set and let

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$$\Sigma: \ \mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \ \ (u,v) \in D,$$

be a surface in $\mathbf{R}^3$. Assume that Σ is different from a cone with the vertex at the origin, i.e.,

$$(\mathbf{r}, \mathbf{r}_u, \mathbf{r}_v) \neq 0, \tag{1.1}$$

where $\mathbf{r}$ denotes the position vector of an arbitrary point on the surface. In this case, the surface Σ can be defined by the following PDE system

$$\begin{cases} \mathbf{r}_{uu} = a\mathbf{r}_u + b\mathbf{r}_v + c\mathbf{r} \\ \mathbf{r}_{uv} = a'\mathbf{r}_u + b'\mathbf{r}_v + c'\mathbf{r} \\ \mathbf{r}_{vv} = a''\mathbf{r}_u + b''\mathbf{r}_v + c''\mathbf{r}, \end{cases} \tag{1.2}$$

where $a, a', a'', ...$ are nine functions of u and v, and for which the conditions of completely integrability

$$(\mathbf{r}_{uu})_v = (\mathbf{r}_{uv})_u, \quad (\mathbf{r}_{uv})_v = (\mathbf{r}_{vv})_u \tag{1.3}$$

must be satisfied. Note that (1.2) defines a surface leaving a centroaffinity aside, so that, the coefficients $a, a', a'', ...$ are called *centroaffine invariants.* Recall that the asymptotic lines are different for a surface which is not developable. Next, if the surface Σ is related to the asymptotic lines then we have $c = c'' = 0$ in (1.2), and so Σ is given by

$$\begin{cases} \mathbf{r}_{uu} = a\mathbf{r}_u + b\mathbf{r}_v \\ \mathbf{r}_{uv} = a'\mathbf{r}_u + b'\mathbf{r}_v + c'\mathbf{r} \\ \mathbf{r}_{vv} = a''\mathbf{r}_u + b''\mathbf{r}_v. \end{cases} \tag{1.4}$$

Theorem 1 (Tzitzeica). *Let Σ be a surface defined by the PDE system (1.4).Then the ratio $I = \frac{K}{d^4}$ is constant if and only if $a' = b' = 0$.*

According to the above theorem, the following PDE system

$$\begin{cases} \mathbf{r}_{uu} = a\mathbf{r}_u + b\mathbf{r}_v \\ \mathbf{r}_{uv} = h\mathbf{r} \\ \mathbf{r}_{vv} = a''\mathbf{r}_u + b''\mathbf{r}_v, \end{cases} \tag{1.5}$$

defines a Tzitzeica surface (here denote $c' = h$). In this case, the integrability conditions (3) turn into

$$\begin{aligned} ah = h_u, \quad a_v = ba'' + h, \quad b_v + bb'' = 0, \\ h_v = b''h, \quad a''_u + aa'' = 0, \quad h = b''_u + a''b. \end{aligned} \tag{1.6}$$

Particular cases:
1. if $b = 0$ or $a'' = 0$ then Σ is a simply ruled surface;
2. if $b = 0$, $a'' \neq 0$ then the coordinates curves $v = v_0$ are straight lines;
3. if $b \neq 0$, $a'' = 0$ then the coordinates curves $u = u_0$ are straight lines;
4. if $b = a'' = 0$, then Σ is a double ruled surface (a quadric surface).

The PDE system (1.5) takes two different forms if Σ is a Tzitzeica ruled surface or not. Thus, the *Tzitzeica ruled surfaces* are given by the PDE system

$$(1.7) \qquad \begin{cases} \mathbf{r}_{uu} = \frac{h_u}{h}\mathbf{r}_u + \frac{\varphi(u)}{h}\mathbf{r}_v \\ \mathbf{r}_{uv} = h\mathbf{r} \\ \mathbf{r}_{vv} = \frac{h_v}{h}\mathbf{r}_v, \end{cases}$$

where h is a solution of the Liouville-Tzitzeica PDE

$$(1.8) \qquad (\ln h)_{uv} = h.$$

The *Tzitzeica surfaces which are not ruled surfaces* are defined by the system

$$(1.9) \qquad \begin{cases} \mathbf{r}_{uu} = \frac{h_u}{h}\mathbf{r}_u + \frac{1}{h}\mathbf{r}_v \\ \mathbf{r}_{uv} = h\mathbf{r} \\ \mathbf{r}_{vv} = \frac{1}{h}\mathbf{r}_u + \frac{h_v}{h}\mathbf{r}_v, \end{cases}$$

with h a solution of the Tzitzeica equation

$$(1.10) \qquad (\ln h)_{uv} = h - \frac{1}{h^2}.$$

It can be proved that (1.5) is related to the scalar PDE system

$$(1.11) \qquad \begin{cases} \theta_{uu} = a\theta_u + b\theta_v \\ \theta_{uv} = h\theta \\ \theta_{vv} = a''\theta_u + b''\theta_v, \end{cases}$$

for which (1.6) holds, through the condition: three independent solutions of (1.11) and (1.6) define a Tzitzeica surface. Every linear combination of three independent solutions of (1.11) is also a solution of this system. Some recent results related to the Liouville-Tzitzeica equation and the Tzitzeica equation can be found in [6] and [32].

The purpose of this paper is to apply the classical symmetry approach to the PDE systems arising in Tzitzeica surfaces theory, and to make the connection of our study to the already known results. The symmetry analysis of PDEs, introduced by Sophus Lie at the end of the 19-th century [18] has been proven to be a powerful tool in studying ODEs and PDE systems arising in geometry, mechanics and physics (see e.g., [2] - [5], [7], [8], [13], [17], [19], [21], [22], [28], [29] and [32]). Lie's method, known today as the *classical Lie method*, is based on the notion of the *symmetry group*. This is a local group of transformations acting on the space of the independent variables and the space of the dependent variables of a studied PDE system with the property that it leaves the set of its solutions invariant. Since the classical Lie method is an algorithmic procedure, many symbolic manipulation programs have been designed for this purpose [14]. Unfortunately, in the case of the PDE systems (1.5) and (1.11), none of these packages can be used due to the form of these systems.

The paper is structured as follows: in Section 2 we present the classical symmetry approach for a PDE system (see for details Olver's book [21]). Classical symmetries associated with the PDE system (1.5) and respectively, with the PDE system (1.11) are studied in Section 3. Variational symmetries and conservation laws for the Liouville-Tzitzeica PDE and Tzitzeica PDE are given in the last section.

2 Symmetry group of a PDE system

Consider the PDE system

$$\Delta_\nu(x, u^{(n)}) = 0, \quad \nu = 1, ..., l, \tag{2.12}$$

where $x = (x^1, ..., x^p)$ are the independent variables, $u = (u^1, ..., u^q)$ are the dependent variables and

$$\Delta(x, u^{(n)}) = (\Delta_1(x, u^{(n)}), ..., \Delta_l(x, u^{(n)}))$$

is a differentiable function. Denote $u^{(n)}$ all the partial derivatives of the function u to 0 up to the order n. For any function $u = h(x)$, where

$$h : D \subset \mathbf{R}^p \to U \subset \mathbf{R}^q, \quad h = (h^1, ..., h^q),$$

we can define its *prolongation of order* n,

$$pr^{(n)}h : D \to U^{(n)},$$

where $u^{(n)} = pr^{(n)}h$, $u_J^\alpha = \partial_J h^\alpha$, so that, for each $x \in D$, $pr^{(n)}h$ is a vector whose $qp^{(n)} = C_{p+n}^n$ entries represent the values of h and its derivatives up to order n at the point x.

The space $D \times U^{(n)}$, whose coordinates represent the independent variables, the dependent variables and the derivatives of the dependent variables up to order n, is called the *jet space of order* n of the underlying space $D \times U$. In this sense, Δ is a map from the jet space $D \times U^{(n)}$ to $\mathbf{R}^l$, and moreover, the PDE system (2.12) defines the subvariety

$$\mathcal{S} = \{(x, u^{(n)}) \,|\Delta(x, u^{(n)}) = 0\}$$

of the total jet space $D \times U^{(n)}$. In that follows, (2.12) is identified with $\mathcal{S}$.

Consider $M \subset D \times U$ an open set. A *symmetry group associated with the PDE system (2.12)* is a local group of transformations G acting on M with the property that whenever $u = f(x)$ is a solution of (2.12), and whenever $g \cdot f$ is defined for $g \in G$, then $u = g \cdot f(x)$ is also a solution of the system. Then the system (2.12) is called *invariant with respect to G*.

Let X be a vector field on M. Assume that X is the infinitesimal generator of the symmetry group of the PDE system (2.12), which is the (local) one-parameter group $\exp(\varepsilon X)$. Then its associated prolongation of order n is the one parameter group $pr^{(n)}[\exp(\varepsilon X)]$ with the infinitesimal generator

$$pr^{(n)}X|_{(x,u^{(n)})} = \frac{d}{d\varepsilon} pr^{(n)}[\exp(\varepsilon X)](x, u^{(n)})|_{\varepsilon=0}$$

where $(x, u^{(n)}) \in M^{(n)}$. This is a vector field on $M^{(n)}$ called the *prolongation of order n of X*.

The PDE system (2.12) is a *maximal rank system* if the Jacobi matrix

$$J_\Delta(x, u^{(n)}) = \left(\frac{\partial \Delta_\nu}{\partial x^i}, \frac{\partial \Delta_\nu}{\partial u_J^\alpha} \right)$$

of the function Δ satisfies the condition $rank J_\Delta = l$ on $\mathcal{S}$.

Theorem 2. *Let*

$$X = \sum_{i=1}^{p} \zeta^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

be a vector field on open set $M \subset D \times U$. The prolongation of order n of X is given by the vector field

$$pr^{(n)}X = X + \sum_{\alpha=1}^{q}\sum_{J} \phi_\alpha^J(x,u^{(n)})\frac{\partial}{\partial u_J^\alpha}, \tag{2.13}$$

defined on the corresponding jet space $M^{(n)} \subset D \times U^{(n)}$, where

$$\phi_\alpha^J(x,u^{(n)}) = D_J\left(\phi_\alpha - \sum_{i=1}^{p}\zeta^i u_i^\alpha\right) + \sum_{i=1}^{p}\zeta^i u_{J,i}^\alpha,$$

with $u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i}$, $u_{J,i}^\alpha = \frac{\partial u_J^\alpha}{\partial x^i}$ (the second summation in (2.13) is over all multi-indices $J = (j_1, ...j_k)$ with $1 \leq j_k \leq p$, $1 \leq k \leq n$).

The coefficient functions of X, i.e., ξ^i and ϕ_α, are called *infinitesimals.*

Theorem 3 (Criterion for infinitesimal invariance). *Let (2.12) be a PDE system of maximal rank on $M \subset D \times U$. If G is a local group of transformations acting on M and*

$$pr^{(n)}X[\Delta_\nu(x,u^n)] = 0, \quad \nu = 1,...,l, \tag{2.14}$$

whenever $\Delta_\nu(x,u^{(n)}) = 0$, for every infinitesimal generator X of G, then G is a symmetry group of the PDE system (2.12).

The classical Lie method: consider X a vector field on M and write the criterion for infinitesimal invariance for the system (2.12); eliminate any dependence between the partial derivatives of the functions u^α, determined by the PDE system itself; write the condition (2.14) like polynomials in the partial derivatives of u^α; equate to zero the coefficients of partial derivatives of u^α. The resulting over-determined linear PDE system for the infinitesimals ζ^i, ϕ_α is called the *determining equations of the symmetry group G.*

3 Symmetries of the Tzitzeica Surfaces PDE systems

3.1 Classical symmetries of the PDE system (1.5)

In the first part of this subsection we discuss the symmetries of the PDE system (1.5), in the case when Σ admits two real asymptotic lines. Note that this system can be written as

$$\begin{cases} x_{uu} = ax_u + bx_v \\ x_{uv} = hx \\ x_{vv} = a''x_u + b''x_v \\ y_{uu} = ay_u + by_v \\ y_{uv} = hy \\ y_{vv} = a''y_u + b''y_v \\ z_{uu} = az_u + bz_v \\ z_{uv} = hz \\ z_{vv} = a''z_u + b''z_v, \end{cases} \tag{3.15}$$

for which the conditions (1.1) and (1.6) must be satisfied. The relation (1.1) is equivalent to

$$(y_u z_v - z_u y_v)x - (x_u z_v - x_v z_u)y + (x_u y_v - x_v y_u)z = f, \tag{3.16}$$

where $f = f(u,v)$ is a nonzero function. Let $D \times U^{(2)}$ be the jet space of second order associated with the PDE system (3.15) whose coordinates represent the independent variables u, v, the dependent variables x, y, z and the derivatives of the dependent variables up to the order two. Consider $M \subset D \times U$ an open set and let

$$X = \zeta\partial_u + \eta\partial_v + \phi\partial_x + \lambda\partial_y + \psi\partial_z \tag{3.17}$$

be the infinitesimal generator of the symmetry group G of the PDE system (3.15) and (3.16) (here the infinitesimals ζ, η, ϕ, λ and ψ are functions of u, v, x, y and z).

3.1.1 Symmetries acting on the space of the dependent variables

In that follows, we seek for a symmetry subgroup G_1 (of the symmetry group G), which acts on the space of the dependent variables x, y and z of the system (3.15). Let Y be its associated infinitesimal generator. For that, set $\zeta = 0$, $\eta = 0$, $\phi = \phi(x,y,z)$, $\lambda = \lambda(x,y,z)$ and $\psi = \psi(x,y,z)$ in (3.17). The second prolongation of the vector field Y (see (2.13)) is defined by the following functions

$$\begin{aligned}
\Phi^u &= \phi_x x_u + \phi_y y_u + \phi_z z_u,\\
\Phi^v &= \phi_x x_v + \phi_y y_v + \phi_z z_v,\\
\Phi^{uu} &= \phi_{xx}x_u^2 + \phi_{yy}y_u^2 + \phi_{zz}z_u^2 + 2\phi_{xy}x_u y_u + 2\phi_{xz}x_u z_u + 2\phi_{yz}y_u z_u\\
&\quad + \phi_x x_{uu} + \phi_y y_{uu} + \phi_z z_{uu},\\
\Phi^{uv} &= \phi_{xx}x_u x_v + \phi_{xy}x_v y_u + \phi_{xz}x_v z_u + \phi_{xy}x_u y_v + \phi_{yy}y_u y_v + \phi_{yz}y_v z_u\\
&\quad + \phi_{xz}z_v x_u + \phi_{yz}y_u z_v \phi_{zz}z_u z_v + \phi_x x_{uv} + \phi_y y_{uv} + \phi_z z_{uv},\\
\Phi^{vv} &= \phi_{xx}x_v^2 + 2\phi_{xy}x_v y_v + 2\phi_{xz}x_v z_v + 2\phi_{yz}y_v z_v + \phi_{yy}y_v^2 + \phi_{zz}z_v^2\\
&\quad + \phi_x x_{vv} + \phi_y y_{vv} + \phi_z z_{vv},
\end{aligned}$$

and respectively, by the functions Λ^u, Λ^v, Λ^{uu}, Λ^{uv}, Λ^{vv}, and Ψ^u, Ψ^v, Ψ^{uu}, Ψ^{uv}, Ψ^{vv} that have a similar form obtained by substituting ϕ by λ, and respectively, ϕ by ψ. Note that the PDE system (3.15) and (3.16) is of maximal rank. Writing the criterion for infinitesimal invariance (2.14) for the system (3.15) we obtain the relations

$$\begin{cases}
a\Phi^u + b\Phi^v - \Phi^{uu} = 0\\
h\Phi - \Phi^{uv} = 0\\
a''\Phi^u + b''\Phi^v - \Phi^{vv} = 0\\
\dots\dots\dots\dots\dots\dots .
\end{cases} \tag{3.18}$$

Substituting Φ^u, Φ^v and Φ^{uu} into the first relation of (3.18) we get

$$\begin{gathered}
a(\phi_x x_u + \phi_y y_u + \phi_z z_u) + b(\phi_x x_v + \phi_y y_v + \phi_z z_v) - \phi_{xx}x_u^2 - 2\phi_{xy}x_u y_u - 2\phi_{xz}x_u z_u\\
-2\phi_{yz}y_u z_u - \phi_{yy}y_u^2 - \phi_{zz}z_u^2 - \phi_x x_{uu} - \phi_y y_{uu} - \phi_z z_{uu} = 0.
\end{gathered}$$

Eliminate the dependencies between the derivatives of x, y and z by substituting into the above relation

$$x_{uu} = ax_u + bx_v, \quad y_{uu} = ay_u + by_v, \quad z_{uu} = az_u + bz_v,$$

so that, we obtain

$$\phi_{xx}x_u^2 + \phi_{yy}y_u^2 + \phi_{zz}z_u^2 + 2\phi_{xy}x_u y_u + 2\phi_{xz}x_u z_u + 2\phi_{yz}y_u z_u = 0.$$

Equate to zero the coefficients of the remaining unconstrained partial derivatives of x, y and z. It follows the PDE system

$$\phi_{xx} = 0, \quad \phi_{yy} = 0, \quad \phi_{zz} = 0, \quad \phi_{xy} = 0, \quad \phi_{yz} = 0, \quad \phi_{xz} = 0,$$

with the solution given by

$$\phi(x, y, z) = a_{11}x + a_{12}y + a_{13}z + k,$$

where a_{11}, a_{12}, a_{13} and k are real numbers. Substituting the function ϕ into the next two relations of the system (3.18) we get $k = 0$, and thus

$$\phi(x, y, z) = a_{11}x + a_{12}y + a_{13}z.$$

Similarly, from the next six relations of the system (3.18) we have

$$\lambda(x, y, z) = a_{21}x + a_{22}y + a_{23}z,$$

and

$$\psi(x, y, z) = a_{31}x + a_{32}y + a_{33}z,$$

where a_{ij} are real numbers. Writing the criterion for infinitesimal invariance (2.14) in the case of the equation (3.16), we get the relation

$$\phi(y_u z_v - z_u y_v) + \lambda(x_v z_u - x_u z_v) + \psi(x_u y_v - x_v y_u) + \Phi^u(zy_v - yz_v) + \Phi^v(yz_u - zy_u)$$
$$+\Lambda^u(xz_v - zx_v) + \Lambda^v(zx_u - xz_u) + \Psi^u(yx_v - xy_v) + \Psi^v(xy_u - yx_u) = 0.$$

The substitution of the functions $\Phi^u, \Phi^v, ...$ into the above relation yields

$$(x_u y_v - x_v y_u)(\psi - x\psi_x - y\psi_y + z\phi_x + z\lambda_y) + (x_v z_u - x_u z_v)(\lambda - x\lambda_x - z\lambda_z + y\phi_x$$
$$+y\psi_z) + (y_u z_v - y_v z_u)(\phi - y\phi_y - z\phi_z + x\lambda_y + x\psi_z) = 0.$$

Any dependencies between the derivatives of x, y and z is eliminated by using the relation (3.16). It follows the relation

$$\phi_x + \lambda_y + \psi_z = 0$$

which is equivalent to

$$a_{33} + a_{11} + a_{22} = 0.$$

Thus, the infinitesimal generator Y of the symmetry subgroup G_1 is defined by the following functions

$$\begin{aligned}\phi(x,y,z) &= a_{11}x + a_{12}y + a_{13}z,\\ \lambda(x,y,z) &= a_{21}x + a_{22}y + a_{23}z,\\ \psi(x,y,z) &= a_{31}x + a_{32}y - (a_{11}+a_{22})z,\end{aligned}$$

and so, this has the form

$$Y = a_{11}\left(x\partial_x - z\partial_z\right) + a_{22}\left(y\partial_y - z\partial_z\right) + a_{12}y\partial_x + a_{13}z\partial_x$$

$$+a_{21}x\partial_y + a_{23}z\partial_y + a_{31}x\partial_z + a_{32}y\partial_z.$$

Theorem 4. *The Lie algebra of the subgroup G_1 of the symmetry group G of the PDE system (3.15) and (3.16) (G_1 acts on the space of the dependent variables) is described by the vector fields*

$$Y_1 = x\partial_x - z\partial_z,\;\; Y_2 = y\partial_y - z\partial_z,\;\; Y_3 = y\partial_x,\;\; Y_4 = z\partial_x \tag{3.19}$$

$$Y_5 = x\partial_y,\;\; Y_6 = z\partial_y,\;\; Y_7 = x\partial_z,\;\; Y_8 = y\partial_z.$$

The subgroup G_1 is the unimodular subgroup of the group of centroaffine transformations.

Knowledge of the symmetry subgroup G_1 allows us to find the group-invariant solutions of the PDE system (3.15) and (3.16). For example, consider the subalgebra described by the vector fields Y_1 and Y_2. A function F invariant with respect to these vector fields satisfies $Y_1(F) = 0$, and $Y_2(F) = 0$. Therefore, we get $F = \varphi(u, v, xyz)$. In this case we obtain the well-known Tzitzeica surfaces

$$z = \frac{C}{xy},\;\; C \in \mathbf{R}^*. \tag{3.20}$$

3.1.2 Symmetries acting on the space of the independent variables

Let G_2 be the symmetry subgroup of the symmetry group G acting only on the space of the independent variables u and v of the system (3.15) and (3.16). Suppose Z is its associated infinitesimal generator, for which we have $\zeta = \zeta(u,v)$, $\eta = \eta(u,v)$, $\phi = 0$, $\lambda = 0$, and $\psi = 0$ in (3.17). Similarly, applying the classical Lie method we get

Theorem 5. *The symmetry subgroup G_2 acting on the space of the independent variables of the system (3.15) is generated by the vector field*

$$Z = \zeta(u)\partial_u + \eta(v)\partial_v, \tag{3.21}$$

where the infinitesimals ζ and η satisfy the PDE system:

$$\begin{cases}\zeta a_u + \eta a_v + a\zeta_u + \zeta_{uu} = 0\\ \zeta b_u + \eta b_v - b\eta_v + 2b\zeta_u = 0\\ \zeta h_u + \eta h_v + h(\zeta_u + \eta_v) = 0\\ \zeta a''_u + \eta a''_v - a''\zeta_u + 2a''\eta_v = 0\\ \zeta b''_u + \eta b''_v + b''\eta_v + \eta_{vv} = 0,\end{cases} \tag{3.22}$$

and the functions a, b, h, a'' and b'' satisfy the integrability conditions (1.6).

In that follows, we discuss the PDE system (3.22) for the Tzitzeica surfaces defined by (1.7), and respectively by (1.9).

I. If Σ is a ruled Tzitzeica surface (1.7) then the completely integrability conditions (1.6) are given by

$$a = \frac{h_u}{h}, \ \ b = \frac{\varphi(u)}{h}, \ \ a'' = 0, \ \ b'' = \frac{h_v}{h},$$

where h is a solution of the Liouville-Tzitzeica equation (1.8). In this case, the relations (3.22) can be written as

$$\begin{cases} \zeta h_u + \eta h_v + h(\zeta_u + \eta_v) = 0 \\ \zeta^3 = \frac{k}{\varphi}, \\ h h_{uv} - h_u h_v = h^3. \end{cases}$$

Consider the change of variables $\zeta = \frac{1}{U'}$ and $\eta = -\frac{1}{V'}$, where $U = U(u)$ and $V = V(v)$. The first equation implies $h = U'V'\mu(U + V)$, and by substituting it into the last PDE (that is the Liouville-Tzitzeica equation), we get the second order ODE

$$\mu\mu'' - \mu'^2 = \mu^3,$$

with the general solution given by

$$\mu(t) = \begin{cases} \frac{2}{(t+C)^2}, & k = 0 \\ \\ \frac{l^2}{2\cos^2(\frac{l}{2}t+C)}, & k = -l^2 \\ \\ \frac{l^2}{2\sinh^2(\frac{l}{2}t+C)}, & k = l^2, \ \ l > 0, \end{cases}$$

where $t = U + V$. Consider the following changes of functions:
- for $k = 0$ set $\tilde{U} = F(U)$ and $\tilde{V} = G(V)$, where $\tilde{U} = U + C$ and $\tilde{V} = V$;
- $k = l^2$ then $\tilde{U} = \tanh\frac{l}{2}(U + C)$ and $\tilde{V} = \tanh\frac{l}{2}V$;
- $k = -l^2$ set $\tilde{U} = \cot(\frac{l}{2}U + C)$ and $\tilde{V} = \tan\frac{l}{2}V$.

We obtain the general solution of the Liouville-Tzitzeica equation (see [15] and [25]) written as

$$h(u, v) = \frac{2\tilde{U}'\tilde{V}'}{(\tilde{U} + \tilde{V})^2}. \tag{3.23}$$

II. If Σ is a Tzitzeica surface which is not a ruled surface then the conditions (1.6) turn into

$$a = \frac{h_u}{h}, \ \ b = a'' = \frac{1}{h}, \ \ b'' = \frac{h_v}{h},$$

where h is a solution of the Tzitzeica equation (1.10). If we substitute these functions into the system (3.22) then we get the PDE system

$$\zeta_u = 0, \quad \eta_v = 0, \quad \zeta h_u + \eta h_v + h(\zeta_u + \eta_v) = 0,$$

with the solution $\zeta = C_1$, $\eta = C_2$ and $h = \mu(C_1 v - C_2 u)$. Then the infinitesimal generator of G_2 has the form

$$Z = C_1\partial_u + C_2\partial_v. \tag{3.24}$$

Substituting the function $h = \mu(C_1 v - C_2 u)$ into the Tzitzeica equation we obtain the following second order ODE

$$-C_1 C_2(\mu\mu'' - \mu'^2) = \mu^3 - 1. \tag{3.25}$$

Case 1. If $C_1C_2 = 0$ then $\mu = 1$, and so $h = 1$. In this case we get the Tzitzeica solution [25].
Case 2. If $C_1C_2 \neq 0$ then denote $k = -\frac{1}{C_1C_2}$. The ODE (3.25) becomes

$$\mu\mu'' - \mu'^2 = k(\mu^3 - 1).$$

Consider $k = 1$. Then (3.25) can be reduced to the following first order ODE

$$\mu'^2 = 2\mu^3 + C\mu^2 + 1, \quad C \in \mathbf{R},$$

and after the change of function $\mu = \frac{1}{2}g$, this turns into

$$g'^2 = g^3 + Cg^2 + 4. \tag{3.26}$$

Let λ be the real root of the polynomial written in the right hand side of the ODE (3.26). Since that $\lambda \neq 0$ and λ cannot be a triple solution, the ODE is equivalent to

$$g'^2 = (g - \lambda)\left(g^2 - \frac{4}{\lambda^2}g - \frac{4}{\lambda}\right). \tag{3.27}$$

Case 2.1. If $\lambda = -1$ then $C = -3$ and (3.27) becomes

$$g'^2 = (g+1)(g-2)^2.$$

If $g = w^{-2} + 2$ then we get the ODE

$$w'^2 = \frac{1}{4}\left(3w^2 + 1\right),$$

with the general solution given by

$$w(t) = \frac{1}{\sqrt{3}}\sinh\left(\frac{t\sqrt{3}}{2} + C_1\right), \quad C_1 \in \mathbf{R},$$

where $t = u + v$. It follows that the Tzitzeica equation has the solution

$$h = \frac{1}{2w^2} + 1,$$

and for $C_1 = 0$ this can be written as follows

$$h(t) = \frac{3}{2\sinh^2\left(\frac{t\sqrt{3}}{2}\right)} + 1, \quad t = u + v. \tag{3.28}$$

Case 2.2. Assume $\lambda \neq -1$. Then for $\lambda > -1$ or $C < -3$, the roots of the polynomial from the right hand side of (3.26) are three different real numbers. For $\lambda < -1$ or

$C > -3$ then the polynomial has a real root and the other two are complex. Note that, in this case, the integral

$$J = \int \frac{dg}{\sqrt{(g-\lambda)\left(g^2 - \frac{4}{\lambda^2}g - \frac{4}{\lambda}\right)}}$$

can be reduced to a first genus elliptical integral [10]

$$J = \int \frac{d\varphi}{\sqrt{1 - k^2 \sin\varphi}}.$$

In conclusion, for $C \neq -3$, the solutions of the Tzitzeica equation of the form $h = \mu(u+v)$ are written in the terms of the elliptical functions.

Proposition 1. *The solution (3.28) defines a revolution Tzitzeica surface. Moreover, there is also an associated ruled Tzitzeica surface.*

Proof. Studying the revolution surfaces defined by (1.11), Tzitzeica (see [25], pp. 164–174) showed that $h_u = h_v$. From that we get $h = \mu(u+v)$ which must satisfy the ODE

$$\mu\mu'' - \mu'^2 = \mu^3 - 1. \tag{3.29}$$

Tzitzeica did not integrate the equation (3.29) but he proved that, by using the notation

$$\frac{\mu'^2 - 2\mu^3 - 1}{4\mu^2} = -k^2,$$

the solution of the system (1.11) is the following

$$\theta(u,v) = k_1 e^{\int \frac{h'-1}{2h} d\alpha} \cos k\beta + k_2 e^{\int \frac{h'-1}{2h} d\alpha} \sin k\beta + k_3 e^{\int \frac{h^2}{h'+1} d\alpha}, \tag{3.30}$$

for $k \neq 0$, and respectively,

$$\theta(u,v) = e^{\int \frac{h'-1}{2h} d\alpha} \left[k_1 \left(\beta^2 + \int \frac{4\mu}{\mu'+1} d\alpha \right) + k_2\beta + k_3 \right],$$

for $k = 0$, where $\alpha = u+v$, $\beta = u-v$ and k_i are real numbers. According to our results, we have $k^2 = -\frac{C}{4}$. It results that the function (3.28) defines a revolution surface (3.30). Moreover, after the change of functions

$$\tilde{U} = \tanh \frac{\sqrt{3}}{2}(U + C_1), \quad \tilde{V} = \tanh \frac{\sqrt{3}}{2} V,$$

the function h takes the form

$$h(u,v) = \frac{2\tilde{U}'\tilde{V}'}{(\tilde{U}+\tilde{V})^2} + 1 = H(u,v) + 1.$$

Notice that the function H is in fact a solution of Liouville-Tzitzeica equation (1.8) and this defines a ruled Tzitzeica surface.

Proposition 2. *The solution (3.30) given by Tzitzeica is invariant under the symmetry subgroup generated by (3.24).*

3.2 Classical symmetries of the PDE system (1.11)

Consider the PDE system (1.11) with the integrability conditions (1.6). Let $D \times \bar{U}^{(2)}$ be the second order jet space associated with this system, whose coordinates are the independent variables u, v, the dependent variable θ and the derivatives of the dependent variable up to the order two. Consider $\bar{M} \subset D \times \bar{U}$ an open set and let

$$\bar{X} = \zeta\partial_u + \eta\partial_v + \alpha\partial_\theta$$

be the infinitesimal generator of the symmetry group $\bar{G}$ of this system, where the infinitesimals ζ, η and α are functions of u, v and θ.

Similarly, we are interested in finding the symmetry subgroups $\bar{G}_1$, and respectively, $\bar{G}_2$ of the symmetry group $\bar{G}$, with the property that they act on the space of the dependent variable θ, and respectively, the space of the independent variables u and v of (1.11). It this case, it can be proved that

Theorem 6. *The symmetry subgroup $\bar{G}_1$, acting on the space of the dependent variable of the system (1.11) is generated by the vector field*

$$\bar{Y}_1 = \theta\partial_\theta. \tag{3.31}$$

Theorem 7. *The vector field*

$$\bar{Z} = \zeta(u)\partial_u + \eta(v)\partial_v, \tag{3.32}$$

where ζ and η satisfy the relations (3.22), generates the symmetry subgroup $\bar{G}_2$ of $\bar{G}$ (G_2 acts on the space of the independent variables of the system (1.11)).

3.3 Symmetries for Liouville-Tzitzeica PDE and Tzitzeica PDE

In order to study the symmetries of the Liouville-Tzitzeica equation (1.8) and the Tzitzeica equation (1.10), consider the change of function $\ln h = \omega$. Then

$$\omega_{uv} = e^\omega \tag{3.33}$$

is equivalent to the Liouville-Tzitzeica PDE, and respectively, the equation

$$\omega_{uv} = e^\omega - e^{-2\omega} \tag{3.34}$$

is equivalent to the Tzitzeica PDE. Note that (3.33) and (3.34) belong to the same class of second order PDEs

$$\omega_{uv} = H(\omega), \tag{3.35}$$

that has been studied by Sophus Lie, and recently by Pucci, Saccomandi and Mansfield [19].

Theorem 8. *If $\zeta = \zeta(u, v, \omega)$, $\eta = \eta(u, v, \omega)$ and $\phi = \phi(u, v, \omega)$ satisfy the PDE system*

$$\zeta_v = 0, \quad \zeta_\omega = 0, \quad \eta_u = 0, \quad \eta_\omega = 0, \quad \phi_{\omega\omega} = 0, \quad \phi_{u\omega} = 0, \quad \phi_{v\omega} = 0, \tag{3.36}$$

$$\phi_{uv} + (\phi_\omega - \zeta_u - \eta_v - \phi)H - H'\phi = 0,$$

where $H = H(\omega)$, then

$$X = \zeta\partial_u + \eta\partial_v + \phi\partial_\omega$$

is the infinitesimal generator of the symmetry group associated with (3.35).

In particular, in the case of the equations (3.33) and (3.34) we get

Theorem 9. *The vector field*

$$W = f\partial_u + g\partial_v - (f' + g')\partial_\omega, \tag{3.37}$$

where $f = f(u)$ and $g = g(v)$, generates the symmetry group of the Liouville-Tzitzeica equation (3.33).

Theorem 10. *There is a 3D Lie algebra associated with the symmetry group of the Tzitzeica equation (3.34) and this is described by*

$$U_1 = u\partial_u - v\partial_v,\ \ U_2 = \partial_u,\ \ U_3 = \partial_v. \tag{3.38}$$

Notice that for any $\omega = f(u,v)$ solution of the Tzitzeica PDE (3.33) the following functions

$$\omega^{(1)} = f(e^\varepsilon u, e^{-\varepsilon}v),\ \ \omega^{(2)} = f(u-\varepsilon, v),\ \ \omega^{(3)} = f(u, v-\varepsilon),$$

are also solutions of the equation (here ε is a real number).

Using the adjoint representation of the symmetry group of the Tzitzeica PDE (3.33) given by the following table

Ad	U_1	U_2	U_3
U_1	U_1	$e^\varepsilon U_2$	$e^{-\varepsilon}U_3$
U_2	$U_1 - \varepsilon U_2$	U_2	U_3
U_3	$U_1 + \varepsilon U_3$	U_2	U_3

Table 1

the one-dimensional subalgebras of the Lie algebra associated with Tzitzeica equation (1.10) can be classified. The optimal system of these subalgebras is described by U_2, U_3 and respectively, by $U_2 - U_3$.

1. For U_2 and U_3, the group-invariant solution is $\omega = 0$. In this case, we get the Tzitzeica solution $h = 1$.
2. The group-invariant solutions with respect to $U_2 - U_3$ have the form $\omega = f(u+v)$ (and respectively, we have $h = \mu(u+v)$ for equation (1.10)). Note that this case was studied in Section 3.1.2.

Theorem 11. *The second order PDE invariant with respect to the symmetry group of the Tzitzeica PDE, has the form*

$$H(\omega, \omega_u\omega_v, \omega_{uv}, \omega_{uu}\omega_{vv}) = 0. \tag{3.39}$$

Proof. Consider the following maximal chain of Lie subalgebras

$$\{U_2\} \subset \{U_2, U_3\} \subset \{U_1, U_2, U_3\}.$$

of the Lie algebra of the symmetry group related to (3.33). Let $F(u, v, \omega^{(2)}) = 0$ be a second order PDE for the unknown function $\omega = \omega(u,v)$. If this is an equation invariant under the considered symmetry group, then the criterion for infinitesimal invariance must be satisfied by the vector fields U_i. If $pr^{(2)}U_2(F) = 0$ then we get $F = F_1(v, \omega^{(2)})$. From $pr^{(2)}U_3(F) = 0$ it results $F = F_2(\omega^{(2)})$, and $pr^{(2)}U_1(F) = 0$ leads us to the PDE

$$U_1(F_2) - \omega_u \frac{\partial F_2}{\partial \omega_u} + \omega_v \frac{\partial F_2}{\partial \omega_v} - 2\omega_{uu} \frac{\partial F_2}{\partial F_2 \omega_{uu}} + 2\omega_{vv} \frac{\partial F_2}{\partial \omega_{vv}} = 0,$$

with the general solution given by $F_2 = H$ in (3.39).

4 Lagrangians associated with Tzitzeica PDEs

4.1 Euler-Lagrange equations and Tzitzeica PDEs

If a PDE is an Euler-Lagrange equation then the classical Lie symmetries lead us to variational symmetries for the associated variational problem. Moreover, by using the Noether Theorem, we can determine conservation laws for the studied PDE (see for more details [1], [7], [16], [21], [22], [29], [32] and references therein). In this subsection we study the inverse problem for the equations (3.33) and (3.34).

Remind that a second order PDE

$$\Delta(u, v, \omega^{(2)}) = 0, \tag{4.40}$$

for the unknown function $\omega = \omega(u, v)$ is said to be *identically to an Euler-Lagrange equation* if and only if the integrability Helmholtz conditions

$$\begin{cases} \frac{\partial \Delta}{\partial \omega_u} & = & D_u \left(\frac{\partial \Delta}{\partial \omega_{uu}} \right) + D_v \left(\frac{1}{2} \frac{\partial \Delta}{\partial \omega_{uv}} \right) \\ \frac{\partial T}{\partial \omega_v} & = & D_u \left(\frac{1}{2} \frac{\partial \Delta}{\partial \omega_{uv}} \right) + D_v \left(\frac{\partial \Delta}{\partial \omega_{vv}} \right) \end{cases} \tag{4.41}$$

are satisfied, where denote D the total derivatives. In this case, we can find a function L called *Lagrangian* for which the Euler-Lagrange equation, i.e.,

$$E(L) = \frac{\partial L}{\partial \omega} - D_u \left(\frac{\partial L}{\partial \omega_u} \right) - D_v \left(\frac{\partial L}{\partial \omega_v} \right) = 0$$

is equivalent to (4.40) – in the sense that every solution of (4.40) is a solution of the Euler-Lagrange equation and conversely.

On the other hand, the equation (4.40) is said to be *equivalent to an Euler-Lagrange equation* if there is a nonzero function

$$f = f(u, v, \omega, \omega_u, \omega_v),$$

called *variational integrant factor*, such that $f \cdot \Delta = E(L)$.

Theorem 12. *The Liouville-Tzitzeica equation (3.33) and the Tzitzeica equation (3.34) are Euler-Lagrange equations, and their associated Lagrangians are given by*

$$L_1(u, v, \omega^{(1)}) = -\frac{1}{2}\omega_u \omega_v - e^{\omega}, \tag{4.42}$$

and respectively,

(4.43) $$L_2(u,v,\omega^{(1)}) = -\frac{1}{2}\omega_u\omega_v - e^{\omega} - \frac{1}{2}e^{-2\omega}.$$

Proof. Since the Helmholtz integrability conditions (4.41) are satisfied, we write the Euler-Lagrange equations for L_1, and respectively, for L_2, and so, we get the PDE (3.33), and respectively (3.34).

Notice that the equations (1.8) and (1.10) are equivalent to Euler-Lagrange equations, and they admit h^{-3} as variational integrant factor.

4.2 Variational symmetries and conservation laws

In this subsection, the theory of variational symmetry groups is briefly presented (for more details see [21] and [22]). Consider the functional

(4.44) $$\mathcal{L}[\omega] = \int\int_{\Omega_0} L(u,v,\omega^{(1)})dudv,$$

where Ω_0 is a domain in $\mathbf{R}^2$. Let $D \subset \Omega_0$ be a subdomain, U an open set in $\mathbf{R}$, and M an open set in $D \times U$. Consider $\omega \in C^2(D)$, $\omega = f(u,v)$ such that its graph $\Gamma_\omega = \{(u,v,\omega(u,v))|(u,v) \in D\} \subset M$. A local group of transformations G acting on M is called *variational symmetry group* of the functional (4.44), if for any $g_\varepsilon \in G$, $g_\varepsilon(u,v,\omega) = (\bar{u},\bar{v},\bar{\omega})$, then the function $\bar{\omega} = \bar{f}(\bar{u},\bar{v}) = (g\cdot f)(\bar{u},\bar{v})$ is defined on $\bar{\Omega} \subset \Omega_0$ and

$$\int\int_{\bar{D}} L(\bar{u},\bar{v},pr^{(1)}\bar{f}(\bar{u},\bar{v}))d\bar{u}d\bar{v} = \int\int_{D} L(u,v,pr^{(1)}f(u,v))dudv.$$

Theorem 13 (Infinitesimal criterion for the variational problem). *A connected group of transformations G acting on $M \subset \Omega_0 \times U$ is a variational symmetries group of the functional (4.44) if and only if the condition*

(4.45) $$pr^{(1)}X(L) + L\ Div\ \xi = 0$$

holds for any $(u,v,\omega^{(2)}) \in M^{(2)} \subset D \times U^{(2)}$ and for any infinitesimal generator

$$X = \zeta(u,v,\omega)\partial_u + \eta(u,v,\omega)\partial_v + \phi(u,v,\omega)\partial_\omega$$

of G (here $\xi = (\zeta,\eta)$ and $Div\ \xi = D_u\zeta + D_v\eta$ is the total divergence).

Theorem 14. *If G is a variational symmetry group of the functional (4.44), then G is a symmetry group of the Euler-Lagrange equation.*

In general, the converse of Theorem 14 is false.

A *conservation law* associated with the equation (4.40) is a divergence expression of the form

$$Div\ P = 0$$

that is identically zero on the set of the solutions $u = f(x)$ of the equation. If $P = (P^1, P^2)$ then $Div\ P = D_uP^1 + D_vP^2$ is the total divergence. The function P^1 is called the *associated flow* and P^2 is called the *conserved density* of the conservation law. It can be proved that there is a function Q such that

(4.46) $$Div\ P = Q\cdot\Delta.$$

The above relation is called the *characteristic form of the conservation law*, and Q is called the *characteristic of the conservation law.* The *vector field of evolution* associated with a vector field

$$X = \zeta(u,v,\omega)\partial_u + \eta(u,v,\omega)\partial_v + \phi(u,v,\omega)\partial_\omega \tag{4.47}$$

is given by

$$X_Q = Q\partial_u, \quad Q = \phi - \zeta\omega_u - \eta\omega_v,$$

where Q is called the *characteristic of* X.

Theorem 15 (Noether Theorem). *Let (4.47) be the infinitesimal generator of the symmetry group G of the variational problem (4.44). Then the characteristic Q of the field X is also a characteristic of the conservation law for the associated Euler-Lagrange equation $E(L) = 0$.*

If $L = L(u,v,\omega^{(1)})$ is a first order Lagrangian, then ([21], p. 356)

$$P = -(A + L\xi) = -(A^1 + L\zeta, A^2 + L\eta) = (P^1, P^2), \tag{4.48}$$

where $A = (A^1, A^2)$ is given by

$$A^1 = Q \cdot E^{(u)}(L), \quad A^2 = Q \cdot E^{(v)}(L).$$

In this case,

$$E^{(u)}(L) = \frac{\partial L}{\partial \omega_u} \quad \text{and} \quad E^{(v)}(L) = \frac{\partial L}{\partial \omega_v}$$

are called *first order Euler operators.*

4.3 Variational symmetries and conservation laws for the Liouville-Tzitzeica PDE and Tzitzeica PDE

The variational problems related to the first order Lagrangians (4.42) and (4.43) are the following

$$\mathcal{L}[\omega] = \int\int_D L_1(u,v,\omega^{(1)})dudv, \tag{4.49}$$

and respectively,

$$\bar{\mathcal{L}}[\omega] = \int\int_D L_2(u,v,\omega^{(1)})dudv, \tag{4.50}$$

where D is a domain in $\mathbf{R}^2$ and $\omega \in C^2(D)$.

Theorem 16. *The Lie algebra of the variational symmetry group of the functional (4.49) is described by the vector fields*

$$W_1 = u\partial_u - \partial_\omega, \quad W_2 = v\partial_v - \partial_\omega, \quad W_3 = \partial_u, \quad W_4 = \partial_v. \tag{4.51}$$

Proof. According to Theorem 13 and Theorem 14, the condition (4.45) must be satisfied only for certain vector fields that generate the symmetry group of the equation (3.33). Consider the vector field (3.37) and its second order prolongation (see Theorem 9) given by

$$pr^{(2)}W = W - (f'' + f'\omega_u)\frac{\partial}{\partial\omega_u} - (g'' + g'\omega_v)\frac{\partial}{\partial\omega_v}.$$

Substituting $\xi = (f, g)$ and $Div\ \xi = f' + g'$ into (4.45) we get the relation $f''\omega_v + g''\omega_u = 0$. Equate to zero the coefficients of the partial derivatives of the function ω. It follows $f'' = g'' = 0$, and so $f = C_1 u + C_3$ and $g = C_2 v + C_4$. Thus the variational symmetry group is generated by the vector field

$$W = C_1 (u\partial_u - \partial_\omega) + C_2 (v\partial_v - \partial_\omega) + C_3 \partial u + C_4 \partial_v.$$

Theorem 17. *The following vector fields*

$$U_1 = u\partial_u - v\partial_v, \quad U_2 = \partial_u \quad U_3 = \partial_v. \tag{4.52}$$

generate the variational symmetry group of the functional (4.50).

Proposition 3. *The associated flows and the conserved densities related to the Liouville-Tzitzeica equation (3.33), and respectively, of the Tzitzeica equation (3.34) are given by*

$-W_i$	P^1	P^2
$-W_1$	$\frac{1}{2}\omega_v - ue^\omega$	$\frac{1}{2}\omega_u(1 + u\omega_u)$
$-W_2$	$\frac{1}{2}\omega_v(1 + v\omega_v)$	$\frac{1}{2}\omega_u - ve^\omega$
$-W_3$	$-e^\omega$	$\frac{1}{2}\omega_u^2$
$-W_4$	$\frac{1}{2}\omega_v^2$	$-e^\omega$

Table 2

$-U_i$	P^1	P^2
$-U_1$	$-\frac{1}{2}ue^{-2\omega} - \frac{1}{2}v\omega_v^2 - ue^\omega$	$\frac{1}{2}u\omega_u^2 + ve^\omega + \frac{1}{2}ve^{-2\omega}$
$-U_2$	$-e^\omega - \frac{1}{2}e^{-2\omega}$	$\frac{1}{2}\omega_u^2$
$-U_3$	$\frac{1}{2}\omega_v^2$	$-e^\omega - \frac{1}{2}e^{-2\omega}$

Table 3

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