1. ARTÍCULOS DE ESTADÍSTICA

PROBABILISTIC VALUATION OF CERTAIN UNIT-LINKED CONTRACTS

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Abstract

We consider the fair prize of insurance contracts with benefit received either at the insurer's demise or at maturity. Explicit formulas are given for a Brennan & Schwartz contract with benefit contingent on decease.

Keywords: Risk-neutral fair-price, unit-linked insurance contracts, martingale probability.

1. Introduction

A unit-linked insurance contract is a life insurance product where the benefit depends upon the value of some reference stock which is traded in some associated market. Valuation of these products is a natural area for probabilistic and statistical applications. Even though not so widespread as that of pricing purely financial products, this topic is becoming a classical one in the actuarial literature; it was first discussed by Brennan & Schwartz (1976) and Boyle & Schwartz (1977). See also Bacinello & Ortu (1993), Aase & Persson(1994), Ekern & Persson (1996), Boyle & Hardy (1997), Moeller (2001) and Bacinello (2005). We remark that the interplay between probability theory on the one hand, and insurance theory on the other, has been recognized for a long time (see for example Li and Ma, 2008). However, the problem of prizing insurance contracts linked to stocks considered here goes a step beyond, requiring new ideas and techniques as it involves actuarial considerations as well as financial ones.

Two sources of randomness are intrinsically associated to any probabilistic model that aims to describe these products. The first stems from the uncertainty in mortality. To model this, let $\tau : \bar{\Omega} \to \mathbb{R}$ be the random time at which decease occurs for a given individual aged d at time 0. Mathematically, τ is defined on a probability space $(\bar{\Omega}, \bar{\mathcal{G}}, \bar{\mathbb{P}})$. Under natural assumptions the "life" conditional distribution function (the *survival function* ${}_{\hat{T}}p_{t_1+d}$) and conditional density h read

$$ {}_{\hat{T}}p_{t_1+d} \equiv \bar{\mathbb{P}}\Big(\tau > T | \tau > t_1\Big) = e^{-\int_{t_1}^{T} \mu(s+d;s)ds} \tag{1.1}$$

and

$$ h(T) = {}_{\hat{T}}\, p_{t_1+d}\ \mu(T+d;T) \tag{1.2}$$

where $\mu(t_1 + d, t)$ is the mortality intensity and $\hat{T} \equiv T - t_1 > 0$.

The second source of uncertainty arises from the random nature of the stock markets. Here we consider a simple model of financial market consisting of two securities: a "savings account" whose value B_t at time t evolves via $B_t = B_0 e^{rt}$, where r is the (constant) instantaneous interest rate of the market. The second instrument in the market, *to which the insurance policy is linked*, is a given stock whose price at time t, X_t, varies according to a definite stochastic dynamics. The prototype model for stocks-price evolution after the seminal work of Black and Scholes (1973) and Merton (1973) assumes that X_t evolves via the Itô's stochastic differential equation

$$ dX_t = \kappa X_t dt + \sigma X_t dW_t \tag{1.3}$$

Here κ is the mean return rate, σ the stock's volatility while W_t is a Brownian motion on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$, the "real world" probability. Note that we assume that r, κ and σ are constants.

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We suppose that the contract considered here is "written" at a certain time t_1 and will expire or "mature" at a time $T > t_1$. At maturity T the contract holder is entitled to a benefit denoted by $\Psi^{1)}_T$ where $\Psi^{1)}_T = \Psi^{1)}(T; X_s, t_1 \leq s \leq T)$ is a functional of the future evolution of the stock up to maturity time. However, if the insurer's demise happens at a time τ before maturity, then the policy entitles the beneficiaries to a payment $\Psi^{2)}_\tau$ at the decease time. (This situation corresponds to "endowment insurance contracts"-see section 3.2 below- By contrast, for "pure endowment contracts", benefit is received only at maturity). Let Δ be the demise time or maturity, whichever comes first: $\Delta = \min\{T, \tau\}$. Given the benefit function $\Psi_\Delta \equiv \Psi^{1)}_T \mathbf{1}_{\{\tau \geq T\}} + \Psi^{2)}_\tau \mathbf{1}_{\{t_1 \leq \tau < T\}}$, we determine in this paper explicit formulas for the *fair price* υ_{t_1} required by the insurance firm at time t_1 to hedge the exposure to the evolution of the risky asset i.e., we determine the map $\Psi_\Delta \to \upsilon_{t_1}$.

2. Contract characteristics and valuation

In a unit-linked contract the premium υ_{t_1} paid at time t_1 is invested in a stock. Let $X_t, 0 \leq t \leq T$ be the value at time t of an unit of the stock and let $\mathcal{F}_t \equiv \sigma\big(X_s, 0 \leq s \leq t\big)$ be the filtration generated by the stock, which contains all information on past history and take $\mathcal{G} \equiv \mathcal{F}_\infty$. We shall also assume that our market is efficient, i.e., the existence of the Harrison and Pliska (1981) risk-neutral probability $\mathbb{P}^*$ on $(\Omega, \mathcal{F}_\infty)$ under which relative prices[1] of *all financial products* $\upsilon'_t \equiv \upsilon_t / B_t$ are martingales[2] with respect to the history of the process up to time t. No further reference will be made to the original, *real world* probability, so to ease notation we shall drop the symbol* and denote the martingale probability $\mathbb{P}^*$ simply as $\mathbb{P} \equiv \mathbb{P}^*$.

We assume that the risk stemming from the market has no influence on the mortality risk. Thus, we can assume independence of the filtrations $\mathcal{F}_{t_1}, \bar{\mathcal{F}}_{t_1} \equiv \sigma\Big(\mathbf{1}_{\{\tau > s\}}, 0 \leq s \leq t_1\Big)$.

Let $\mathbb{E} \equiv \mathbb{E}^{\mathbb{P}\times\bar{\mathbb{P}}}$ denote the expectation with respect to the product measure $\mathbb{P}\times\bar{\mathbb{P}}$, $\mathbb{E}^{\mathbb{P}}$ expectation respect to the probability measure $\mathbb{P}$ and so forth.

We next note that the price process must satisfy:

$$\text{(i)} \quad \lim_{t_1 \uparrow T} \upsilon_{t_1} = \Psi^{1)}_T. \tag{2.1}$$

$$\text{(ii)} \quad \upsilon'_{t_1} = \mathbb{E}\Big(\upsilon'_t \Big| \mathcal{F}_{t_1} \vee \bar{\mathcal{F}}_{t_1}\Big) \text{ or}$$

$$\upsilon_{t_1} e^{-rt_1} = e^{-rt} \mathbb{E}\Big(\upsilon_t \Big| \mathcal{F}_{t_1} \vee \bar{\mathcal{F}}_{t_1}\Big)\ \mathcal{F}t_1 \leq t \leq T. \tag{2.2}$$

(i) states that we "pay just what we receive" if contract matures right after it is written. (ii) is the martingale property, alluded to above. Using Doob's optional stopping theorem it can be extended to random times[3]. We also require property (i) to hold if T is replaced by the random time τ. Hence

$$\upsilon'_{t_1} = \mathbb{E}\Big(\upsilon'_\Delta \Big| \mathcal{F}_{t_1} \vee \bar{\mathcal{F}}_{t_1}\Big) = \mathbb{E}\Big(\Psi'_\Delta \Big| \mathcal{F}_{t_1} \vee \bar{\mathcal{F}}_{t_1}\Big) =$$
$$\mathbb{E}\Big(\Psi'^{1)}_T \mathbf{1}_{\{\tau \geq T\}} + \Psi'^{2)}_\tau \mathbf{1}_{\{t_1 \leq \tau < T\}} \Big| \mathcal{F}_{t_1} \vee \bar{\mathcal{F}}_{t_1}\Big) =$$
$$e^{-rT} \mathbb{E}\Big(\Psi^{1)}_T \mathbf{1}_{\{\tau \geq T\}} \Big| \mathcal{F}_{t_1} \vee \bar{\mathcal{F}}_{t_1}\Big) +$$
$$\mathbb{E}\Big(\Psi^{2)}_\tau e^{-r\tau} \mathbf{1}_{\{t_1 \leq \tau < T\}} \Big| \mathcal{F}_{t_1} \vee \bar{\mathcal{F}}_{t_1}\Big) \tag{2.3}$$

Recalling that $\mathcal{F}_{t_1}, \bar{\mathcal{F}}_{t_1}$ are independent $\sigma-$ fields we have

$$\mathbb{E}\Big(\Psi^{1)}_T \mathbf{1}_{\{\tau \geq T\}} \Big| \mathcal{F}_{t_1} \vee \bar{\mathcal{F}}_{t_1}\Big) =$$
$$\mathbb{E}^{\bar{\mathbb{P}}}\Big(\mathbf{1}_{\{\tau \geq T\}} \Big| \bar{\mathcal{F}}_{t_1}\Big) \mathbb{E}^{\mathbb{P}}\Big(\Psi^{1)}_T \Big| \mathcal{F}_{t_1}\Big) =$$
$$_{\hat{T}}p_{t_1+d} \quad \mathbb{E}^{\mathbb{P}}\Big(\Psi^{1)}_T \Big| \mathcal{F}_{t_1}\Big). \tag{2.4}$$

Similarly, using the tower property for conditional expectations we have in terms of the conditional density h of τ (see eq. (1.1))

$$\mathbb{E}\Big(\mathbf{1}_{\{t_1 \leq \tau < T\}} \Psi^{2)}_\tau e^{-r\tau} \Big| \mathcal{F}_{t_1} \vee \bar{\mathcal{F}}_{t_1}\Big) =$$
$$\mathbb{E}\Big[\mathbb{E}\Big(\mathbf{1}_{\{t_1 \leq \tau < T\}} \Psi^{2)}_\tau e^{-r\tau} \Big| \mathcal{F}_{t_1} \vee \bar{\mathcal{F}}_{t_1}, \tau\Big) \Big| \mathcal{F}_{t_1} \vee \bar{\mathcal{F}}_{t_1}\Big] =$$
$$\mathbb{E}\Big(\int_{t_1}^T h(s) e^{-rs} \Psi^{2)}_s ds \Big| \mathcal{F}_{t_1} \vee \bar{\mathcal{F}}_{t_1}\Big) =$$
$$\mathbb{E}^{\mathbb{P}}\Big(\int_{t_1}^T h(s) e^{-rs} \Psi^{2)}_s ds \Big| \mathcal{F}_{t_1}\Big). \tag{2.5}$$

[1] similarly, the relative benefit $\Psi'_t \equiv \Psi_t / B_t$ compares our benefit with that investing the money in a savings account.

[2] It implies that one can not expect a better deal by delaying the "signature" of the contract to the future.

[3] Note that the optional stopping theorem can be applied since $\bar{\mathbb{P}}(\Delta < \infty) = 1, \mathbb{E}^{\mathbb{P}}\Big(\sup_{0 \leq s \leq T} \Psi^{i)}_s\Big) < \infty$.

Thus the risk-neutral fair-price at time t_1 is made up of two terms, in correspondence with the benefits at maturity or at decease:

$$v_{t_1} \equiv v(T|t_1, x) = v_{t_1}^{1)} + v_{t_1}^{2)} \tag{2.6}$$

where

$$v_{t_1}^{1)} = {}_{\hat{T}}\, p_{t_1+d}\; e^{-r(T-t_1)} \mathbb{E}^{\mathbb{P}}\Big(\Psi_T^{1)}\Big|\mathcal{F}_{t_1}\Big), \tag{2.7}$$

$$v_{t_1}^{2)} \equiv \mathbb{E}^{\mathbb{P}}\Big(\int_{t_1}^{T} h(s) e^{-r(s-t_1)} \Psi_s^{2)} ds \Big|\mathcal{F}_{t_1}\Big). \tag{2.8}$$

We shall find it convenient to use the notation $v(T|t_1, X_{t_1}) \equiv v_{t_1}$ to stress that v_{t_1} depends not only on the "starting" values of time and of stock (t_1 and X_{t_1}) but also on the maturity time T.

In a generic case, explicit evaluation of $v_{t_1}^{1)}$ and, mainly, of $v_{t_1}^{2)}$ given by (2.8) can be a difficult matter. The case when $\Psi_\tau^{2)}$ can be represented as $\Psi_\tau^{2)} = \Psi_2(X_\tau)$ in terms of a given increasing function $\Psi_2 : \mathbb{R} \to \mathbb{R}$ is of particular interest in applications; further, in this case Eq. (2.8) can be simplified further since the Markovian nature of X_t yields

$$v_{t_1}^{2)} = \Big[\int_0^{\hat{T}} \frac{h(s+t_1)}{B_s} \mathbb{E}_x^{\mathbb{P}}\big(\Psi_s^{2)}\big) ds\Big]_{x=X_{t_1}}. \tag{2.9}$$

For guaranteed unit-linked contracts the benefit at maturity depends on the value of the associated stock but there is a minimum guaranteed amount if the stock price falls below a fixed level; this can be taken to correspond to the capital accrued at a fixed interest rate δ, the "technical rate". The simplest example corresponds to a linear dependence on X in the benefit: suppose that both $\Psi_\Delta^{1,2)} = X_\Delta + X_{t_1}\big(e^{\delta(\Delta-t_1)} - 1\big)$, say; here, contingent on an insurance event happening (maturity $\Delta = T$ or death $\Delta = \tau$) the insured receives the stock plus the interest accrued with rate δ. By substitution in eqs. (2.7) and (2.8) and using that X_t' is also a martingale the fair price is found in explicit way as

$$v_{t_1}^{1)} = {}_{\hat{T}}\, p_{t_1+d}\; X_{t_1}\Big(1 + \frac{B_{t_1}}{B_T}\big(e^{\delta\hat{T}} - 1\big)\Big),$$

$$v_{t_1}^{2)} = \; X_{t_1}\Big(1 - {}_{\hat{T}}\, p_{t_1+d} +$$

$$\int_{t_1}^{T} h(s) \frac{B_{t_1}}{B_s}\big(e^{\delta(s-t_1)} - 1\big) ds\Big). \tag{2.10}$$

When the mortality rate μ is constant one recovers the result of Shen & Xu (2005). A case of more interest, non-linear in X_T, is considered in the next section.

3. Valuation of endowment insurance contracts of Brennan & Schwartz type

We first review the case considered by Brennan & Schwartz (1976) and Boyle & Schwartz (1977) where $\Psi_T^{1)} = \max\{X_T, X_{t_1}e^{\delta\hat{T}}\}, \Psi_\tau^{2)} = 0$. Here, the insured receives *at maturity* the value of the associated stock; further, the initial capital, accrued at a fixed interest rate δ, is guaranteed, in case the stock price does not reach this level. In case of decease before maturity, beneficiaries are *not entitled to any extra benefit.*

3.1. Pure endowment case

Evaluation of the premium involves determining the corresponding expectation given by eq. (2.7); it can be obtained in closed form, as follows. Recall first that Girsanov's theorem implies that the evolution of X_t under the martingale probability is given by Eq. (1.1) where the drift κ is replaced by the instantaneous interest rate r of the market. The solution that at t_1 equals X_{t_1} is

$$X_t = X_{t_1} e^{\sigma(W_t - W_{t_1}) + \left(r - \frac{\sigma^2}{2}\right)(t-t_1)} \overset{\text{Law}}{=}$$

$$X_{t_1} e^{\sigma W_{\hat{t}} + q\hat{t}},\; t_1 \leq t \leq T$$

where we use that independence of Brownian increments implies that $W_t - W_{t_1} \overset{\text{Law}}{\sim} W_{\hat{t}}$ and define $q \equiv r - \frac{\sigma^2}{2}, \hat{t} \equiv t - t_1$. Say $X_{t_1} = x$. Using the Markovian nature of the benefit and the well known Gaussian distribution of Brownian motion $W_t \sim \mathcal{N}(0, \sqrt{t})$ we obtain

$$\mathbb{E}^{\mathbb{P}}\Big(\Psi_T^{1)}\Big|\mathcal{F}_{t_1}\Big) =$$

$$\mathbb{E}^{\mathbb{P}}\Big(X_{t_1} \max\{e^{\sigma W_{\hat{T}} + q\hat{t}}, e^{\delta\hat{T}}\} | X_{t_1} = x\Big) =$$

$$\int \frac{x}{\sqrt{2\pi\hat{T}}} e^{-y^2/(2\hat{T})}\; \max\{e^{q\hat{T} + \sigma y}, e^{\delta\hat{T}}\} dy =$$

$$x\Big(e^{r\hat{T}}\Phi\big(m_+\sqrt{\hat{T}}\big) + e^{\delta\hat{T}}\Phi\big(-m_-\sqrt{\hat{T}}\big)\Big) \tag{3.1}$$

where

$$m_\pm \equiv \frac{\hat{r}}{\sigma} \pm \frac{\sigma}{2}, \quad \hat{r} \equiv r - \delta, \quad \hat{T} \equiv T - t \qquad (3.2)$$

and Φ is the standard normal distribution function

$$\Phi(x) = \int_{-\infty}^{x} e^{-z^2/2} \frac{dz}{\sqrt{2\pi}}. \qquad (3.3)$$

Hence $v^{1)}_{t_1} \equiv v^{1)}(T|t_1, x) =$

$$_{\hat{T}}p_{t_1+d}\, x\Big(\Phi\big(m_+\sqrt{\hat{T}}\big) + e^{-\hat{r}\hat{T}}\Phi(-m_-\sqrt{\hat{T}})\Big). \qquad (3.4)$$

The limit behavior for long values of $\hat{T}$ is interesting; depending on whether $\hat{r} > 0$, $\hat{r} = 0$ or $\hat{r} < 0$ three different possibilities are found: $v^{1)}(T|t_1, x)$ tends, respectively, to either $_{\hat{T}}p_{t_1+d}x \equiv v^{1)}(T|T, x)$, to $2_{\hat{T}}p_{t_1+d}x$ or to ∞. The result is easy to understand. The higher the guaranteed rate the more interesting the contract becomes. Further, when δ is higher than the market rate r, the relative benefit tends to ∞ asymptotically in time and so it does the fair price.

3.2. Endowment insurance contract

In the sequel we study valuation for a case that generalizes the result of Brennan & Schwartz to a contract paying $\Psi^{1)}_T = \max\{X_T, X_{t_1}e^{\delta(T-t_1)}\}$ at expiry or $\Psi^{2)}_\tau = \max\{X_\tau, X_{t_1}e^{\delta(\tau-t_1)}\}$ should decease occur before maturity. It turns out that with this demise benefit an *analytical, exact* formula for the fair price can be *also* derived if an exponential distribution is assumed for the decease rate: $h(t) = \mu e^{-\mu(t-t_1)}$ where the parameter μ^{-1} is the mean life. In this case, using (2.8) and (3.1) we find $v^{2)}_{t_1} \equiv v^{2)}(T|t_1, x) =$

$$x\mu \int_0^{\hat{T}} \Big(e^{-\mu s}\Phi(m_+\sqrt{s}) + e^{-(\mu+\hat{r})s}\Phi(-m_-\sqrt{s})\Big) ds \qquad (3.5)$$

and hence it involves the integral

$$I \equiv \int_0^{\hat{T}} e^{-\alpha s}\Phi(m\sqrt{s}) ds \qquad (3.6)$$

By interchanging integrals we find

$$I = \frac{1}{2\alpha}\Big[1 - \frac{|m|}{\sqrt{m^2+2\alpha}} + \frac{2|m|}{\sqrt{m^2+2\alpha}}\Phi\big(\frac{|m|}{m}\sqrt{(m^2+2\alpha)\hat{T}}\big) - 2e^{-\alpha\hat{T}}\Phi\big(m\sqrt{\hat{T}}\big)\Big]. \qquad (3.7)$$

Thus, the demise contribution to the price is

$$v^{2)}_{t_1} = \frac{x}{2}\Big[1 - \frac{m_+}{\sqrt{\eta}} + \frac{2m_+}{\sqrt{\eta}}\Phi\big(\sqrt{\eta\hat{T}}\big) - 2e^{-\mu\hat{T}}\Phi\big(m_+\sqrt{\hat{T}}\big)\Big] + \frac{x\mu}{2(\mu+\hat{r})}\Big[1 - \frac{|m_-|}{\sqrt{\eta}} + \frac{2|m_-|}{\sqrt{\eta}}\Phi\big(-\sqrt{\eta\hat{T}}\text{sign } m_-\big) - 2e^{-(\mu+\hat{r})\hat{T}}\Phi\big(-m_-\sqrt{\hat{T}}\big)\Big] \qquad (3.8)$$

where $\eta \equiv m_+^2 + 2\mu$ and $m_\pm$ are defined in Eq. (3.2). The premium simplifies when the insurance company is committed to pay a technical interest rate δ satisfying $\delta = r$. In this case $m_\pm = \pm\frac{\sigma}{2}, \eta = \frac{\sigma^2}{4} + 2\mu$ and the full price $v_{t_1} \equiv v^{1)}_{t_1} + v^{2)}_{t_1}$ for such a contract reads

$$v_{t_1} = x\Big[1 + \frac{\sigma}{\sqrt{\eta}}\Big(\Phi\big(\sqrt{\eta\hat{T}}\big) - \frac{1}{2}\Big)\Big]. \qquad (3.9)$$

In Fig. 1 we plot $v_{t_1} \equiv v(T|t_1, x) = v(T-t_1|0, x)$ as a function of $T - t_1$. Notice how it starts from $v_{t_1} = x$ when $T - t_1 = 0$ and then it increases towards $x\Big(1 + \frac{\sigma}{2\sqrt{\eta}}\Big)$.

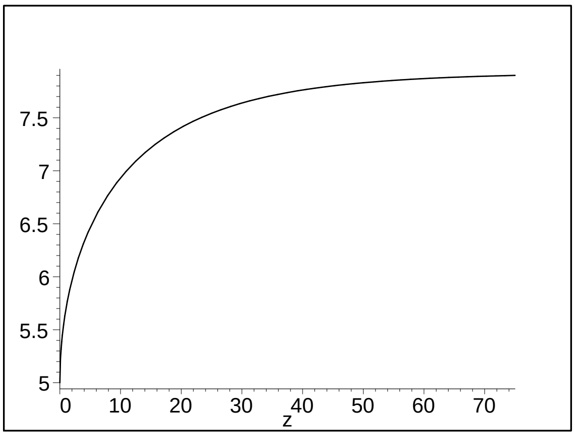

Figure 1: Fair price as a function of time to maturity given in years corresponding to a constant annual interest rate $r = 4.5\%$. Other parameters are: $X_{t_1} \equiv x = 5, \sigma = 25\%, \mu = 0.015$

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