Boletín de Estadística e Investigación Operativa
Vol. 26, No. 1, Febrero 2010, pp. 19-24

Estadística

Some commentaries on confidence intervals of the mean in a Normal population

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Abstract

In this paper, a study on the length of confidence intervals of the mean in a Normal population when the variance is either known or unknown is carried out. It is also proved that ignoring the population variance always leads to a worse interval than when such a variance is taken into account.

Keywords: Student's t population, Chi-squared population.
AMS Subject classifications: 62F99.

1. Introduction

Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed (i.i.d.) random variables (a sample) from an $N(\mu, \sigma)$ population, and let α be a specified value such that $0 < \alpha < 1$. If σ is known then the shortest length $1-\alpha$ confidence interval based on the pivotal quantity[1] (pivot) $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$ of the mean μ is

$$\left(\overline{X} - \frac{1}{\sqrt{n}}\sigma\, z_{1-\alpha/2}\,,\, \overline{X} + \frac{1}{\sqrt{n}}\sigma\, z_{1-\alpha/2}\right) \tag{1.1}$$

where $\overline{X}$ is the mean of the sample and $z_{1-\alpha/2}$ is the $(1-\alpha/2)$th quantile of the $N(0,1)$ population. If σ is unknown then the shortest length $1-\alpha$ confidence interval based on the pivot $\frac{\overline{X}-\mu}{S/\sqrt{n-1}} \sim t_{n-1}$ of the mean μ is

$$\left(\overline{X} - \frac{S}{\sqrt{n-1}}\, t_{n-1,1-\alpha/2}, \overline{X} + \frac{S}{\sqrt{n-1}}\, t_{n-1,1-\alpha/2}\right) \tag{1.2}$$

where S is the standard deviation of the sample and $t_{n-1,1-\alpha/2}$ is the $(1-\alpha/2)$th quantile of the Student's t population with $n-1$ degrees of freedom (see for instance [1]).

[1] The symbol "$\sim$" must be read as "distributed as".

Let us consider a toy example: suppose that $x_1 = 0.95$ and $x_2 = 1.05$ is a sample from an $N(\mu, \sigma)$ population and two researchers want to build a 0.90 confidence interval of the unknown parameter μ. The first researcher knows that $\sigma = 1$ and therefore, from (1.1), the confidence interval of μ is $I_1 = (-1.630\,,\, 2.163)$. But for the second researcher, σ is unknown and therefore, from (1.2), the confidence interval will be $I_2 = (-0.684\,,\, 1.316)$. In view of these intervals the following question arises: how can I_2 be more accurate (shorter) than I_1 when the latter needs less information? The main aim of this paper is to answer this question.

2. Ratio of lengths

Intervals (1.1) and (1.2) have the same center ($\overline{X}$) although different radii. The length of the interval (1.1) is $\frac{2}{\sqrt{n}}\sigma\, z_{1-\alpha/2}$, which only depends on the size of the sample, whereas the length of the interval (1.2) is $\frac{2}{\sqrt{n-1}} S\, t_{n-1,\,1-\alpha/2}$, that depends on the sample because $S = S(X_1, \cdots, X_n)$ is a statistic. To compare these two lengths, we are going to study the random ratio between them which will be denoted by $C = C(X_1, \cdots, X_n)$, that is, $C = \frac{t_{n-1,1-\alpha/2}}{z_{1-\alpha/2}} \frac{\sqrt{n}}{\sqrt{n-1}} \frac{S}{\sigma}$.

Let us recall two well-known results (see for instance [1]): *i*) If $X_1, \cdots, X_n$ are i.i.d. random variables from an $N(\mu, \sigma)$ population, then $\frac{nS^2}{\sigma^2} \sim \chi^2_{n-1}$; and *ii*) if $Y \sim \chi^2_p$ then $E[Y^r] = \frac{2^r \Gamma(r + \frac{p}{2})}{\Gamma(\frac{p}{2})}$, where $\Gamma(x)$ is the gamma function and r and p are positive. Hence, the probability density function (pdf) of C, denoted by f_C, is given as

$$\begin{aligned} f_C(c) = & \left(\frac{z_{1-\alpha/2}}{t_{n-1,1-\alpha/2}}\right)^{n-1} \frac{(n-1)^{\frac{n-1}{2}}}{2^{\frac{n-3}{2}}\,\Gamma((n-1)/2)}\, c^{n-2} \\ & \exp\left(-\frac{n-1}{2}\left(\frac{z_{1-\alpha/2}}{t_{n-1,1-\alpha/2}} c\right)^2\right), \quad c > 0. \end{aligned} \tag{2.1}$$

This function is depicted in Figure 1, and it can be seen that its maximum is near to 1 and the tails are tighter as n increases. It is also straightforward to calculate the expectation $\mu_c = E[C]$ and the variance $\sigma_c^2 = Var[C]$:

$$\begin{aligned} \mu_c &= \frac{t_{n-1,1-\alpha/2}}{z_{1-\alpha/2}} \frac{\sqrt{2}\ \Gamma(n/2)}{\sqrt{n-1}\ \Gamma((n-1)/2)}, \qquad \text{and} \\ \sigma_c^2 &= \left(\frac{t_{n-1,1-\alpha/2}}{z_{1-\alpha/2}}\right)^2 \left(1 - \left(\frac{\sqrt{2}\ \Gamma(n/2)}{\sqrt{n-1}\ \Gamma((n-1)/2)}\right)^2\right). \end{aligned}$$

Furthermore, if n is large then[2] (see for instance [3]) $t_{n\,,\,p} \approx z_p$ for any $0 < p < 1$ and $\lim\limits_{x \to +\infty} \frac{\Gamma(x+1/2)}{\sqrt{x}\ \Gamma(x)} = 1$ (see for instance [4]), and hence

[2]The symbol $\approx$ must be read as "approximately equal to".

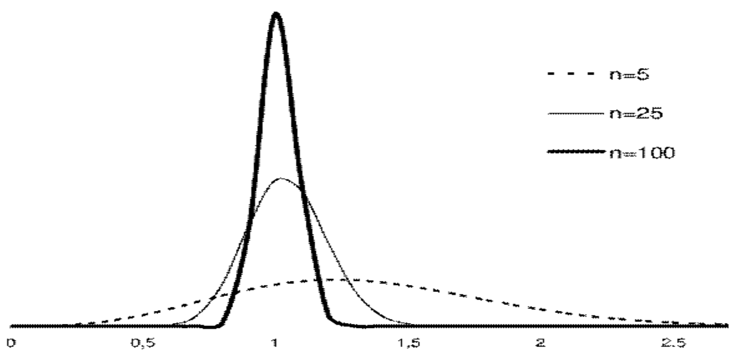

Figure 1: Some pdfs of C with $\alpha = 0.05$.

$\lim_{n\to+\infty} \frac{\sqrt{2}\,\Gamma(\frac{n}{2})}{\sqrt{n-1}\,\Gamma(\frac{n-1}{2})} = 1$. Therefore $\lim_{n\to+\infty} \mu_c = 1$ and $\lim_{n\to+\infty} \sigma_c^2 = 0$, which imply the convergence in probability of the random variable C to 1 (see for instance [2]), that is, the intervals (1.1) and (1.2) are asymptotically equivalent.

But if n is not very large then these two intervals can widely differ. To show it, Table 1 and Table 2 have been built for several values of α, n and k. So for

Table 1: Values of μ_c and σ_c for some n and α.

	$\alpha = 0.10$		$\alpha = 0.05$		$\alpha = 0.01$	
n	μ_c	σ_c	μ_c	σ_c	μ_c	σ_c
2	3.0627	2.3139	5.1726	3.9079	19.7179	14.8971
3	1.5733	0.8224	1.9455	1.0170	3.4147	1.7850
4	1.3182	0.5563	1.4960	0.6313	2.0891	0.8816
5	1.2183	0.4422	1.3316	0.4834	1.6801	0.6099
10	1.0840	0.2588	1.1226	0.2680	1.2272	0.2930
25	1.0294	0.1493	1.0421	0.1512	1.0746	0.1559
100	1.0069	0.0716	1.0098	0.0719	1.0171	0.0724
500	1.0014	0.0317	1.0019	0.0317	1.0033	0.0318

instance, Table 2 shows that the event considered in the toy example, that is, that the interval (1.2) is shorter than the interval (1.1) for a sample with $n = 2$ and $\alpha = 0.10$, has a probability equal to 0.2055.

Table 2: Values of $P(C < k)$ for some n, k and α.

	$\alpha = 0.10$			$\alpha = 0.05$			$\alpha = 0.01$		
n	k=0.75	k=0.9	k=1	k=0.75	k=0.9	k=1	k=0.75	k=0.9	k=1
2	0.1549	0.1854	**0.2055**	0.0921	0.1104	0.1226	0.0242	0.0291	0.0323
3	0.1635	0.2267	0.2719	0.1102	0.1547	0.1874	0.0372	0.0531	0.0651
4	0.1564	0.2439	0.3098	0.1128	0.1798	0.2321	0.0454	0.0751	0.0998
5	0.1454	0.2511	0.3340	0.1091	0.1938	0.2630	0.0492	0.0924	0.1305
10	0.0937	0.2471	0.3885	0.0759	0.2087	0.3375	0.0433	0.1307	0.2260
25	0.0260	0.1955	0.4317	0.0221	0.1749	0.3995	0.0146	0.1303	0.3236
100	0.0001	0.0659	0.4664	0.0001	0.0612	0.4503	0.0001	0.0507	0.4113
500	0.0000	0.0006	0.4850	0.0000	0.0005	0.4779	0.0000	0.0005	0.4603

3. Correction of confidence

It still remains the previous question: if σ is known and the interval (1.2) is shorter than the interval (1.1), would it be better to choose the shortest one as the confidence interval of μ? The answer is negative. For such a purpose we are proving that if $C = c$ is known then the confidence coefficient of the interval (1.2) is not longer $1 - \alpha$ but $2\Phi(z_{1-\alpha/2}\, c) - 1$, where Φ is the cumulative distribution function (cdf) of the $N(0, 1)$ distribution, value which is smaller than $1-\alpha$ when $c < 1$.

To see this, we begin by noting that the interval (1.2) will not cover the true parameter μ just if the following event takes place

$$\{\left|\overline{X} - \mu\right| \geq \frac{1}{\sqrt{n-1}} S\, t_{n-1\,,\,1-\alpha/2}\}.$$

Denoting $Z = \frac{\sqrt{n}(\overline{X}-\mu)}{\sigma}$, this event can be written as $\{|Z| \geq z_{1-\alpha/2}\, c\}$. Its probability depends on c and shall be written as:

$$p(c) = P\{|Z| \geq z_{1-\alpha/2}\, c\}. \tag{3.1}$$

Taking into account that $Z \sim N(0, 1)$ no matters the value of c (since $\overline{X}$ and S are independent), the expression on the right-hand side of the inequality of the event given in (3.1) for each fixed value of c will be the $(1 - p(c)/2)$th quantile of the $N(0, 1)$ distribution, that is

$$z_{1-p(c)/2} = z_{1-\alpha/2}\, c. \tag{3.2}$$

By applying the transformation Φ in (3.2) then:

$$\Phi(z_{1-p(c)/2}) = \Phi(z_{1-\alpha/2}\, c) \Rightarrow 1 - p(c)/2 = \Phi(z_{1-\alpha/2}\, c)$$

since $\Phi(z_p) = p$, and hence

$$1 - p(c) = 2\Phi(z_{1-\alpha/2}\, c) - 1. \tag{3.3}$$

That is, if σ is known then $2\Phi(z_{1-\alpha/2}\, c) - 1$ is the true confidence of the interval (1.2) as we wanted to prove. Furthermore, as Φ is a strictly increasing function (since it is the cdf of the N(0,1) distribution) then $1-p(c)$ is a strictly increasing function too, and hence, if $c < 1$ then

$$1 - p(c) < 1 - p(1) = 2\Phi(z_{1-\alpha/2}) - 1 = 2(1 - \alpha/2) - 1 = 1 - \alpha.$$

But as c can be any positive number, the corrected confidence of the interval (1.2) can be also considered a random variable $1 - p(C)$, and it can be worth noting that its expectation is $1-\alpha$ in coherence with the initial confidence coefficient. To see it, let ϕ and f_C the pdfs of Z and C respectively. Thus, $\phi(z)f_C(c)$ is the joint pdf of the random vector (Z, C), since both random variables are independent, and therefore

$$\begin{aligned} E[1-p(C)] &= \int_0^\infty P(|Z| < z_{1-\alpha/2}c)\, f_C(c)\, dc \\ &= \int_0^\infty \left(\int_{-z_{1-\alpha/2}c}^{z_{1-\alpha/2}c} \phi(z)\, dz \right) f_C(c)\, dc \\ &= P\left(|Z| < z_{1-\alpha/2}C\right) = P\left(\frac{|\overline{X}-\mu|}{S}\sqrt{n-1} < t_{n-1,1-\alpha/2} \right) \\ &= 1-\alpha. \end{aligned}$$

Considering still C as a random variable, let us study how often the confidence $1-p(C)$ of the random interval (1.2) will be considerably smaller than $1-\alpha$, and to this end, we are going to evaluate the probability of the $\left\{ \frac{1-p(C)}{1-\alpha} < k \right\}$ event for several values of k. It follows from (3.3) that

$$P\left(\frac{1-p(C)}{1-\alpha} < k \right) = P\left(C < \frac{\Phi^{-1}\left((1+k(1-\alpha))/2\right)}{z_{1-\alpha/2}} \right)$$

and from this result, Table 3 has been obtained for different values of n, k and α. This table suggests that the probability of a substantial correction decreases

Table 3: Values of $P\left(\frac{1-p(C)}{1-\alpha} < k\right)$ for some n, k and α.

	$\alpha = 0.10$			$\alpha = 0.05$			$\alpha = 0.01$		
n	k=0.75	k=0.9	k=0.95	k=0.75	k=0.9	k=0.95	k=0.75	k=0.9	k=0.95
2	0.1239	0.1644	0.1826	0.0667	0.0913	0.1038	0.0142	0.0201	0.0236
3	0.1074	0.1825	0.2205	0.0593	0.1084	0.1379	0.0129	0.0257	0.0354
4	0.0866	0.1819	0.2351	0.0467	0.1103	0.1537	0.0097	0.0267	0.0423
5	0.0687	0.1754	0.2402	0.0355	0.1061	0.1601	0.0068	0.0250	0.0450
10	0.0218	0.1323	0.2294	0.0084	0.0721	0.1511	0.0008	0.0119	0.0368
25	0.0009	0.0552	0.1696	0.0001	0.0197	0.0938	0.0000	0.0008	0.0103
100	0.0000	0.0012	0.0436	0.0000	0.0001	0.0096	0.0000	0.0000	0.0000
500	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

when α tends to zero and n tends to infinity, but however when n is not very large then there is a reasonably high probability of a certain loss in the re-evaluated confidence. A significant correction of the confidence is more likely to happen when the initial confidence is not high.

In order to complete this study, let us obtain the pdf of the corrected confidence, which is denoted by f_{Co}. From (2.1), it can be written that for $0 < u < 1$:

$$f_{Co}(u) = \frac{\left(\Phi^{-1}\right)'\left(\frac{u+1}{2}\right)}{2z_{1-\alpha/2}}\, f_C\left(\frac{\Phi^{-1}\left(\frac{u+1}{2}\right)}{z_{1-\alpha/2}} \right)$$

where $\left(\Phi^{-1}\right)'$ denotes the derivative of Φ^{-1}, and since $\left(\Phi^{-1}\right)'(u) = \sqrt{2\pi}\exp(z_u^2/2)$, then by substitution,

$$f_{Co}(u) = \frac{(n-1)^{\frac{n-1}{2}}\sqrt{\pi}}{2^{\frac{n}{2}-1}\,\Gamma((n-1)/2)\left(t_{n-1,1-\alpha/2}\right)^{n-1}}\left(z_{(u+1)/2}\right)^{n-2} \exp\left(\frac{1}{2}\left(z_{(u+1)/2}\right)^2\right)\exp\left(-\frac{(n-1)\left(z_{(u+1)/2}\right)^2}{2\left(t_{n-1,1-\alpha/2}\right)^2}\right).$$

Conclusions

In this paper it has been shown that ignoring σ leads always to a worse confidence interval because the interval (1.2) will be longer in general, but if even it were shorter then the gained accuracy is false since there is a loss of the real confidence.

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