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An Introduction to the Theory of Stochastic Orders

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Abstract

In this paper we make a review of some of the main stochastic orders that we can find in the literature. We show some of the main relationships among these orders and properties and we point out applications in several fields.

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1. Introduction

One of the main objectives of statistics is the comparison of random quantities. These comparisons are mainly based on the comparison of some measures associated to these random quantities. For example it is a very common practice to compare two random variables in terms of their means, medians or variances. In some situations comparisons based only on two single measures are not very informative. For example let us consider two Weibull distributed random variables X and Y with distribution functions $F(x) = 1 - \exp x^3$ and $G(x) = 1 - \exp x^{\frac{1}{2}}$ for $x \geq 0$, respectively. In this case we have that $E[X] = 0.89298 < E[Y] = 2$. If X and Y denote the random lifetimes of two devices or the survival lifetimes of patients under two treatments then, if we look only at mean values, we can say that X has a less expected survival time than Y. However if we consider the probability to survive for a fixed time point $t \geq 0$ we have that $P[X > t] \geq P[Y > t]$ for any $t \in [0, 1]$ and $P[X > t] \leq P[Y > t]$ for any $t \in [1, +\infty)$. Therefore the comparison of the means does not ensure that the probabilities to survive any time t are ordered in the same sense. The need to provide a more detailed comparison of two random quantities has been the origin of the theory of stochastic orders that has grown significantly during the last 40 years (see Shaked and Shanthikumar (2007)). The purpose of this review article is to provide the reader with an introduction to some of the most popular stochastic orders including some properties of interest and applications of these stochastic orders. The organization of this paper is the following, in Section 2 we provide definitions of some stochastic orders. These definitions are based on some functions associated to the random variables. We will describe some applications of these functions and therefore several situations where these stochastic orders can be applied. Some properties of these orders are also described. In Section 3 we describe some multivariate extensions of stochastic orders and to finish, in Section 4 we recall further applications and comments of stochastic orders. Some notation that will be used along the paper is the following: Given a distribution function F, the survival function will be denoted by $\overline{F}(t) = 1 - F(t)$ and the quantile function will be denoted by $F^{-1}(p) = \inf\{x \in \mathbb{R} : F(x) \geq p\}$ for any $p \in (0, 1)$. General references for the theory of stochastic orders are Shaked and Shanthikumar (1994), Müller and Stoyan (2002) and Shaked and Shanthikumar (2007).

2. Definitions and properties of some univariate stochastic orders

As mentioned in the introduction the use of stochastic orders arises when the comparisons of single measures is not very informative. For example if the random variable X denotes the random lifetime of a device or a living organism a function of interest in this context is the survival function $\overline{F}(t)$, that is, the probability to survive any fixed time point $t \geq 0$. This function has been studied extensively in the context of reliability and survival analysis. If we have another random lifetime Y with survival function $\overline{G}$ then it is of interest to study whether one of the two survival functions lies above or below the other one. This basic idea is used to define the usual stochastic order. The formal definition is the following.

Definition 2.1. *Given two random variables X and Y, with survival functions $\overline{F}$ and $\overline{G}$, respectively, we say that X is smaller than Y in the stochastic order, denoted by $X \leq_{\text{st}} Y$, if*

$$\overline{F}(t) \leq \overline{G}(t) \text{ for all } t \in \mathbb{R}. \tag{2.1}$$

Clearly this is a partial order in the set of distribution functions and is reflexive and transitive. The definition of the stochastic order is a way to formalize the idea that the random variable X is less likely than Y to take on large values. However, given two distribution functions, they can cross as in the example provided in the introduction, and one of the main fields of research in the area of stochastic orders is to study under which conditions we can ensure the stochastic order. When considering two samples a first step, for the em-

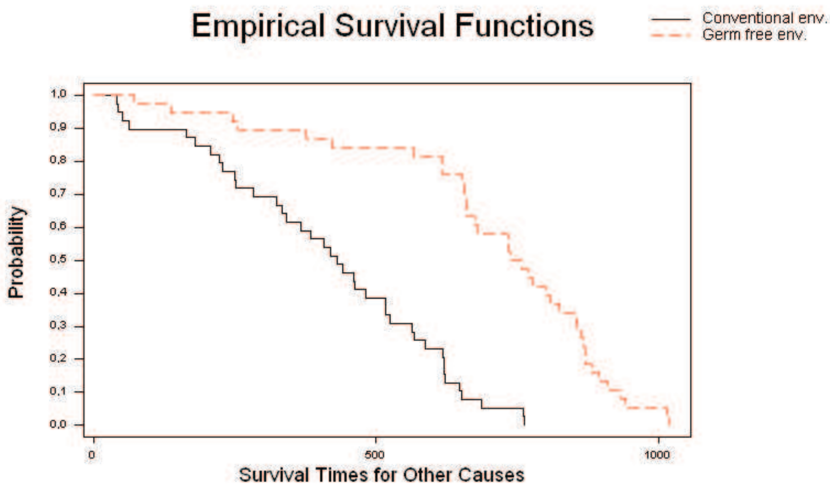

Figure 1: Estimation of survival functions for survival times of male mice.

pirical validation of the stochastic order of the parent populations, is to plot the empirical survival functions. Given a random sample $X_1, X_2, \ldots, X_n$, if we denote by $F_n(x) = \sum_{i=1}^{n} I_x(X_i)/n$ the empirical distribution function, where I_x denotes the indicator function in the set $(-\infty, x]$, the empirical survival function is defined as $\overline{F}_n(x) \equiv 1 - F_n(x)$. The Glivenko-Cantelli's theorem shows the uniform convergence of $\overline{F}_n$ to $\overline{F}$. Let us consider the following example for two data sets taken from Hoel (1972). The data set consists of two groups of survival times of male mice. Hoel (1972) considers three main groups depending on the cause of death. We consider here the group where the cause of death was different of thymic linfoma and cell sarcoma. The group was labeled "other causes". This group was divided in two subgroups. The first subgroup lived in a conventional laboratory environment while the second subgroup was in a germ free environment. Figure 1, provides empirical evidence for the stochastic order among these two subgroups.

Clearly another alternative is to plot the curve $(F(x), G(x))$, that is a P-P plot, and if $X \leq_{\text{st}} Y$, then the points of the P-P plot should lie below the diagonal $x = y$. It is interesting to note that the stochastic order is related to an important measure in risk theory, the value at risk notion. Given a random variable X with distribution function F the value at risk at a point $p \in (0, 1)$ is given by $VaR[X; p] \equiv F_X^{-1}(p)$, that is, it is the quantile function at point p. If the random variable is the risk associated to some action, like the potential loss in a portfolio position, then $VaR[X; p]$ is the larger risk for the $100p\%$ of the situations. In terms of the VaR notion we have that $X \leq_{\text{st}} Y$, if and only if, $VaR[X; p] \leq VaR[Y; p]$ for all $p \in (0, 1)$. In terms of a plot of the points

$(VaR[X;p], VaR[Y;p])$, that is a Q-Q plot, we have that $X \leq_{\text{st}} Y$ if the points of the Q-Q plot lie above the diagonal $x = y$. A useful characterization of the stochastic order is the following.

Theorem 2.1. *Given two random variables X and Y, then $X \leq_{\text{st}} Y$, if and only if,*

$$E[\phi(X)] \leq E[\phi(Y)],$$

for all increasing function ϕ for which previous expectations exist.

This result is of interest from both, a theoretical and an applied point of view. In some theoretical situations it is easier to provide a comparison for X and Y rather than a comparison for increasing transformations of the random variables. On the other hand in some situations we are more interested in some transformations of the random variables. For example $\phi(X)$ can be the benefit of a mechanism, which depends increasingly on the random lifetime X of the mechanism. Previous characterization also highlights a way to compare random variables. The idea is to compare two random variables in terms of expectations of transformations of the two random variables when the transformations belong to some specific family of functions of interest in the context we are working. For a general theory of this approach the reader can look at Müller (1997). Among the different families of distributions that can be considered one of the most important families is the family of increasing convex functions which lead to the consideration of the so called increasing convex order.

Definition 2.2. *Given two random variables X and Y, we say that X is smaller than Y in the increasing convex order, denoted by $X \leq_{\text{icx}} Y$, if*

$$E[\phi(X)] \leq E[\phi(Y)],$$

for all increasing convex function ϕ for which previous expectations exist.

This partial order can be characterized by the stop-loss function. Given a random variable X the stop-loss function is defined as

$$E[(X-t)^+] = \int_x^{+\infty} \overline{F}(u)du \text{ for all } x \in \mathbb{R},$$

where $(x)^+ = x$ if $x \geq 0$ and $(x)^+ = 0$ if $x < 0$. The stop-loss function is well known in the context of actuarial risks. If the random variable X denotes the random risk for an insurance company, it is very common that the company pass on pars of it to a reinsurance company. In particular the first company bears the whole risk, as long as it less than a fixed value t (called retention) and if $X > t$ the reinsurance company will take over the amount $X - t$. This is called a stop-loss contract with fixed retention t. The expected cost for the reinsurance company $E[(X-t)^+]$ is called the net premium. In terms of the stop-loss function we have that $X \leq_{\text{icx}} Y$ if and only if,

$$E[(X-t)^+] \leq E[(Y-t)^+] \text{ for all } t \in \mathbb{R}.$$

The icx order is of interest not only in risk theory but also in several situations where the stochastic order does not hold. From the definition it is clear that the icx order is weaker than the st order. Also if the survival functions cross just one time and the survival function of X is less than the survival function of Y after the crossing point, as in the example considered in the introduction, then we have that $X \leq_{\text{icx}} Y$.

The comparison of the survival functions can be made in several ways. For example we can examine the behaviour of the ratio of the two survival functions. For example we can study whether $\overline{F}(x)/\overline{G}(x)$ is decreasing, or equivalently, to avoid problems with zero values in the denominator, whether

$$\overline{F}(x)\overline{G}(y) \geq \overline{F}(y)\overline{G}(x) \text{ for all } x \leq y. \tag{2.2}$$

This condition leads to the following definition.

Definition 2.3. *Given two random variables X and Y, with distribution functions F and G, respectively, we say that X is smaller than Y in the hazard rate order, denoted by $X \leq_{\text{hr}} Y$, if* (2.2) *holds.*

The hazard rate order can be characterized, in the absolutely continuous case, in terms of the hazard rate functions. Given a random variable X with absolutely continuous distribution F and density function f, the hazard rate function is defined as $r(t) = f(t)/\overline{F}(t)$ for any t such that $\overline{F}(t) > 0$. The hazard rate measures, in some sense, the "probability" of instant failure at any time t when X denotes the random lifetime of a unit or a system. Given two random variables X and Y with hazard rates r and s respectively, then $X \leq_{\text{hr}} Y$ if and only if $r(t) \geq s(t)$ for all t such that $\overline{F}(t)$, $\overline{G}(t) > 0$.

In some situations it is not possible to provide an explicit expression for the distribution function and therefore is not possible to check some of the previous orders. An alternative is to use the density function (or the probability mass function in the case of discrete random variables) to compare two random variables. We consider the absolutely continuous case, the discrete case is similar replacing the density function by the probability mass function.

Definition 2.4. *Given two random variables X and Y with density functions f and g, respectively, we say that X is smaller than Y in the likelihood ratio order, denoted by $X \leq_{\text{lr}} Y$, if*

$$f(x)g(y) \geq f(y)g(x) \textit{ for all } x \leq y.$$

For example let us consider two gamma distributed random variables X and Y, with density functions given by:

$$f(x) = \frac{a^p}{\Gamma(p)} x^{p-1} \exp(-ax) \text{ for } x > 0 \text{ where } a, p > 0,$$

and

$$g(x) = \frac{b^q}{\Gamma(q)} x^{q-1} \exp(-bx) \text{ for } x > 0 \text{ where } b, q > 0.$$

It is not difficult to show that $X \leq_{\text{lr}} Y$ if $p \leq q$ and $a \geq b$. However it is not so easy to check whether the stochastic order, for example, holds for X and Y.

For all the previous orders we have the following chain of implications:

$$X \leq_{\text{lr}} Y \Rightarrow X \leq_{\text{hr}} Y \Rightarrow X \leq_{\text{st}} Y \Rightarrow X \leq_{\text{icx}} Y,$$

therefore the likelihood ratio order is stronger than the other ones and can be used as a sufficient condition for the rest of the stochastic orders.

Another context where the stochastic orders arise is in the comparison of variability of two random variables. It is usual to compare the variability in terms of the variance, the coefficient of variation and related measures. However it is possible to provide a more detailed comparison of the variability in terms of some stochastic orders. One of the most important orders in this context is the dispersive order.

Definition 2.5. *Given two random variables X and Y, with distribution functions F and G respectively, we say that X is smaller than Y in the dispersive order, denoted by $X \leq_{\text{disp}} Y$, if*

$$G^{-1}(q) - G^{-1}(p) \geq F^{-1}(q) - F^{-1}(p) \text{ for all } 0 < p < q < 1.$$

Therefore we compare the distance between any two quantiles, and we require to the quantiles of X to be less separated than the corresponding quantiles for Y. A characterization of the dispersive order that reinforces this idea is the following based on dispersive trasnformations. A real valued function ϕ is said to be dispersive if for any $x \leq y$ then $\phi(y) - \phi(x) \geq y - x$.

Theorem 2.2. *Given two random variables X and Y, then $X \leq_{\text{disp}} Y$ if and only if there exists a dispersive function ϕ, such that $Y =_{\text{st}} \phi(X)$.*

Another important characterization of the dispersive order is the following.

Theorem 2.3. *Given two random variables X and Y, then $X \leq_{\text{disp}} Y$ if and only if, $(X - F^{-1}(p))^+ \leq_{\text{st}} (Y - G^{-1}(p))^+$ for all $p \in (0, 1)$.*

This characterization provides a useful interpretation in risk theory. As we can see we compare $(X - t)^+$ and $(Y - t)^+$ when we replace t by $Var[X; p]$ and

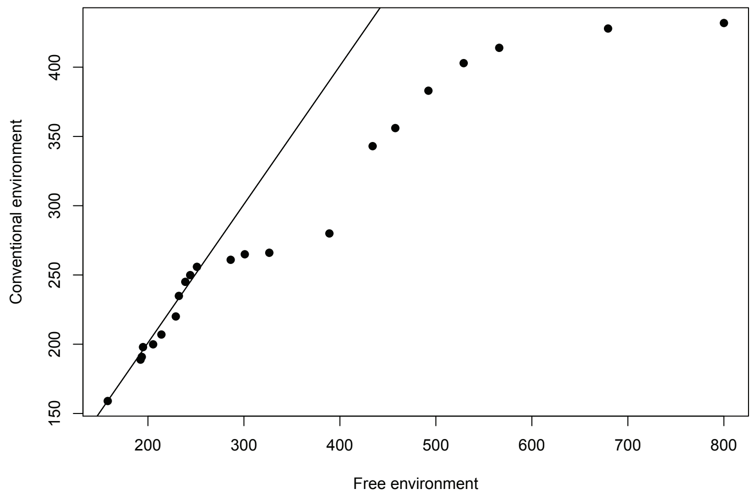

Figure 2: Q-Q plot for survival times of male mice (reference line at $x = y$).

$VaR[Y;p]$, respectively, in the previous expressions, and therefore we compare the so called "shortfalls" of the two random variables.

The comparison of the variability in terms of the dispersive order, is compatible with the comparison of variances, that is, if $X \leq_{\text{disp}} Y \Rightarrow Var[X] \leq Var[Y]$. The dispersive order can be verified in terms of the Q-Q plot of the two random variables. If $X \leq_{\text{disp}} Y$, then the slope of the Q-Q plot should be less than or equal to one between any pair of points. In some situations the dispersive order is too restrictive. For example let us consider the empirical Q-Q plot of two samples taken from the data set from Hoel (1972) considered previously. The data set consists of two groups of survival times of RFM strain male mice. The cause of death was thymic lymphoma. The first group lived in a conventional laboratory environment while the second group was in a germ free environment. Based on the Q-Q plot (see figure 2) it is not clear that the data are ordered according the dispersive order (for a discussion see Kuczmarski and Rosenbaum (1999)).

From Theorem 2.1 and the characterization provided in Theorem 2.3 it is clear that a more general criteria to compare the variability is to compare the expected values of $(X - F^{-1}(p))^+$ and $(Y - G^{-1}(p))^+$ for all $p \in (0,1)$. This condition leads to the so called right-spread order introduced independently by Fernández-Ponce, Kochar and Muñoz-Pérez (1998) and Shaked and Shanthikumar (1998).

Definition 2.6. *Given two random variables X and Y, with distribution func-*

tions F and G respectively, we say that X is smaller than Y in the right-spread order, denoted by $X \leq_{\rm rs} Y$, if

$$E[(X - F^{-1}(p))^+] \leq E[(Y - G^{-1}(p))^+] \text{ for all } p \in (0,1).$$

From previous comments clearly $X \leq_{\rm disp} Y \Rightarrow X \leq_{\rm rs} Y$. Let us study if the two groups considered previously are ordered according to the more general right spread order. A first approach, would be to consider nonparametric estimators of $E\left[\left(X - F^{-1}(p)\right)^+\right]$ and $E\left[\left(Y - G^{-1}(p)\right)^+\right]$ for all p, and to compare them. Noting that, for a non negative random variable X, $E\left[\left(X - F^{-1}(p)\right)^+\right] = E[X] - \int_0^{F_X^{-1}(p)} \overline{F}(x)dx$ and following Barlow *et al.* (1972) (pp. 235-237), a nonparametric estimator of $E\left[\left(X - F^{-1}(p)\right)^+\right]$ given a random sample X_1, X_2, ..., X_n of X ($F_X(0) = 0$), is given by

$$RS_n(p) \equiv \overline{X} - H_n^{-1}(p),$$

where $\overline{X}$ is the sample mean and $H_n^{-1}(p) = nX_{(1)}p$ for $0 \leq p < \frac{1}{n}$, and

$$H_n^{-1}(p) = \frac{1}{n}\sum_{j=1}^{i}(n-j+1)(X_{(j)} - X_{(j-1)}) + \left(p - \frac{i}{n}\right)(n-i)(X_{(i+1)} - X_{(i)})$$

for $\frac{i}{n} \leq p < \frac{i+1}{n}$, where $X_{(i)}$ denotes the $i-th$ order statistic of a sample of size n from X and $X_{(0)} \equiv 0$. The plot of the nonparametric estimators of the right spread functions (see Figure 3) clearly suggests that the survival times in the germ free environment are more dispersed, in right spread order, than that of the laboratory environment.

As in the case of the dispersive order the right spread order can be interpreted and used in the context of risk theory. The $VaR[X;p]$ only provides local information about the distribution, but the measure $E[(X - F^{-1}(p))^+]$, provides more information about the thickness of the upper tail.

3. Some multivariate extensions

In this section we recall some multivariate extension of some of the stochastic orders considered in the previous section. We start considering an extension of the usual stochastic order.

Definition 3.1. *Given two n-dimensional random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$, we say that $\boldsymbol{X}$ is less than $\boldsymbol{Y}$ in the usual multivariate stochastic order, denoted by*

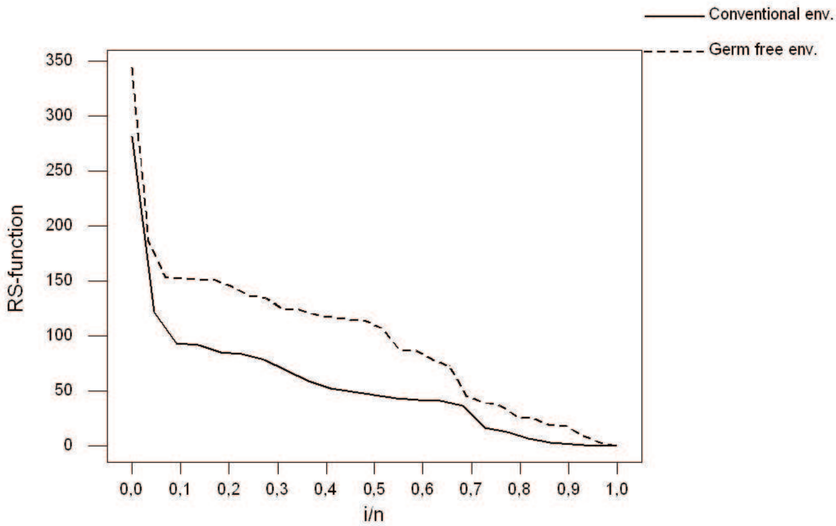

Figure 3: RS estimations for survival times of male mice.

$\boldsymbol{X} \leq_{\text{st}} \boldsymbol{Y}$, *if*

$$E[\phi(\boldsymbol{X})] \leq E[\phi(\boldsymbol{Y})], \tag{3.1}$$

for all increasing function $\phi : \mathbb{R}^n \mapsto \mathbb{R}$, *for which the previous expectations exist.*

Clearly this is an extension based on the characterization provided in Theorem 2.1, for the univariate case. However it is not possible to characterize the multivariate stochastic order in terms of the comparison of multivariate survival or distribution functions, like in the univariate case (see (2.1)). In the multivariate case these comparisons lead to the definitions of some orders that compares the degree of dependence of the random vectors. For the multivariate stochastic order we have that if $\boldsymbol{X} \leq_{\text{st}} \boldsymbol{Y}$ then, for any increasing function $\phi : \mathbb{R}^n \mapsto \mathbb{R}$, we have that $\phi(\boldsymbol{X}) \leq_{\text{st}} \phi(\boldsymbol{Y})$. This is a very interesting property. For example let us consider that $\boldsymbol{X} = (X_1, \ldots, X_n)$ and $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ are vectors of returns for investments under two different scenarios. In the first case investing one unit of money into stock i yields a return X_i and in the second case yields a return Y_i. If we invest $a_i > 0$ units in stock i, then the returns, for the two different scenarios, are $\sum_{i=1}^n a_i X_i$ and $\sum_{i=1}^n a_i Y_i$. Clearly if $\boldsymbol{X} \leq_{\text{st}} \boldsymbol{Y}$ then $\sum_{i=1}^n a_i X_i \leq_{\text{st}} \sum_{i=1}^n a_i Y_i$.

The multivariate extension of the stochastic order in terms of comparisons of expectations can be used to provide multivariate extensions of univariate stochastic orders based on comparisons of expectations. For example we can consider the increasing convex order. In the multivariate case there are several possibilities to extend this concept, depending on the kind of convexity that we consider.

Definition 3.2. *Given two random vectors* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *we say that* $\boldsymbol{X}$ *is less than* $\boldsymbol{Y}$ *in the multivariate increasing convex order, denoted by* $\boldsymbol{X} \leq_{\text{icx}} \boldsymbol{Y}$*, if*

$$E[\phi(\boldsymbol{X})] \leq E[\phi(\boldsymbol{Y})], \tag{3.2}$$

for all increasing convex function $\phi : \mathbb{R}^n \mapsto \mathbb{R}$*, for which the previous expectations exist.*

If (3.2) holds for all increasing componentwise convex function ϕ, then we say that $\boldsymbol{X}$ is less than $\boldsymbol{Y}$ in the increasing componentwise convex order, denoted by $\boldsymbol{X} \leq_{\text{iccx}} \boldsymbol{Y}$. Some other appropriate classes of functions defined on $\mathbb{R}^n$ can be considered to extend convex orders to the multivariate case, by means of a difference operator. Let Δ_i^ϵ tbe the ith difference operator defined for a function $\phi : \mathbb{R}^n \to \mathbb{R}$ as

$$\Delta_i^\epsilon \phi(\mathbf{x}) = \phi(\mathbf{x} + \epsilon \mathbf{1}_i) - \phi(\mathbf{x})$$

where $\mathbf{1}_i = (0, \ldots, 0, \overbrace{1}^{i}, 0, \ldots, 0)$. A function ϕ is said to be directionally convex if $\Delta_i^\epsilon \Delta_j^\delta \phi(\mathbf{x}) \geq 0$ for all $1 \leq i \leq j \leq n$ and $\epsilon, \delta \geq 0$. Directionally convex functions are also known as ultramodular functions (see Marinacci and Montrucchio (2005)). A function ϕ is said to be supermodular if $\Delta_i^\epsilon \Delta_j^\delta \phi(\mathbf{x}) \geq 0$ for all $1 \leq i < j \leq n$ and $\epsilon, \delta \geq 0$. If ϕ is twice differentiable then, it is directionally convex if $\partial^2 \phi / \partial x_i \partial x_j \geq 0$ for every $1 \leq i \leq j \leq n$, and it is supermodular if $\partial^2 \phi / \partial x_i \partial x_j \geq 0$ for every $1 \leq i < j \leq n$. Clearly a function ϕ is directionally convex if it is supermodular and it is componentwise convex.

When we consider increasing directionally convex functions in (3.2) then we say that $\boldsymbol{X}$ is less than $\boldsymbol{Y}$ in the increasing directionally convex order, denoted by $\boldsymbol{X} \leq_{\text{idir-cx}} \boldsymbol{Y}$. If we consider increasing supermodular functions in (3.2) then we say that $\boldsymbol{X}$ is less than $\boldsymbol{Y}$ in the increasing supermodular order, denoted by $\boldsymbol{X} \leq_{\text{ism}} \boldsymbol{Y}$.

The supermodular order is a well known tool to compare dependence structures of random vectors whereas the directionally convex order not only compares the dependence structure but also the variability of the marginals. Again the comparison of random vectors under these different criteria can be used for the comparison of loss or benefits of portfolios.

Now we consider some multivariate stochastic orders in the absolutely continuous case. We start considering the hazard rate order. In the multivariate case it is possible to provide several extensions. We first consider the time-dynamic definition of the multivariate hazard rate order introduced by Shaked and Shanthikumar (1987).

Let us consider a random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ where the X_i's can be considered as the lifetimes of n units. For $t \geq 0$ let h_t denotes the list of units which have failed and their failure times. More explicitly, a history h_t will denote

$$h_t = \{\boldsymbol{X}_I = \boldsymbol{x}_I, \boldsymbol{X}_{\overline{I}} > t\boldsymbol{e}\},$$

where $I = \{i_1, \dots, i_k\}$ is a subset of $\{1, \dots, n\}$, $\overline{I}$ is its complement with respect to $\{1, \dots, n\}$, $\boldsymbol{X}_I$ will denote the vector formed by the components of $\boldsymbol{X}$ with index in I and $0 < x_{i_j} < t$ for all $j = 1, \dots, k$ and $\boldsymbol{e}$ denotes vectors of 1's, where the dimension can be determined from the context.

Now we proceed to give the definition of the multivariate hazard rate order. Given the history h_t, as above, let $j \in \overline{I}$, its multivariate conditional hazard rate, at time t, is defined as follows:

$$\eta_j(t|h_t) = \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} P[t < X_j \le t + \Delta t | h_t]. \tag{3.3}$$

Clearly $\eta_j(t|h_t)$ is the "probability" of instant failure of component j, given the history h_t.

Definition 3.3. *Given two n-dimensional random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ with hazard rate functions $\eta.(\cdot|\cdot)$ and $\lambda.(\cdot|\cdot)$, respectively. We say that $\boldsymbol{X}$ is less than $\boldsymbol{Y}$ in the dynamic multivariate hazard rate order, denoted by $\boldsymbol{X} \le_{\text{dyn-hr}} \boldsymbol{Y}$, if, for every $t \ge 0$,*

$$\eta_i(t|h_t) \ge \lambda_i(t|h'_t)$$

where

$$h_t = \{\boldsymbol{X}_{I \cup J} = \boldsymbol{x}_{I \cup J}, \boldsymbol{X}_{\overline{I \cup J}} > t\boldsymbol{e}\} \tag{3.4}$$

and

$$h'_t = \{\boldsymbol{Y}_I = \boldsymbol{y}_I, \boldsymbol{Y}_{\overline{I}} > t\boldsymbol{e}\}, \tag{3.5}$$

whenever $I \cap J = \emptyset$, $\boldsymbol{0} \le \boldsymbol{x}_I \le \boldsymbol{y}_I \le t\boldsymbol{e}$, and $\boldsymbol{0} \le \boldsymbol{x}_J \le t\boldsymbol{e}$, where $i \in \overline{I \cup J}$.

Given two histories as above, we say that h_t is more severe than h'_t.

The multivariate hazard rate order is not necessarily reflexive. In fact if a random vector $\boldsymbol{X}$ satisfies $\boldsymbol{X} \le_{\text{dyn-hr}} \boldsymbol{X}$, then it is said to have the HIF property (hazard increasing upon failure) and it can be considered as a positive dependence property. Also the HIF notion can be considered as a mathematical formalization of the *default contagion* notion in risk theory. Loosely speaking, the default contagion notion means that the conditional probability of default for a non-defaulted firm increases given the information that some other firms has defaulted. In particular, concerning the HIF notion, we have that if the information become worst, that is, the number of defaulted firms is larger and the default times are earlier, then the probability of default for a non-defaulted firm increases.

Another extension, from a mathematical point of view, is the one provided by Hu, Khaledi and Shaked (2003).

Definition 3.4. *Given two n-dimensional random vectors with multivariate survival functions $\overline{F}(\boldsymbol{x}) = P[\boldsymbol{X} > \boldsymbol{x}]$ and $\overline{G}(\boldsymbol{x}) = P[\boldsymbol{Y} > \boldsymbol{x}]$ for $\boldsymbol{x} \in \mathbb{R}^n$, respectively, we say that $\boldsymbol{X}$ is smaller than Y in the multivariate hazard rate order, denoted by $\boldsymbol{X} \leq_{\text{hr}} \boldsymbol{Y}$, if*

$$\overline{F}(\boldsymbol{x} \wedge \boldsymbol{y})\overline{G}(\boldsymbol{x} \vee \boldsymbol{y}) \geq \overline{F}(\boldsymbol{x})\overline{G}(\boldsymbol{y}) \text{ for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n,$$

where $(\boldsymbol{x} \wedge \boldsymbol{y}) = (\min(x_1, y_1), \dots, \min(x_n, y_n))$ *and* $(\boldsymbol{x} \vee \boldsymbol{y}) = (\max(x_1, y_1), \dots, \max(x_n, y_n))$.

As we can see, this definition is based on the Definition 2.3 and the previous one is based on the characterization, in the univariate case, in terms of the hazard rate function. In a similar way based on the Definition 2.4 we can define the following multivariate extension of the likelihood ratio order.

Definition 3.5. *Given two n-dimensional random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$, with joint densities f and g, respectively, we say that $\boldsymbol{X}$ is less than $\boldsymbol{Y}$ in the multivariate likelihood ratio order, denoted by $\boldsymbol{X} \leq_{\text{lr}} \boldsymbol{Y}$, if*

$$f(\boldsymbol{x} \wedge \boldsymbol{y})g(\boldsymbol{x} \vee \boldsymbol{y}) \geq f(\boldsymbol{x})g(\boldsymbol{y}), \text{ for all } \boldsymbol{x}, \ \boldsymbol{y} \in \mathbb{R}^n.$$

Again the multivariate likelihood ratio order is not reflexive. In fact given an n-dimensional random vector $\boldsymbol{X}$ with density f, we say that $\boldsymbol{X}$ is MTP_2 (multivariate totally positive of order 2) if

$$f(\boldsymbol{x} \wedge \boldsymbol{y})f(\boldsymbol{x} \vee \boldsymbol{y}) \geq f(\boldsymbol{x})f(\boldsymbol{y}), \text{ for all } \boldsymbol{x}, \ \boldsymbol{y} \in \mathbb{R}^n, \tag{3.6}$$

that is, if $\boldsymbol{X} \leq_{\text{lr}} \boldsymbol{X}$.

To finish we provide some multivariate extensions of variability orders. Recall that the definitions given in the univariate case are given in terms of the quantile funciton. In the multivariate case given an n-dimensional random vector $\mathbf{X}$ and $\mathbf{u} = (u_1, \cdots, u_n)$ in $[0,1]^n$, a possible definition for the multivariate $\mathbf{u}$-quantile for $\mathbf{X}$, denoted as $\hat{x}(\mathbf{u}) = (\hat{x}_1(u_1), \hat{x}_2(u_1, u_2), \dots, \hat{x}_n(u_1, \dots, u_n))$, is as follows

$$\hat{x}_1(u_1) \equiv F_1^{-1}(u_1)$$

and

$$\hat{x}_i(u_1, \dots, u_i) \equiv F_{i|1,\dots,i-1}^{-1}(u_i) \text{ for } i = 2, \dots, n,$$

where $F_{i|1,\dots,i-1}^{-1}(\cdot)$ is the quantile function of the r.v. $[X_i | X_1 = \hat{x}_1(u_1), \dots, X_{i-1} = \hat{x}_{i-1}(u_1, \dots, u_{i-1})]$.

This transformation is widely used in simulation theory, and it is named the *standard construction*, and satisfy most of the important properties of the quantile function in the univariate case.

Given two n-dimensional random vectors $\mathbf{X}$ and $\mathbf{Y}$, with standard constructions $\hat{x}(\mathbf{u})$ and $\hat{y}(\mathbf{u})$, respectively, Shaked and Shantikumar (1998) studied the following condition

$$\hat{y}(\mathbf{u}) - \hat{x}(\mathbf{u}) \text{ is increasing in } \mathbf{u} \in (0,1)^n, \tag{3.7}$$

as a multivariate generalization of the dispersive order, and we will say that $\mathbf{X}$ is smaller than $\mathbf{Y}$ in the variability order, denoted by $\mathbf{X} \leq_{Var} \mathbf{Y}$, if (3.7) holds.

Another multivariate generalization based on the standard construction was given by Fernández-Ponce and Suárez-Llorens (2003). We will say that $\mathbf{X}$ is less than $\mathbf{Y}$ in the dispersive order, denoted by $\mathbf{X} \leq_{disp} \mathbf{Y}$, if

$$\| \hat{x}(\mathbf{v}) - \hat{x}(\mathbf{u}) \| \leq \| \hat{y}(\mathbf{v}) - \hat{y}(\mathbf{u}) \|$$

for all $\mathbf{u}, \mathbf{v} \in [0,1]^n$.

Shaked and Shanhtikumar (1998) introduce condition (3.7) to identify pairs of multivariate functions of random vectors that are ordered in the st:icx order.

Definition 3.6. *Given two random variables X and Y, we say that X is smaller than Y in the st:icx order, denoted by $X \leq_{st:icx} Y$, if $E[h(X)] \leq E[h(Y)]$ for all increasing functions h for which the expectations exists (i.e. if $X \leq_{st} Y$), and if*

$$Var[h(X)] \leq Var[h(Y)] \text{ for all increasing convex functions } h,$$

provided the variances exist.

The result was given for CIS random vectors. A random vector $(X_1, ..., X_n)$ is said to be *conditionally increasing in sequence* (CIS) if, for $i = 2, 3, \ldots, n$,

$$\begin{aligned}(X_i | X_1 = x_1, X_2 = x_2, ..., X_{i-1} = x_{i-1}) \\ \leq_{\mathrm{st}} (X_i | X_1 = x'_1, X_2 = x'_2, ..., X_{i-1} = x'_{i-1}) \\ \text{whenever } x_j \leq x'_j,\ j = 1, 2, \ldots, i-1.\end{aligned}$$

Shaked and Shanthikumar (1998) prove that given two nonnegative n-dimensional random vectors $\mathbf{X}$ and $\mathbf{Y}$, with the CIS property, if $\mathbf{X} \leq_{\mathrm{Var}} \mathbf{Y}$ then

$$\phi(\mathbf{X}) \leq_{\mathrm{st:icx}} \phi(\mathbf{Y}),$$

for all increasing directionally convex function ϕ. Therefore we can apply this result for the comparisons of portfolios in the sense described previously.

4. Further comments and remarks

In this section I would like to point out some other applications and fields where stochastic orders can be applied. The main applications that we have discussed previously are in the context of failure or survival times, which provides a lot of applications in reliability theory (see Chapters 15 and 16 in Shaked and Shanthikumar (1994)) and in the context of risk theory (see Chapter 12 in Shaked and Shanthikumar (1994) and Denuit *et al.* (2005)). It is clear that they can be used also to compare the times at which some event occurs. Therefore the comparison of stochastic processes are of interest in general, and in particular in several contexts like epidemiology, ecology, biology (see for example Belzunce *et al.* (2001) for general results and Chapter 11 in Shaked and Shanthikumar (1994) for applications in epidemics). The comparison of ordered data is also an increasing field of research and the reader can look at Boland, Shaked and Shanthikumar (1998), Boland *et al.* (2002), Belzunce, Mercader and Ruiz (2005), Belzunce *et al.* (2007) for several results in this direction. Also the stochastic orders can be used in several contexts in operations research for example can be used to the scheduling of jobs in machines and in Jackson networks (see Chapter 13 and 14, respectively, in Shaked and Shanthikumar (1994)).

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