3. ARTÍCULOS DE APLICACIÓN

SOME PROBABILITY APPLICATIONS TO THE RISK ANALYSIS IN INSURANCE THEORY

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Abstract

This work concerns with some probability applications in insurance theory. The problem to determine the ruin probability of an insurance company is considered. We show that by using some stochastic models and considering several probabilistic procedures such a probability can be approached. As illustration, an application to car insurance is provided.

Keywords: Risk analysis, Stochastic modelling, Insurance theory applications.

1. Introduction

Risk analysis in insurance theory is an important area for probability and statistical applications. In particular, the probabilistic modelling of the surplus evolution in an insurance company has received some attention in the specialized literature. In fact, denoting by $U(t)$ the surplus of an insurance company at time t, the classical model establishes that:

$$U(t) = u + ct - S(t),\ t \geq 0, \tag{1.1}$$

where $u > 0$ is the initial surplus, c is the constant rate at which the premiums are received per unit time and $S(t)$ is the aggregate amount concerning the claims reported in the time interval $[0, t]$. It is assumed that $\{S(t), t \geq 0\}$ is a compound Poisson process,

$$S(t) = \sum_{i=1}^{N(t)} X_i,\ t \geq 0$$

$\{N(t), t \geq 0\}$ being a Poisson process and $\{X_n, n = 1, 2, \ldots\}$ a sequence of independent and identically distributed random variables, both assumed to be independent. The variable $N(t)$ represents the number of reported claims in $[0, t]$ and X_i is the amount corresponding to the ith claim.

In general, from model (2.1), some results in risk theory have been derived but such a model it is not flexible enough in order to describe the probabilistic evolution of $U(t)$ in more complex real situations. In an attempt to contribute some solution to this problem, several classes of stochastic models have been introduced and some theory and applications about them developed. We will quote, for example:

(a) Stochastic models considering claim arrivals governed by non-Poissonian processes. See e.g. [4] where it is assumed that the number of claims is described through a Cox process.

(b) Stochastic models allowing a rate $c(\cdot)$ which change through a Markov process. See e.g. [1] or [6] where it is considered that the rate changes modulated by an underlying irreducible Markov chain or a premium rate in a Markovian environment, respectively.

(c) Stochastic models including a diffusion component which represents uncertainties in both the premium income and the costs. Firstly studied by Gerber, such models have been applied in several risk problems, see e.g. [2], [3] or [8].

In this work, we will focus our interest in a class of risk models which allows premium rate and

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diffusion component depending on an underlying continuous-time Markov chain. Moreover, the occurrence of claims is assumed to be well-described by a Cox process. It is usually referred as *the class of risk models with Markov modulated speed.* We will center our attention in the determination of the ruin probability for an insurance company. This problem is an important objective in many research developed in actuarial risk theory. We will show by using some classical probabilistic techniques that it is possible to obtain the ruin probability.

The paper is organized as follow: In Section 2, we provide the probabilistic descriptions of such a class of risk models. In Section 3, we develop a probabilistic procedure to determine the ruin probability. Finally, Section 4 is devoted to considering an application in car insurance.

2. Stochastic modelling

Let us consider the following stochastic modelling for $U(t)$, $t \geq 0$:

$$U(t) = u + \int_0^t c(I(s))ds - \sum_{i=1}^{N(t)} X_i + \int_0^t \sigma(I(s))dW_s \tag{2.1}$$

where:

(a) $\{N(t), t \geq 0\}$, $N(0) = 0$, is a Cox point process. $N(t)$ represents the number of claims received for the insurance company during the interval $[0, t]$.

(b) $\{X_n, n = 0, 1, \ldots\}$ is a sequence of independent and identically distributed random variables which represents the amount corresponding to the successive claims received by the company in $[0, t]$. Let us write $F(x) = P(X_1 \leq x)$, $x > 0$, $F(0) = 0$, and $\mu = E[X_1]$.

(c) $\{W(t), t \geq 0\}$ is a standard Wiener process with diffusion coefficient $\sigma(\cdot)$. This process represents the disturbances originated from several tiny stochastic factors.

(d) $\{I(t), t \geq 0\}$ is an underlying continuous-time homogeneous Markov chain with state space $S = \{1, \ldots, n\}$ assumed to be irreducible.

Notice that in (2.1) the premium rate $c(\cdot)$ and the diffusion coefficient $\sigma(\cdot)$ depend on the current state of the Markov chain $\{I(t), t \geq 0\}$. In fact, if at time t it is verified that $I(t) = i$ then, by simplicity, $c(I(t))$ and $\sigma(I(t))$ will be denoted, respectively, as c_i and σ_i, assumed to be positive. On the other hand, we are considering that the number of reported claims is governed by a Cox process, hence if $I(s) = i$, $s \in [0, t]$ then the number of claims received in $[0, t]$ has a Poisson distribution with mean $\lambda_i > 0$.

According to [5], we shall denote by q_i the rate at which the Markov chain $\{I(t), t \geq 0\}$ leaves the state i, and by q_{ij} and p_{ij}, respectively, the transition intensity and the transition probability that it leaves the state i for the first time and enter into the state j immediately. Assuming that $p_{ii} = 0$, $i \in S$, one deduces that $q_{ij} = q_i p_{ij}$ for $i \neq j$ and $q_{ii} = -q_i$. Since all the states communicate,

$$\pi_i q_i = \sum_{j=1}^{n} \pi_j q_j p_{ji}. \tag{2.2}$$

where $\pi_1, \pi_2, ..., \pi_n$ denotes a stationary distribution corresponding to $\{I(t), t \geq 0\}$.

Also, we will assume that the named safety loading is positive, namely $c - \lambda\mu > 0$ where:

$$c = \sum_{i=1}^{n} \pi_i c_i \quad \text{and} \quad \lambda = \sum_{i=1}^{n} \pi_i \lambda_i.$$

3. Ruin probability

In this section we are interested in the determination of the ruin probability defined by:

$$\psi(u) = \sum_{i=1}^{n} \pi_i \psi_i(u) \tag{3.1}$$

where $\psi_i(u) = P(U(t) \leq 0 \mid U(0) = u, I(0) = i)$ for some $t \geq 0$.

First, by considering the evolution of $U(t)$ in a short interval $[0, h)$, $h > 0$, we shall determine a system of equations for $R_i(u) = 1 - \psi_i(u)$, $i \in S$. In fact, assuming that $I(0) = i$, we can consider the following possibilities during the interval $[0, h)$:

(a) There is not reported claims and $I(s) = i$ for $s \in [0, h)$.

(b) One claim is produced but the amount to be paid for such a claim does not cause ruin and $I(s) = i$ for $s \in [0, h)$.

(c) There is not reported claims and, from the state i a change to other state it is produced.

(d) At least one claim is reported in and at least one change of state, from i, is produced.

Consequently, see for more details [2], on has for $t \in [0, h)$,

$$\begin{aligned} R_i(u) &= (1-\lambda_i h - q_i h + o(h))E[R_i(\varphi_i(u,h)] \\ &+(\lambda_i h + o(h))(1-q_i h + o(h)) \\ &E[\int_0^{\varphi_i(u,h)} R_i(\varphi_i(u,h)-x)dF(x)] \\ &+(1-\lambda_i h + o(h))(q_i h + o(h)) \\ &\sum_{j=1}^{n} p_{ij} E[R_j(\varphi_i(u,h))] + o(h). \end{aligned} \tag{3.2}$$

where $\varphi_i(u,h) = u + c_i h + \sigma_i W_h$.

By considering the Taylor expression in u of $E[R_i(\varphi_i(u,h))]$, dividing by h, and taking limit as $h \downarrow 0$, it is matter of some straightforward calculation to deduce,

$$\begin{aligned} \frac{\sigma_i^2}{2} R_i^{''}(u) + c_i R_i^{'}(u) &= (\lambda_i + q_i) R_i(u) \\ &- \lambda_i \int_0^u R_i(u-x)dF(x) \\ &- q_i \sum_{j=1}^{n} p_{ij} R_j(u). \end{aligned} \tag{3.3}$$

By integration of (3.3) on $[0,t]$ and using the fact that $R_i(0) = 0$,

$$\begin{aligned} \frac{\sigma_i^2}{2} R_i^{'}(t) + c_i R_i(t) &= \frac{\sigma_i^2}{2} R_i^{'}(0) \\ &+ (\lambda_i + q_i) \int_0^t R_i(u)du \\ &- \lambda_i \int_0^t \int_0^u R_i(u-x)dF(x)du \\ &- q_i \sum_{j=1}^{n} p_{ij} \int_0^t R_j(u)du. \end{aligned}$$

and, taking into account that

$$\int_0^t \int_0^u R_i(u-x)dF(x)du = \int_0^t R_i(u)du + \int_0^t R_i(t-x)F^*(x)dx.$$

where $F^*(x) = 1 - F(x)$, considering the fact that $\psi_i(t) = 1 - R_i(t)$, $i = 1, \ldots, n$,

$$\begin{aligned} &\frac{\sigma_i^2}{2} \psi_i^{'}(t) + c_i \psi_i(t) = \\ &c_i + \frac{\sigma_i^2}{2} \psi_i^{'}(0) + \lambda_i \int_0^t F^*(x)dx \\ &-\lambda_i \int_0^t \psi_i(t-x)F^*(x)dx \\ &+q_i \int_0^t \psi_i(u)du - q_i \sum_{j=1}^{n} p_{ij} \int_0^t \psi_j(u)du. \end{aligned} \tag{3.4}$$

Finally, taking limit as $t \uparrow \infty$, we deduce the following system of equations for $\psi_i(u)$, $i = 1, \ldots, n$:

$$\begin{aligned} \psi_i^{'}(0) &= \frac{2}{\sigma_i^2} \, (-c_i - \lambda_i \mu - q_i \int_0^\infty \psi_i(u)du \\ &+q_i \sum_{j=1}^{n} p_{ij} \int_0^\infty \psi_j(u)du \,). \end{aligned} \tag{3.5}$$

Using the numerical solutions of (3.5), from (3.1) we may determine the corresponding ruin probability.

Note that when $c_i = c$ and $\sigma_i = \sigma$, $i = 1, \ldots, n$, by (3.4) and (3.5), taking into account (2.2) we deduce:

$$\frac{\sigma^2}{2} \psi^{'}(t) + c\psi(t) = c + \frac{\sigma^2}{2} \psi^{'}(0) + \lambda \int_0^t F^*(x)dx$$

$$- \sum_{i=1}^{n} \pi_i \lambda_i \int_0^t \psi_i(t-x)F^*(x)dx,$$

and

$$\psi^{'}(0) = \frac{-2}{\sigma^2}(c + \lambda\mu)$$

respectively.

4. Application to car insurance

It is well-known the influence that certain factors, for example the inclemency of the weather, the conditions of the roads, and so on, have in the occurrence of traffic accidents. Next we shall consider an application of the model (2.1) in car insurance.

In a first approximation, we will assume that the underlying Markov chain $\{I(t), t \geq 0\}$ has a two-states space, namely $S = \{1, 2\}$, where:

- The state 1 represents the risk under normal conditions.
- The state 2 represents the risk under bad conditions (for e.g. slippery roads, foggy days or high traffic volume.

We refer the reader to [5] and [7] for more details. Also, we will consider that the variable X_i has exponential distribution with mean μ and that $p_{12} = p_{21} = 1$, $p_{11} = p_{22} = 0$. Then,

$$q_{11} = -q_1, \quad q_{22} = -q_2, \quad q_{12} = q_1, \quad q_{21} = q_2$$

and

$$\pi_1 = q_2(q_1 + q_2)^{-1}, \quad \pi_2 = q_1(q_1 + q_2)^{-1}.$$

Let us denote by

$$\phi_i(s) = \int_0^\infty e^{-st}\psi_i(t)dt$$

and

$$\phi^*(s) = \int_0^\infty e^{-st}F^*(t)dt.$$

Taking into account that

$$\int_0^\infty e^{-st}\psi_i^{'}(t)dt = s\phi_i(s) - 1,$$

and

$$\int_0^\infty e^{-st}\int_0^t \psi_i(u)dudt = \frac{1}{s}\phi_i(s).$$

by using the Laplace transformation in (3.4), ones deduces, for $i = 1, 2$,

$$\left(c_i + \frac{\sigma_i^2}{2}s - \frac{q_i}{s} + \lambda_i\phi^*(s)\right)\phi_i(s) + \frac{q_i}{s}\sum_{j=1}^n p_{ij}\phi_j(s) =$$

$$\frac{\sigma_i^2}{2} + \frac{1}{s}(c_i + \frac{\sigma_i^2}{2}\psi_i^{'}(0)) + \frac{\lambda_i}{s}\phi^*(s)$$

hence,

$$\left(c_1 + \frac{\sigma_1^2}{2}s - \frac{q_1}{s} + \frac{\lambda_1\mu}{s\mu + 1}\right)\phi_1(s) + \frac{q_1}{s}\phi_2(s) =$$

$$\frac{\sigma_1^2}{2} + \frac{1}{s}\left(c_1 + \frac{\sigma_1^2}{2}\psi_1^{'}(0)\right) + \frac{\lambda_1\mu}{s(s\mu + 1)}$$

and

$$\left(c_2 + \frac{\sigma_2^2}{2}s - \frac{q_2}{s} + \frac{\lambda_2\mu}{s\mu + 1}\right)\phi_2(s) + \frac{q_2}{s}\phi_1(s) =$$

$$\frac{\sigma_2^2}{2} + \frac{1}{s}\left(c_2 + \frac{\sigma_2^2}{2}\psi_2^{'}(0)\right) + \frac{\lambda_2\mu}{s(s\mu + 1)}.$$

Conclusion:
The surplus evolution corresponding to an insurance company could be suitably described in terms of the general model given in (2.1). From a practical point of view, by solving the system given in (3.5) and taking into account expression (3.1), it is possible to determine the ruin probability. This parameter plays a crucial role in research about risk analysis in insurance theory.

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