Monotonous stability for neutral fixed points

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Abstract

We give subtle, simple and precise results about the convergence or the divergence of the sequence (x_n), where $x_j = f(x_{j-1})$ for every integer j, when the initial element x_0 is in the neighbourhood of a neutral fixed point, i.e. a point x^* such that $f(x^*) = x^*$ with $|f'(x^*)| = 1$ (where f is a C^∞ function defined on a subset of $\mathbb{R}$).

1 Introduction

Throughout this paper, we consider a C^∞ function f defined on a subset $S = dom\ f$ of $\mathbb{R}$ and a *fixed point* x^* for f, i.e. a point x^* which will be supposed in the interior of S and such that $f(x^*) = x^*$.

Given a point $x_0 \in S$, we define the *orbit* of x_0 under f to be the infinite sequence of points $x_0, x_1, x_2, \ldots$, where $x_0 = f^0(x_0),\quad x_1 = f(x_0) = f^1(x_0),\quad x_2 = f(x_1) = f^2(x_0), \ldots, x_{n+1} = f(x_n) = f^{n+1}(x_0), \ldots :$ the point x_0 is called the *seed* of this orbit which will be denoted by $\mathcal{O}(f; x_0)$ [3, 4].

The aim of this note is to very simply study the asymptotic behavior (i.e. the convergence or divergence) of an orbit the seed of which is in a suitable neighbourhood of a fixed point.

The situation is clear and well-known when x^* is *hyperbolic*, i.e. when $|f'(x^*)| \neq 1$ [3]. Indeed, if $|f'(x^*)| < 1$, then x^* is *stable* or *attracting*; this means that there exists an open interval I which contains x^* and such that $f(I) \subset I$ and $\lim_{n\to\infty} f^n(x) = x^*$ for every $x \in I$ [3, p. 43] [7, p. 45]. Moreover, if $|f'(x^*)| > 1$, then x^* is *unstable* or *repelling*; this means that there exists an open interval I which contains x^* and for which the following condition is satisfied : if $x \in I \setminus \{x^*\}$, there exists an integer $n > 0$ such that $f^n(x) \notin I$ [3, p. 44] [6, p. 20].

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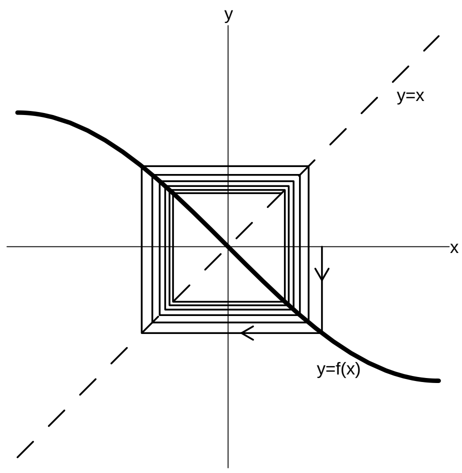
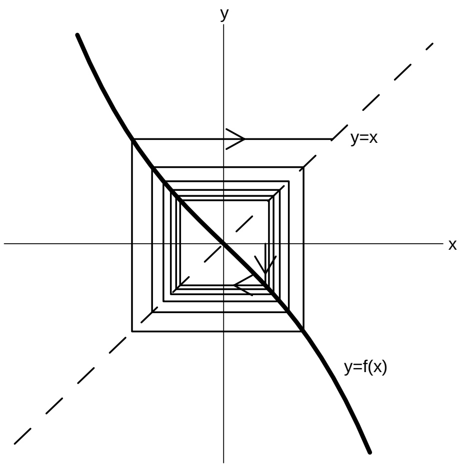
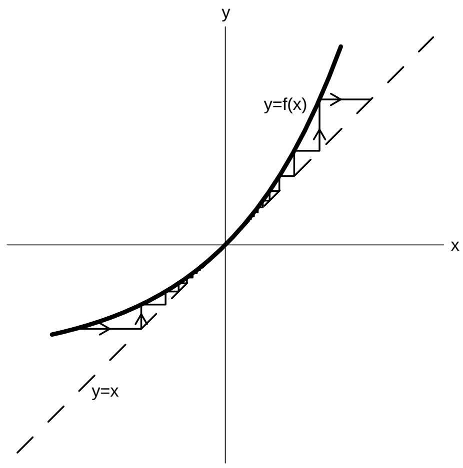
When x^* is *neutral* (i.e. when x^* is not hyperbolic), Holmgren says that "nothing definitive can be said about the behavior of points near x^*" [5, p. 53]; nevertheless, easy examples show that several typical situations are possible : x^* may be stable, unstable, "semistable from above", "semistable from below" as it can be seen on these figures which give the orbit analysis [3] in classical cases.



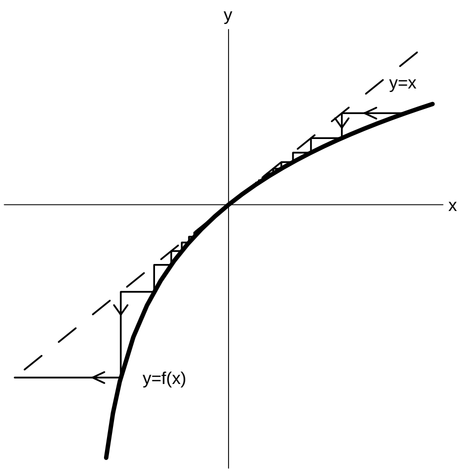
(a) stable
$f'(x^*) = -1$



(b) unstable
$f'(x^*) = -1$



(c) semistable from below
$f'(x^*) = 1$



(d) semistable from above
$f'(x^*) = 1$

These curves suggest that, even if the first derivative is "inconclusive"[6, p. 160] for neutral fixed points, some interesting results can nevertheless be found in this case.

It is necessary to consider separately the cases where x^* is *positively neutral* (i.e. $f'(x^*) = 1$) and *negatively neutral* (i.e. $f'(x^*) = -1$). The fundamental reason for this distinction is the following : in the first case, the function f is increasing in a neighbourhood U of x^* so that $f(U \cap (-\infty, x^*)) \subset (-\infty, x^*)$ and $f(U \cap (x^*, +\infty)) \subset (x^*, +\infty)$; in the second case, f is decreasing in a neighbourhood U of x^* so that $f(U \cap (-\infty, x^*)) \subset (x^*, +\infty)$ and $f(U \cap (x^*, +\infty)) \subset (-\infty, x^*)$.

2 Positively neutral fixed points

Because f is increasing near its positively neutral fixed point x^*, the orbit under f with a seed x_0 in a neighbourhood of x^* is often a monotone sequence. This ascertainment leads to these definitions which are slight and appropriate changes of classical ones [3, 4, 6].

- x^* is *monotonously attracting from below* (for f) if there exists a positive real number ε such that, for every $x \in (x^* - \varepsilon, x^*)$, the orbit $\mathcal{O}(f; x)$ is strictly increasing and converges to x^*;
- x^* is *monotonously attracting from above* (for f) if there exists a positive real number ε such that, for every $x \in (x^*, x^* + \varepsilon)$, the orbit $\mathcal{O}(f; x)$ is strictly decreasing and converges to x^*;
- x^* is *monotonously repelling from below* (for f) if there exists a positive real number ε such that, for every $x \in (x^* - \varepsilon, x^*)$, there is a positive integer n such that $f^k(x) < f^{k-1}(x)$ for $k \in \{1, 2, \ldots, n\}$ and $f^n(x) \notin (x^* - \varepsilon, x^* + \varepsilon)$;
- x^* is *monotonously repelling from above* (for f) if there exists a positive real number ε such that, for every $x \in (x^*, x^* + \varepsilon)$, there is a positive integer n such that $f^{k-1}(x) < f^k(x)$ for $k \in \{1, 2, \ldots, n\}$ and $f^n(x) \notin (x^* - \varepsilon, x^* + \varepsilon)$;
- x^* is *monotonously stable* (for f) if it is monotonously attracting from below and from above (for f);
- x^* is *monotonously semistable from below* (for f) if it is monotonously attracting from below, but monotonously repelling from above (for f);
- x^* is *monotonously semistable from above* (for f) if it is monotonously attracting from above, but monotonously repelling from below (for f);
- x^* is *monotonously unstable* (for f) if it is monotonoulsy repelling from below and from above (for f).

It is clear that if x^* is monotonously stable (resp. monotonously unstable) for f, then x^* is also stable (resp. unstable) for f in the preceding sense, but the converse is not true.

If x^* is a positively neutral fixed point for f, then the point $P^* = (x^*, x^*)$ lies on the graph of f and the line with equation $y = x$ is tangent at P^* to this curve. Thus, usually, we have $f(x) < x$ or $f(x) > x$ for every point x belonging to $(x^* - \varepsilon, x^*)$ and to $(x^*, x^* + \varepsilon)$ for a suitable $\varepsilon > 0$. Now we prove that such a condition characterizes the monotonous stability (from below and from above) of x^* (for f).

Proposition 1 *Let x^* be a positively neutral fixed point for f.*

a) x^* is monotonously attracting from below for f if and only if there exists a positive real number ε such that $f(x) > x$ for every $x \in (x^* - \varepsilon, x^*)$;

b) x^* is monotonously attracting from above for f if and only if there exists a positive real number ε such that $f(x) < x$ for every $x \in (x^*, x^* + \varepsilon)$;

c) x^* is monotonously repelling from below for f if and only if there exists a positive real number ε such that $f(x) < x$ for every $x \in (x^* - \varepsilon, x^*)$;

d) x^* is monotonously repelling from above for f if and only if there exists a positive real number ε such that $f(x) > x$ for every $x \in (x^*, x^* + \varepsilon)$.

Proof. a) The condition is clearly necessary.

Conversely, we may suppose the existence of a real positive number ε such that f is strictly increasing on the open interval $I = (x^* - \varepsilon, x^*)$ and such that $f(x) > x \ \ \forall x \in I$. Therefore, for every $x \in I$, the orbit $\mathcal{O}(f; x)$ is increasing and bounded by x^* : so, $\mathcal{O}(f; x)$ converges to a limit $\bar{x} \in (x^* - \varepsilon, x^*]$ and $\bar{x}$ is a fixed point for f. Because of the assumption $f(x) > x$ when $x \in I$, $\ \bar{x} = x^*$.

b) The proof is similar to *a)*.

c) Necessity of the condition is a trivial consequence of the definitions.

Conversely, let ε be a positive real number such that f is increasing on $I = (x^* - \varepsilon, x^*)$ and $f(x) < x \ \ \forall x \in I$. For an arbitrary real number $x_0 \in I$, it is possible to construct the first elements of a decreasing sequence $x_0, x_1, x_2, \ldots$. If the orbit $\mathcal{O}(f; x_0)$ is well-defined (i.e. if $f^n(x_0) \in S$ for every integer n), then there are two possibilities : there exists an integer n such that $x^* > x_{n-1} > x^* - \varepsilon$ and $x_n \leq x^* - \varepsilon$, whence the conclusion, or all the elements of $\mathcal{O}(f; x_0)$ are greater than $x^* - \varepsilon$, whence $\mathcal{O}(f; x_0)$ converges to a limit $\bar{x}$ which belongs to $[x^* - \varepsilon, x^*) \setminus (x^* - \varepsilon, x^*)$: in these conditions, $\bar{x} = x^* - \varepsilon$ and $x^* - \varepsilon$ is a fixed point for f, with $f(x) < x \ \ \forall x \in (x^* - \varepsilon, x^*)$ and $f'(x^* - \varepsilon) \geq 0$, so we can adopt, for the fixed point $x^* - \varepsilon$, the reasoning made in the case *b)* for x^*. In summary, it is always sufficient to take $\frac{\varepsilon}{2}$, instead of ε, in the definition of a monotonously repelling fixed point from below in order to reach to the conclusion.

d) The proof is similar to the preceding one. ■

Proposition 2 *Let x^* be a positively neutral fixed point for f. Denote by n the smallest integer greater or equal to 2 such that $\frac{d^n}{dx^n} f(x^*) \neq 0$.*

a) If n is odd and $\frac{d^n}{dx^n} f(x^*) < 0$, then x^* is monotonously stable for f;

b) If n is odd and $\frac{d^n}{dx^n} f(x^*) > 0$, then x^* is monotonously unstable for f;

c) If n is even and $\frac{d^n}{dx^n} f(x^*) > 0$, then x^* is monotonously semistable from below for f;

d) If n is even and $\frac{d^n}{dx^n}f(x^*) < 0$, then x^* is monotonously semistable from above for f;

e) If f is strictly convex on an open interval I containing x^*, then x^* is monotonously semistable from below for f;

f) If f is strictly concave on an open interval I containing x^*, then x^* is monotonously semistable from above for f.

Proof. By Taylor's Theorem, we know that

$$f(x) = f(x^*) + \sum_{j=1}^{n} \frac{(x-x^*)^j}{j!} \frac{d^j}{dx^j} f(x^*) + R(x),$$

where $R(x) = \frac{(x-x^*)^{n+1}}{(n+1)!}\frac{d^{n+1}}{dx^{n+1}}f(c)$ for a suitable c between x and x^*.

Since $\lim\limits_{x\to x^*} R(x) = 0$, $f(x) - x$ and $\frac{(x-x^*)^n}{n!}\frac{d^n}{dx^n}f(x^*)$ have the same sign for every point x which is sufficiently close to (but different from) x^*.

When n is odd, $\frac{(x-x^*)^n}{n!}$ and $x - x^*$ have the same sign (for $x \neq x^*$). Therefore, if $\frac{d^n}{dx^n}f(x^*) < 0$, then $f(x) > x$ (resp. $f(x) < x$) for every x close to and less than (resp. greater than) x^*. In the same way, if $\frac{d^n}{dx^n}f(x^*) > 0$, then $f(x) < x$ (resp. $f(x) > x$) for every x close to and less than (resp. greater than) x^*.

When n is even, $(x - x^*)^n$ is positive for $x \neq x^*$. Thus, if $\frac{d^n}{dx^n}f(x^*) > 0$ (resp. $\frac{d^n}{dx^n}f(x^*) < 0$), then $f(x) > x$ (resp. $f(x) < x$) for every x in a neighbourhood of x^* (with $x \neq x^*$).

Proposition 1 gives the conclusion for *a), b), c)* and *d)*.

If f is strictly convex on I, then, for every $x \in I \setminus \{x^*\}$:

$$f(x) > f(x^*) + (x - x^*)f'(x^*).$$

Since $f'(x^*) = 1$ and $f(x^*) = x^*$, we also have

$$f(x) > x \quad \forall x \in I \setminus \{x^*\},$$

and proposition 1 can also be applied.

The reasoning is similar for a strictly concave function. ∎

Examples 1 Let $f(x) = x + \alpha x^p$, with $\alpha \in \mathbb{R}\setminus\{0\}$, $p \in \mathbb{N}$ and $p \geq 2$. The point $x^* = 0$ is a positively neutral fixed point such that $\frac{d^j}{dx^j}f(x^*) = 0$ for $2 \leq j < p$ and $\frac{d^p}{dx^p}f(x^*) = \alpha p!$. Thus, if p is odd and $\alpha < 0$ (resp. $\alpha > 0$), then x^* is monotonously stable (resp. unstable) for f; if p is even and $\alpha < 0$ (resp. $\alpha > 0$), then x^* is monotonously semistable from above (resp. below) for f.

3 Negatively neutral fixed points

When $f'(x^*) = -1$, the situation is fundamentally different from the preceding case because the orbits whose seed x_0 is near x^* cannot be monotone, but often alternate around x^* and consist of two monotone subsequences $\mathcal{O}'(f; x_0) = (x_0, x_2, x_4, x_6, \ldots)$ and $\mathcal{O}''(f; x_0) = (x_1, x_3, x_5, \ldots)$, where $x_n = f^n(x_0)$ for every integer n.

So, we introduce this new definition about a fixed point x^* for f : x^* is *alternatively monotonously stable* for f if there exists a positive real number ε such that, for every $x_0 \in (x^* - \varepsilon, x^* + \varepsilon)$, $\mathcal{O}'(f; x_0)$ and $\mathcal{O}''(f; x_0)$ are strictly monotone sequences, one being increasing and the other decreasing, which both converge to x^*.

Note that if a fixed point x^* is alternatively monotonously stable for f, then it is also stable for f, but the converse is not true.

Let x^* be a negatively neutral fixed point for f. It is clear that $f^2(x^*) = x^*$ and $\frac{d}{dx} f^2(x^*) = 1$. Hence, the line with equation $y = x$ is tangent to the graph of f^2 at the point $P^* = (x^*, x^*)$. So, we generally have $f^2(x) > x$ or $f^2(x) < x$ for every x belonging to $(x^* - \varepsilon, x^*)$ and to $(x^*, x^* + \varepsilon)$ for a suitable $\varepsilon > 0$. Precisely, we shall see that such a condition characterizes the alternatively monotonous stability of x^*.

Proposition 3 *Let x^* be a negatively neutral fixed point for f.*
The following propositions are equivalent :

a) x^* is alternatively monotonously stable for f;

b) x^* is monotonously stable for f^2;

c) there exists a positive real number ε such that $f^2(x) > x \;\; \forall x \in (x^* - \varepsilon, x^*)$ and $f^2(x) < x \;\; \forall x \in (x^*, x^* + \varepsilon)$.

Proof. The assertions *b)* and *c)* are equivalent by virtue of proposition 1.

Suppose that there exists $\varepsilon > 0$ such that, for every $x_0 \in (x^* - \varepsilon, x^* + \varepsilon)$, the two subsequences $\mathcal{O}'(f; x_0)$ and $\mathcal{O}''(f; x_0)$ are monotone and converge to x^*; clearly $\mathcal{O}'(f; x_0)$ is increasing and $\mathcal{O}''(f; x_0)$ is decreasing. Of course, x^* is monotonously stable for f^2 because $f^2(x^*) = x^*$, while $\mathcal{O}(f^2; x_0)$ coïncides with $\mathcal{O}'(f; x_0)$.

Conversely, if x^* is monotonously stable for f^2, there exists an open interval I containing x^* such that $f'(x) < 0$ for any $x \in I$ and $\mathcal{O}(f^2; x_0)$ converges to x^* when x_0 is an arbitrary element of I.

Moreover, by proposition 1, we have $f^2(x) > x$ (resp. $f^2(x) < x$) when x is close to and less (resp. greater) than x^*, so one of the subsequences $\mathcal{O}'(f; x_0)$ and $\mathcal{O}''(f; x_0)$ is increasing, and the other decreasing.

On the other hand, because f is continuous, it is possible to find a real $\varepsilon > 0$ such that $x_1 = f(x_0)$ belongs to I for every $x_0 \in J = (x^* - \varepsilon, x^* + \varepsilon)$. Let x_0 be any point of $I \cap J$. The orbit $\mathcal{O}(f^2; x_1)$ converges to x^*. Therefore, $\mathcal{O}(f; x_0)$ also converges to x^*, since this sequence consists of elements of $\mathcal{O}(f^2; x_0)$ and $\mathcal{O}(f^2; x_1)$. ■

As a corollary of this last result, a statement similar to proposition 2 can be given in this case by using the function f^2 instead of f. Nevertheless, it is convenient to work with the given function f itself. For that, the derivatives of f will be replaced by other more complicated notions as the *schwarzian derivative* of f [1],i.e.

$$D_s f(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left(\frac{f''(x)}{f'(x)} \right)^2$$

and Bell's polynomials defined by

$$\mathbb{B}_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum \frac{n!}{c_1! c_2! \ldots (1!)^{c_1} (2!)^{c_2} \ldots} x_1^{c_1} x_2^{c_2} \ldots$$

where the summation goes for every non-negative integers $c_1, c_2, \ldots$ such that $c_1 + 2c_2 + 3c_3 + \ldots = n$ and $c_1 + c_2 + c_3 + \ldots = k$ [2, pp. 144-145]; moreover, we shall denote

$$b_n = \sum_{k=1}^{n} a_k \mathbb{B}_{n,k}(a_1, a_2, \ldots, a_{n-k+1})$$

where, for each k, $a_k = \frac{d^k}{dx^k} f(x^*)$.

Proposition 4 *Let x^* be a negatively neutral fixed point for f.*

a) If $D_s f(x^*) < 0$, then x^* is alternatively monotonously stable for f;

b) If $D_s f(x^*) > 0$, then x^* is unstable for f : more precisely, x^* is monotonously unstable for f^2;

c) When $D_s f(x^*) = 0$, let n be the smallest integer greater than 3 such that $b_n \neq 0$; n is odd; x^* is alternatively monotonously stable for f when $b_n < 0$; x^* is unstable for f and monotonously unstable for f^2 when $b_n > 0$.

Proof. It is clear that

$$\frac{d}{dx} f^2(x^*) = 1, \quad \frac{d^2}{dx^2} f^2(x^*) = 0,$$

$$D_s f(x^*) = \frac{1}{2} \frac{d^3}{dx^3} f^2(x^*) \quad \text{and} \quad b_n = \frac{d^n}{dx^n} f^2(x^*)$$

due to the formula of Faa di Bueno [2, p. 148].

Now, we prove by contradiction that n is odd. Suppose that n is even. When $b_n > 0$ (resp. $b_n < 0$), x^* is monotonously semistable from below (resp. from above) for f^2 by proposition 2; this is impossible because if a sequence $(x_0, f^2(x_0) = x_2,\ f^2(x_2) = x_4, \ldots)$ converges to x^*, then, by continuity of f, $(f(x_0) = x_1,\ f(x_2) = f^2(x_1) = x_3,\ f(x_4) = f^2(x_3) = x_5, \ldots)$ is also converging to $f(x^*) = x^*$.

Therefore, propositions 2 and 3 give the conclusions. ■

Remark. When the schwarzian derivative $D_s f(x^*)$ is equal to 0 for a negatively neutral fixed point x^* for f, it is convenient to successively compute the reals $b_5, b_7, b_9, \ldots$ until obtaining a non-zero number.

Elementary calculations give, for such a point x^* :

$$b_5 = -2\frac{d^5}{dx^5} f(x^*) - 15\frac{d^4}{dx^4} f(x^*) \frac{d^2}{dx^2} f(x^*) + 30 \left[\frac{d^2}{dx^2} f(x^*)\right]^4.$$

Examples 2 Here are some elementary and varied examples of functions for which $x^* = 0$ is a negatively neutral fixed point.

- $f(x) = -\sin x$ and $g(x) = -\mathrm{arctg} x$: 0 is alternatively monotonously stable for f and for g, since $D_s f(0) = -1$ and $D_s g(0) = -2$.

- $f(x) = -\arcsin x$: 0 is unstable for f and monotonously unstable for f^2 because $D_s f(0) = 1$.

- $f(x) = -x + \alpha x^2 - \beta x^3$, with $\alpha \in \mathbb{R} \setminus \{0\}$ and $\beta \in \mathbb{R}$:

  $D_s f(0) = 6(\beta - \alpha^2)$, $b_4 = 24\alpha(\alpha^2 - \beta)$ and $b_5 = 480\alpha^4$. Therefore, if $\beta < \alpha^2$, then 0 is alternatively monotonously stable for f; if $\beta \geq \alpha^2$, then 0 is unstable for f and monotonously unstable for f^2.

- $f(x) = -x + \alpha x^p$, where p is an integer greater than 3 and α is an arbitrary real number which is different from $0 : D_s f(0) = 0$.

  If p is odd, then $n = p$ and $b_n = -2\alpha p! :$ thus, 0 is alternatively monotonously stable for f when $\alpha > 0$; 0 is unstable for f and monotonously unstable for f^2 when $\alpha < 0$.

  If p is even, then $n = 2p - 1$ and $b_n = -\alpha^2 p(2p-1)! :$ 0 is alternatively monotonously stable for f.

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