The Fundamental Theorems of Curves and Hypersurfaces in Centro-affine Geometry

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Abstract

The motivation of this paper is to find formulations of the local rigidity theorems for centro-affine curves and hypersurfaces that are amenable to direct application to problems in control theory. Élie Cartan's method of moving frames develops the solutions in a natural way. The case of centro-affine curves has previously appeared only for certain low dimensions. This is the first time a theory for curves in arbitrary dimensions has appeared.

Preliminaries

The method of moving frames is a technique that is well suited to the study of submanifolds of a homogeneous space. In [Ca], Cartan shows that when a Lie group acts transitively and effectively on a manifold, one can construct a bundle of frames over the manifold. Cartan develops this theory to study two submanifold problems: the problem of contact and the problem of equivalence. The first problem involves determining the order of contact two submanifolds have at a point. The second problem involves determining when there exists an element of the given Lie group that translates one submanifold onto another. Cartan devotes most of his book to specific geometric examples of these problems. Another source for examples is a

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book by Favard [Fa]. A more recent description of the method of moving frames can be found in [Gn] and [Gs]. Green's paper [Gn] carefully points out subtle features in the theory and includes many examples.

Cartan proves two lemmas in [Ca] that are central to many applications of moving frames. We will state them in a form adapted to our applications.

Lemma 1. *Let N be a smooth connected manifold, let G be a Lie group and let π be the right-invariant Maurer-Cartan form on G. Suppose that we are given two smooth mappings X and X' from N to G, then there exists a fixed element $g \in G$ such that $X'(p) = X(p) \cdot g$ for every $p \in N$ if and only if $X'^*(\pi) = X^*(\pi)$.*

Definition. Let G be a Lie group, N a smooth manifold and ϖ a 1-form on N taking values in the Lie algebra of G. For each $p \in N$, we define *the rank* of ϖ at p to be the rank of the linear transformation $\varpi : T_pN \to T_eG$.

Lemma 2. *Let G be a Lie group and let π be the right-invariant Maurer-Cartan form on G. Let N be a smooth manifold and let ϖ be a 1-form on N taking values in the Lie algebra of G. If the exterior derivative of ϖ satisfies the structure equation*

$$d\varpi = \varpi \wedge \varpi$$

then at every point $p \in N$ there exists a neighborhood $U \subset N$ about p and a unique smooth mapping $X : U \to G$ such that $X(p)$ is the identity element in G and $\varpi = X^*(\pi)$. Moreover, X is an immersion if and only if $\varpi|_U$ has constant rank equal to the dimension of N.

Speaking informally, lemma 1 says that two maps into a Lie group differ by a right multiplication if and only if their respective pullbacks of the right-invariant Maurer-Cartan form agree. Lemma 2 says that a Lie algebra valued 1-form on Nsatisfying the necessary structure equation is always (locally) the pullback of the right-invariant Maurer-Cartan form. For our applications, G will be the general linear group $\mathrm{GL}(n+1,\mathbf{R})$ and the right-invariant Maurer-Cartan form will be $dSS^{-1} = (\omega_i{}^j)$. The proofs of the lemmas are straightforward. The *if* part of lemma 1 follows from the fact that a function is constant when its derivative is identically zero. One can prove lemma 2 using the technique of the graph, which is described in Warner [Wa].

We will also need the following standard result:

Lemma 3 (Cartan's lemma). *Let $\{\omega^1, \ldots, \omega^n\}$ be a set of pointwise linearly independent 1-forms. The 1-forms $\{\theta_1, \ldots, \theta_n\}$ satisfy the relation*

$$\omega^1 \wedge \theta_1 + \cdots + \omega^n \wedge \theta_n = 0$$

if and only if

$$\theta_\alpha = h_{\alpha 1}\, \omega^1 + \cdots + h_{\alpha n}\, \omega^n, \quad 1 \le \alpha \le n,$$

where the n^2 functions $h_{\alpha\beta}$ satisfy the symmetry relation $h_{\beta\alpha} = h_{\alpha\beta}$.

The notion of *semi-basic* differential forms will be important in the following sections. Let $\rho : B \to M$ be a surjective submersion. The kernel of $\rho_* : TB \to TM$ determines the subbundle $V \subset TB$ of vertical tangent vectors. Its annihilator $V^\perp \subset T^*B$ generates a family of subbundles $\bigwedge^p \left(V^\perp\right) \subset \bigwedge^p \left(T^*B\right)$.

Definition. Let θ be a differential p-form on B. We say that θ is *semi-basic* for $\rho : B \to M$ if θ is a section of $\bigwedge^p \left(V^\perp\right)$.

To get a feel for semi-basic forms, pick a system of local coordinates $(x^1, \ldots, x^m)$ on M. Lift these functions to B and complete them to a system of local coordinates $(x^1, \ldots, x^m; y^1, \ldots, y^\nu)$ on B. In this system of coordinates, a semi-basic 1-form is represented by $\theta = \sum_i a_i(x, y)\, dx^i$ and, using multi-index notation where $I = \{i_1, \ldots, i_p\}$, a semi-basic p-form is represented by $\theta = \sum_I a_I(x, y)\, dx^I$. Notice that the pull-back $\rho^*\theta$ of any p-form θ on M is semi-basic, but not every semi-basic p-form is the pull-back of a p-form on M. This will be true exactly when the coefficient functions $a_I(x, y)$ are functions of x alone. It is easy to show that a differential form θ on B is a pullback if and only if both θ and $d\theta$ are semi-basic.

1 The centro-affine frame bundle on punctured R^{n+1}

We define *punctured* $(n+1)$*-space* to be the set of nonzero row vectors in $\mathbf{R}^{n+1}$, and we will denote it by $\mathbf{R}_0^{n+1}$. Since the group $\mathrm{GL}(n+1,\mathbf{R})$ acts transitively and effectively on $\mathbf{R}_0^{n+1}$, we can use the method of moving frames to study the $\mathrm{GL}(n+1,\mathbf{R})$-invariants of curves and hypersurfaces in punctured space. A *centro-affine frame* on $\mathbf{R}_0^{n+1}$ will consist of $n+1$ linearly independent vectors in $\mathbf{R}_0^{n+1}$, $(\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n)$. We define the *centro-affine frame bundle*, $\mathcal{F}$, to be the set of all frames on $\mathbf{R}_0^{n+1}$. We may also think of the frame $(\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n)$ as the $\mathrm{GL}(n+1,\mathbf{R})$ matrix $({e_i}^j), 0 \le i, j \le n$, where $\mathbf{e}_i$ is the row vector $\mathbf{e}_i = ({e_i}^0, {e_i}^1, \ldots, {e_i}^n)$. Now $\mathrm{GL}(n+1,\mathbf{R})$ is an open subset of the $(n+1)^2$-dimensional vector space $\mathrm{L}(n+1,\mathbf{R})$ of all $(n+1) \times (n+1)$ matrices. The projections $\mathbf{e}_i : \mathrm{L}(n+1,\mathbf{R}) \to \mathbf{R}^{n+1}$ onto the i^{th} row of the matrices are linear maps, and therefore their restrictions to any open subset are differentiable. In particular, we have $n+1$ differentiable mappings $\mathbf{e}_i : \mathcal{F} \to \mathbf{R}_0^{n+1}$. Expressing the derivatives of these mappings relative to themselves gives the structure equations on $\mathcal{F}$:

$$d\mathbf{e}_i = \sum_{j=0}^{n} {\omega_i}^j \, \mathbf{e}_j \qquad 0 \le i \le n \tag{1.1}$$

$$d{\omega_i}^j = \sum_{k=0}^{n} {\omega_i}^k \wedge {\omega_k}^j \qquad 0 \le i, j \le n. \tag{1.2}$$

In these equations we view $\mathbf{e}_i$ as a function from $\mathcal{F}$ to $\mathbf{R}_0^{n+1}$. The structure equations merely express the derivatives of these maps in terms of the given frame. The set of $(n+1)^2$ 1-forms $\{\, {\omega_i}^j \mid 0 \le i, j \le n \,\}$ forms a basis for the 1-forms on $\mathcal{F}$.

2 Centro-affine hypersurfaces in $\mathbf{R}^{n+1}$

We begin by illustrating the method of moving frames for the case of hypersurfaces in $\mathbf{R}_0^{n+1}$. The centro-affine theory of hypersurfaces in $\mathbf{R}_0^{n+1}$ is similar to the equi-affine theory of hypersurfaces in $\mathbf{R}^n$. This was worked out in our setting in [Ga], and makes a useful comparison to this section. Let $X : N^n \to \mathbf{R}_0^{n+1}$ be a smooth immersion of the n-dimensional manifold N, and form the pullback bundle, $\mathcal{F}_X^{(0)}$, of the centro-affine frame bundle

$$\begin{array}{ccc} \mathcal{F}_X^{(0)} & \longrightarrow & \mathcal{F} \\ \Big\downarrow & & \Big\downarrow \mathbf{e}_0 \\ N & \underset{X}{\longrightarrow} & \mathbf{R}_0^{n+1}. \end{array}$$

By construction, $\mathcal{F}_X^{(0)} = \{\,(u, \mathbf{e}_0, \dots, \mathbf{e}_n) \in N \times \mathcal{F} \mid X(u) = \mathbf{e}_0\,\}$, and we think of $\mathcal{F}_X^{(0)}$ as the set of frames $(\mathbf{e}_0, \dots, \mathbf{e}_n)$ whose zeroth leg is the position vector of N. From equation (1.1) we see that

$$dX = d\mathbf{e}_0 = {\omega_0}^0\,\mathbf{e}_0 + {\omega_0}^1\,\mathbf{e}_1 + \cdots + {\omega_0}^n\,\mathbf{e}_n, \tag{2.1}$$

and thus ${\omega_0}^0, \dots, {\omega_0}^n$ are *semi-basic* 1-forms for $\mathcal{F}_X^{(0)} \to N$. The natural restriction to frames with $\mathbf{e}_1, \dots, \mathbf{e}_n$ tangent to the image of the surface $X(N)$ utilizes the first order information. However, there can be an obstruction to this adaptation. Since the vectors $\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_n$ must be linearly independent, it must be the case that the position vector, $X = \mathbf{e}_0$, does not lie in its tangent space, or equivalently, that the tangent space does not pass through the origin. From equation (2.1) we see that this is equivalent to the condition ${\omega_0}^1 \wedge \cdots \wedge {\omega_0}^n \neq 0$. We will assume that we are in this case. Let $\mathcal{F}_X^{(1)}$ denote the set of frames in $\mathcal{F}_X^{(0)}$ such that $\mathbf{e}_1, \dots, \mathbf{e}_n$ are tangent to $X(N)$. Restricting to $\mathcal{F}_X^{(1)}$, equation (2.1) implies that ${\omega_0}^0 = 0$, and since

$$d\mathbf{e}_0 = \sum_{\alpha=1}^n {\omega_0}^\alpha \mathbf{e}_\alpha,$$

$\{{\omega_0}^1, \dots, {\omega_0}^n\}$ is a basis for the semi-basic 1-forms. (In this section we use the following naming convention for index ranges: greek letters span the tangential range $1 \le \alpha, \beta, \ldots \le n$ and latin letters span the full range $0 \le i, j, \ldots \le n$.)

Differentiation of ${\omega_0}^0$ yields

$$0 = d{\omega_0}^0 = \sum_{\alpha=1}^n {\omega_0}^\alpha \wedge {\omega_\alpha}^0, \tag{2.2}$$

so using Cartan's lemma we have by (2.2),

$${\omega_\alpha}^0 = \sum_{\beta=1}^n h_{\alpha\beta}\,{\omega_0}^\beta \quad \text{with} \quad h_{\alpha\beta} = h_{\beta\alpha}, \qquad 1 \le \alpha \le n.$$

Differentiation of this last formula together with equations (1.2) yields

$$\sum_{\beta,\gamma=1}^n (dh_{\alpha\gamma} - {\omega_\alpha}^\beta h_{\beta\gamma} - h_{\alpha\beta}{\omega_\gamma}^\beta) \wedge {\omega_0}^\gamma = 0, \qquad 1 \le \alpha \le n. \tag{2.3}$$

Letting $H = (h_{\alpha\beta}), \Omega = (\omega_\alpha{}^\beta)$ we may write this last equation in matrix form as

$$dH - \Omega H - H^t\Omega \equiv 0 \qquad \text{mod } (\omega_0{}^1, \ldots, \omega_0{}^n), \tag{2.4}$$

which is the infinitesimal action induced by conjugation. (See pages 40–43 of [Ga1] for a discussion of group actions and how to identify them from their infinitesimal actions.) Equations (2.3) imply that H is the matrix of the quadratic form

$$II_{CA} = \sum_{\alpha,\beta=1}^{n} h_{\alpha\beta}\, \omega_0{}^\alpha\, \omega_0{}^\beta,$$

which by construction is well defined independent of frame and hence drops to the hypersurface N. This form is called the *centro-affine metric*, and is the centro-affine analog to the Blaschke metric in equi-affine geometry [Bl].

If we assume the hypersurface is convex, then H is negative definite and we can use the conjugation to normalize $H = -I$. Then (2.4) becomes

$$\Delta = \frac{1}{2}(\Omega + {}^t\Omega) \equiv 0 \ \text{ mod } (\omega_0{}^1, \ldots, \omega_0{}^n)$$

and utilizing (2.3) and Cartan's lemma we see that

$$\Delta_\alpha{}^\beta = \sum S_\alpha{}^\beta{}_\gamma\, \omega_0{}^\gamma$$

is a tensor symmetric in all three indices. The cubic form

$$P_{CA} = \sum_{\alpha,\beta,\gamma=1}^{n} S_{\alpha\beta\gamma}\, \omega_0{}^\alpha\, \omega_0{}^\beta\, \omega_0{}^\gamma,$$

where $S_{\alpha\beta\gamma} = \sum_\sigma S_\alpha{}^\sigma{}_\gamma\, h_{\sigma\beta}$, is called the *centro-affine Pick form.*

Let $\Phi = \frac{1}{2}(\Omega - {}^t\Omega)$ so that ${}^t\Phi = -\Phi$, then

$$\Omega = \Phi + \Delta.$$

If we differentiate the structure equations

$$d\begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e} \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -{}^t\omega & \Omega \end{pmatrix} \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e} \end{pmatrix} \tag{2.5}$$

we get

$$d\begin{pmatrix} 0 & \omega \\ -{}^t\omega & \Omega \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -{}^t\omega & \Omega \end{pmatrix} \wedge \begin{pmatrix} 0 & \omega \\ -{}^t\omega & \Omega \end{pmatrix} \tag{2.6}$$

and in particular the upper right blocks of (2.6) give

$$d\omega = \omega \wedge \Omega = \omega \wedge \Phi + \omega \wedge \Delta = \omega \wedge \Phi, \tag{2.7}$$

since $\omega \wedge \Delta = 0$ by the symmetry of the indices in $S_\alpha{}^\beta{}_\gamma$.

If $\bar{\Phi}$ were any other matrix of 1-forms satisfying

$$d\omega = \omega \wedge \bar{\Phi} \ \text{ and } \ {}^t\bar{\Phi} = -\bar{\Phi}, \tag{2.8}$$

then subtraction of (2.8) from (2.7) gives

$$\omega \wedge (\Phi - \bar{\Phi}) = 0,$$

and Cartan's lemma with the skew symmetry of $\Phi - \bar{\Phi}$ implies that $\Phi - \bar{\Phi}$ is a three-index tensor symmetric in one pair of indices and skew symmetric in the other pair and hence is identically zero. Thus $\Phi = \bar{\Phi}$, and (2.7) defines Φ uniquely. The forms in Φ depend not only on coordinates on the hypersurface, but also on group coordinates in $\mathrm{SO}(n,\mathbf{R})$ resulting from the frames satisfying the normalization $H = -I$. In fact, this normalization simply means that ${\omega_0}^1, \ldots, {\omega_0}^n$ diagonalizes the centro-affine metric, and therefore equation (2.7) shows that Φ is the metric connection determined by II_{CA}.

In addition to the metric connection of the centro-affine metric, we also have the connection induced by the centro-affine normal, $\mathbf{e}_0$. Equation (2.5) shows this connection is represented by Ω. Equation (2.6) implies that

$$d\Omega - \Omega \wedge \Omega = -{}^t\omega \wedge \omega,$$

which means that Ω is projectively flat. Since $\Omega = \Phi + \Delta$, we see that Δ is the difference tensor of the two connections. Thus, the centro-affine Pick form is determined by the difference tensor of the connections and the centro-affine metric.

It is worth pointing out that the analysis is essentially the same even if the hypersurface is not convex. The only condition we really need is that II_{CA} is non-degenerate, i.e. $\det H \neq 0$. If H is indefinite then Φ will take its values in the Lie algebra of some $\mathrm{SO}(p,q)$ determined by the signature of II_{CA}.

We may now state the fundamental theorems for centro-affine hypersurfaces.

Theorem 1. *Let N be a smooth connected n-dimensional manifold and let X and X' be two smooth immersions of N into $\mathbf{R}_0^{n+1}$ with respective non-degenerate centro-affine metrics II_{CA}, II'_{CA} and respective centro-affine Pick forms P_{CA}, P'_{CA}. Then $X(N)$ and $X'(N)$ are related by a centro-affine motion if and only if $II_{CA} = II'_{CA}$ and $P_{CA} = P'_{CA}$.*

Theorem 2. *Let N be a smooth connected n-dimensional manifold. Let II_{CA} be a smooth non-degenerate quadratic form on N and let $\nabla^\bullet$ be its metric connection. Let P_{CA} be a smooth cubic form on N and let Δ be the tensor of type $(1,2)$ characterized by*

$$II_{CA}(\Delta(v_1, v_2), v_3) = P_{CA}(v_1, v_2, v_3), \textit{for all tangent vector fields } v_1, v_2 \textit{ and } v_3 \textit{ on } N.$$

If the connection $\nabla = \nabla^\bullet + \Delta$ is projectively flat then for each point in N there is an open neighborhood U containing that point and a smooth immersion $X : U \to \mathbf{R}_0^{n+1}$ such that the restriction of II_{CA} to U is the centro-affine metric of X and the restriction of P_{CA} to U is the centro-affine Pick form of X.

The proofs of these theorems are direct applications of lemmas 1 and 2. We will sketch the proofs assuming that II_{CA} and II'_{CA} are negative definite. The general non-singular case is nearly identical.

For theorem 1, the $\mathrm{GL}(n+1,\mathbf{R})$ invariance of the structure equations clearly implies that the respective centro-affine metrics and Pick forms must agree if the surfaces are congruent by a centro-affine motion. It is sufficient to prove the *if* part on connected neighborhoods of N which are small enough to have a basis of orthonormal vector fields $v_1, \ldots, v_n$. We may associate a centro-affine frame to each immersion. The frames associated to X and X' are respectively given by

$$\begin{aligned}&\mathbf{e}_0 = X, \mathbf{e}_1 = X_*(v_1), \ldots, \mathbf{e}_n = X_*(v_n)\\ &\mathbf{e}'_0 = X', \mathbf{e}'_1 = X'_*(v_1), \ldots, \mathbf{e}'_n = X'_*(v_n).\end{aligned}$$

From this construction we see immediately that ${\omega'_0}^{\alpha} = {\omega_0}^{\alpha}$, $1 \leq \alpha \leq n$, where the 1-forms are defined by the equations

$$\begin{aligned}d\mathbf{e}_0 &= \sum_{\alpha=1}^{n} {\omega_0}^{\alpha}\, \mathbf{e}_\alpha\\ d\mathbf{e}'_0 &= \sum_{\alpha=1}^{n} {\omega'_0}^{\alpha}\, \mathbf{e}'_\alpha.\end{aligned}$$

Since the basis $v_1, \ldots, v_n$ is orthonormal for both centro-affine metrics, the dual 1-forms ${\omega_0}^1 \ldots, {\omega_0}^n$ diagonalize both metrics. This implies that the structure equations for the two frame fields are

$$d\begin{pmatrix}\mathbf{e}_0\\ \mathbf{e}\end{pmatrix} = \begin{pmatrix}0 & \omega\\ -{}^t\omega & \Omega\end{pmatrix}\begin{pmatrix}\mathbf{e}_0\\ \mathbf{e}\end{pmatrix} \qquad \text{and} \qquad d\begin{pmatrix}\mathbf{e}'_0\\ \mathbf{e}'\end{pmatrix} = \begin{pmatrix}0 & \omega\\ -{}^t\omega & \Omega'\end{pmatrix}\begin{pmatrix}\mathbf{e}'_0\\ \mathbf{e}'\end{pmatrix}, \tag{2.9}$$

with $\Omega = \Phi + \Delta$ and $\Omega' = \Phi' + \Delta'$. Since Φ and Φ' are the respective centro-affine metric connections relative to the same basis, they must be equal. Further, since Δ and Δ' are determined by the respective Pick forms, they must be equal and we have that $\Omega' = \Omega$. With $\Omega' = \Omega$, equations (2.9) and lemma 1 imply that there is a fixed $\mathrm{GL}(n+1,\mathbf{R})$ matrix A transforming one frame to the other. In particular, we have $\mathbf{e}'_0 = \mathbf{e}_0 \cdot A$ which is equivalent to $X' = X \cdot A$. ∎

To prove theorem 2 we begin as in theorem 1 by picking an orthonormal frame field $v_1, \ldots, v_n$ with dual coframe $\omega = ({\omega_0}^1, \ldots {\omega_0}^n)$. Let Φ be the matrix of 1-forms representing the connection $\nabla^\bullet$, then Φ is determined by the equations

$$d\omega = \omega \wedge \Phi, \qquad {}^t\Phi = -\Phi. \tag{2.10}$$

In this coframe the tensor Δ, which is determined by the cubic Pick form, can be represented by a symmetric matrix of 1-forms ${\Delta_\alpha}^\beta = \sum {S_\alpha}^\beta{}_\gamma\, {\omega_0}^\gamma$. The matrix of 1-forms representing the connection $\nabla = \nabla^\bullet + \Delta$ is given by $\Omega = \Phi + \Delta$. The symmetries of the cubic form imply that $\omega \wedge \Delta = 0$, and from (2.10) this gives

$$d\omega = \omega \wedge (\Phi + \Delta) = \omega \wedge \Omega. \tag{2.11}$$

The condition that ∇ is projectively flat yields the equation

$$d\Omega = \Omega \wedge \Omega - {}^t\omega \wedge \omega. \tag{2.12}$$

Equations (2.11) and (2.12) show that the $(n+1)\times(n+1)$ matrix of 1-forms

$$\begin{pmatrix} 0 & \omega \\ -{}^t\omega & \Omega \end{pmatrix} \tag{2.13}$$

satisfies the structure equations

$$d\begin{pmatrix} 0 & \omega \\ -{}^t\omega & \Omega \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -{}^t\omega & \Omega \end{pmatrix} \wedge \begin{pmatrix} 0 & \omega \\ -{}^t\omega & \Omega \end{pmatrix}$$

Lemma 2 then implies the local existence of a map from N into $\mathrm{GL}(n+1,\mathbf{R})$ for which the pullback of the Maurer-Cartan form on $\mathrm{GL}(n+1,\mathbf{R})$ equals (2.13). If we let X be the first row of this matrix valued function, then standard arguments show that the matrix valued function gives an adapted centro-affine frame and therefore X induces the desired centro-affine metric and centro-affine Pick forms. ∎

A proof in the context of relative differential geometry and vector fields can be found in [SS-SV, §4.12.3]. The current formulation is better suited for applications to control theory. One such application can be found in [GW].

Notice that the vanishing of the Pick form is equivalent to the vanishing of the difference tensor Δ. In this case, we have that $\Omega = \Phi$ and the matrix of 1-forms (2.13) is skew symmetric, so its values lie in the Lie algebra of $\mathrm{O}(n+1)$. Lemma 2 then implies that the frame field itself is $\mathrm{O}(n+1)$ valued. This $\mathrm{O}(n+1)$ valued frame allows us to define an inner product on $\mathbf{R}^n$, and relative to this inner product the first row of the frame lies on the unit sphere. Therefore the image of the immersion lies on a convex quadric centered at the origin. The converse is also clear. This argument easily generalizes to the non-convex case as well, which gives the following corollary.

Corollary 1. *Let N and X be as in theorem 1. Then the image of X lies on a central non-degenerate quadric if and only if P_{CA} vanishes.*

3 Centro-affine curves in R^{n+1}

Our goal in this section is to develop for curves in $\mathbf{R}_0^{n+1}$ the centro-affine analog of the Frenet apparatus. The method of moving frames provides an iterative procedure to use. This section gives a detailed discussion of the procedure, leading finally to a centro-affine invariant framing of the curve. As one might expect, we will uncover an invariant arc-length parameter and n invariant curvature functions. The surprising result will be the order of these curvatures.

Let $I \subset \mathbf{R}$ be an open interval in $\mathbf{R}$ and let $\mathbf{x} : I \to \mathbf{R}_0^{n+1}$ be a smooth immersed curve. As in section 2, we form the zeroth order frame bundle, $\mathcal{F}_{\mathbf{x}}^{(0)} = \{\,(u, \mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n) \in I \times \mathcal{F} \mid \mathbf{e}_0 = \mathbf{x}(u)\,\}$:

$$\begin{tikzcd} \mathcal{F}_{\mathbf{x}}^{(0)} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, "\mathbf{e}_0"] \\ I \arrow[r, "\mathbf{x}"'] & \mathbf{R}_0^{n+1}. \end{tikzcd}$$

Using equation (1.1) we have

$$\mathbf{x}'(u)\,du = d\mathbf{x} = d\mathbf{e}_0 = {\omega_0}^0\,\mathbf{e}_0 + {\omega_0}^1\,\mathbf{e}_1 + \cdots + {\omega_0}^n\mathbf{e}_n,$$

which implies that the 1-forms ${\omega_0}^0, {\omega_0}^1, \ldots, {\omega_0}^n$ are semi-basic and thus are multiples of du. The natural restriction to frames with $\mathbf{e}_1$ pointing in the same direction as $\mathbf{x}'(u)$ utilizes the first order information, but this is not always possible. Since $\mathbf{e}_0$ and $\mathbf{e}_1$ must be linearly independent, and since $\mathbf{e}_0 = \mathbf{x}(u)$, it will only be possible to choose $\mathbf{e}_1$ to be parallel to $\mathbf{x}'(u)$ if $\mathbf{x}(u)$ and $\mathbf{x}'(u)$ are linearly independent. Since

$$\mathbf{x}(u) \wedge \mathbf{x}'(u)\,du = \mathbf{e}_0 \wedge d\mathbf{e}_0 = {\omega_0}^1\,\mathbf{e}_0 \wedge \mathbf{e}_1 + \cdots + {\omega_0}^n\,\mathbf{e}_0 \wedge \mathbf{e}_n,$$

we see that $\mathbf{x}(u)$ and $\mathbf{x}'(u)$ are linearly dependent if and only if $0 = {\omega_0}^1 = \cdots = {\omega_0}^n$. In geometric terms linear dependence means that the tangent line to $\mathbf{x}(u)$ pass through the origin. Suppose now that $\mathbf{e}_0 \wedge d\mathbf{e}_0 = 0$ for every $u \in I$. Then the ray through $\mathbf{e}_0$ is constant and therefore the curve $\mathbf{x}(u)$ lies on a fixed ray.

We will now assume that $\mathbf{e}_0 \wedge d\mathbf{e}_0 \neq 0$ for all $u \in I$. We define the first order frames to be

$$\mathcal{F}_{\mathbf{x}}^{(1)} = \{\,(u, \mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n) \in \mathcal{F}_{\mathbf{x}}^{(0)} \mid \mathbf{e}_1 \text{ points in the same direction as } \mathbf{x}'(u)\,\}.$$

For the first order frames we have that $\mathbf{x}'(u)\,du = d\mathbf{e}_0 = {\omega_0}^1\,\mathbf{e}_1$, which implies that ${\omega_0}^1$ is a nonzero multiple of du and

$${\omega_0}^0 = 0 = {\omega_0}^2 = \cdots = {\omega_0}^n.$$

Differentiating these relations gives

$$0 = d{\omega_0}^0 = {\omega_0}^1 \wedge {\omega_1}^0 \quad \text{and} \quad 0 = d{\omega_0}^j = {\omega_0}^1 \wedge {\omega_1}^j \quad (2 \le j \le n).$$

Thus we have n functions, $h_1, {H_1}^2, \ldots, {H_1}^n$, defined by

$${\omega_1}^0 = h_1\,{\omega_0}^1 \quad \text{and} \quad {\omega_1}^j = {H_1}^j\,{\omega_0}^1 \quad (2 \le j \le n). \tag{3.0}$$

In matrix form, equations (1.1) take the form

$$d\begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_n \end{pmatrix} = \begin{pmatrix} 0 & {\omega_0}^1 & 0 & \ldots & 0 \\ h_1\,{\omega_0}^1 & {\omega_1}^1 & {H_1}^2\,{\omega_0}^1 & \ldots & {H_1}^n\,{\omega_0}^1 \\ {\omega_2}^0 & {\omega_2}^1 & {\omega_2}^2 & \ldots & {\omega_2}^n \\ \vdots & \vdots & \vdots & & \vdots \\ {\omega_n}^0 & {\omega_n}^1 & {\omega_n}^2 & \ldots & {\omega_n}^n \end{pmatrix} \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}. \tag{3.1}$$

From this point on we will use successively higher order derivatives of $\mathbf{x}(u)$ to refine the centro-affine frame. The details of this process are given in the proof of the following lemma. Before stating the lemma, we will need an additional definition.

Definition. Let I be an open interval and let $\mathbf{x} : I \to \mathbf{R}_0^{n+1}$ be a smooth curve. We say that $\mathbf{x}$ is *substantial* if for every $u \in I$

(S1) $\mathbf{x}(u), \mathbf{x}'(u), \ldots, \mathbf{x}^{(n)}(u)$ are linearly independent, and

(S2) $\mathbf{x}'(u), \mathbf{x}''(u), \ldots, \mathbf{x}^{(n+1)}(u)$ are linearly independent.

Lemma 4. *Let $\mathbf{x} : I \to \mathbf{R}_0^{n+1}$ be a substantial curve. Then there is a reduction of the centro-affine frames $\mathcal{F}_{\mathbf{x}}^{(1)}$ for which the Maurer-Cartan matrix in equation* (3.1) *has the form*

$$\begin{pmatrix}
0 & {\omega_0}^1 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & {\omega_1}^1 & {\omega_0}^1 & 0 & \ldots & \ldots & \ldots & \ldots \\
0 & {\omega_2}^1 & 2{\omega_1}^1 & {\omega_0}^1 & 0 & \ldots & \ldots & \ldots \\
0 & {\omega_3}^1 & 3{\omega_2}^1 & 3{\omega_1}^1 & {\omega_0}^1 & 0 & \ldots & \ldots \\
0 & {\omega_4}^1 & 4{\omega_3}^1 & 6{\omega_2}^1 & 4{\omega_1}^1 & {\omega_0}^1 & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & {\omega_{n-1}}^1 & (n-1){\omega_{n-2}}^1 & \ldots & \ldots & \ldots & (n-1){\omega_1}^1 & {\omega_0}^1 \\
h_n\,{\omega_0}^1 & {\omega_n}^1 & {\omega_n}^2 & \ldots & \ldots & \ldots & \ldots & {\omega_n}^n
\end{pmatrix}$$

and the last row determines n functions $h_n, {\ell_n}^2, \ldots, {\ell_n}^n$ where ${\ell_n}^{j+1}$ is defined by the relation

$${\omega_n}^{j+1} = \binom{n}{j} {\omega_{n-j}}^1 + {\ell_n}^{j+1}\,{\omega_0}^1 \qquad (1 \le j \le n-1).$$

Using the Maurer–Cartan equations (1.2) and equations (3.0) we can compute the infinitesimal action on the functions in (3.1). We calculate the action on h_1 in the following way. From (1.2) we have that

$$d{\omega_1}^0 = {\omega_1}^0 \wedge {\omega_0}^0 + {\omega_1}^1 \wedge {\omega_1}^0 + {\omega_1}^2 \wedge {\omega_2}^0 + \cdots + {\omega_1}^n \wedge {\omega_n}^0,$$

thus using equations (3.0) and the fact that ${\omega_0}^0 = 0$ this equation becomes

$$d{\omega_1}^0 = h_1\,{\omega_1}^1 \wedge {\omega_0}^1 + {H_1}^2\,{\omega_0}^1 \wedge {\omega_2}^0 + \cdots + {H_1}^n\,{\omega_0}^1 \wedge {\omega_n}^0. \tag{3.2}$$

Since ${\omega_1}^0 = h_1\,{\omega_0}^1$ we also have

$$\begin{aligned} d{\omega_1}^0 &= d(h_1\,{\omega_0}^1) \\ &= dh_1 \wedge {\omega_0}^1 + h_1\,d{\omega_0}^1. \end{aligned}$$

Once again, we use (1.2) to expand $d{\omega_0}^1$, use our relations, and find that $d{\omega_0}^1 = {\omega_0}^1 \wedge {\omega_1}^1$. Thus we see

$$d{\omega_1}^0 = dh_1 \wedge {\omega_0}^1 + h_1\,{\omega_0}^1 \wedge {\omega_1}^1. \tag{3.3}$$

For the ${H_1}^j$s, we have the similar (and abbreviated) calculations

$$d{\omega_1}^j = {H_1}^j\,{\omega_1}^1 \wedge {\omega_0}^1 + {H_1}^2\,{\omega_0}^1 \wedge {\omega_2}^0 + \cdots + {H_1}^n\,{\omega_0}^1 \wedge {\omega_n}^0 \tag{3.4}$$

and

$$d{\omega_1}^j = d({H_1}^j\,{\omega_0}^1) = d{H_1}^j \wedge {\omega_0}^1 + {H_1}^j\,{\omega_0}^1 \wedge {\omega_1}^1, \quad (2 \le j \le n). \tag{3.5}$$

Subtracting (3.2) from (3.3) and (3.4) from (3.5) we get

$$
\begin{aligned}
0 &= (dh_1 - 2h_1\,\omega_1{}^1 + H_1{}^2\,\omega_2{}^0 + \cdots + H_1{}^n\omega_n{}^0) \wedge \omega_0{}^1 && (3.6)\\
0 &= (dH_1{}^j - 2H_1{}^j\,\omega_1{}^1 + H_1{}^2\,\omega_2{}^j + \cdots + H_1{}^n\omega_n{}^j) \wedge \omega_0{}^1, \quad (2 \le j \le n), && (3.7)
\end{aligned}
$$

In matrix notation, the second equation has the form

$$
0 \equiv d(H_1{}^2, \dots, H_1{}^n) - 2(H_1{}^2, \dots, H_1{}^n)\omega_1{}^1 + (H_1{}^2, \dots, H_1{}^n) \begin{pmatrix} \omega_2{}^2 & \dots & \omega_2{}^n \\ \vdots & & \vdots \\ \omega_n{}^2 & \dots & \omega_n{}^n \end{pmatrix} \pmod{\omega_0{}^1}
$$

which shows that the vector $(H_1{}^2, \dots, H_1{}^n)$ transforms by an arbitrary general linear action and by a multiplication by a square. Suppose that this vector is zero for all $u \in I$, then from equation (3.1) we see that $d(\mathbf{e}_0 \wedge \mathbf{e}_1) = \omega_1{}^1\,\mathbf{e}_0 \wedge \mathbf{e}_1 \equiv 0 \pmod{\mathbf{e}_0 \wedge \mathbf{e}_1}$, which implies that the 2-plane spanned by $\mathbf{e}_0$ and $\mathbf{e}_1$ is constant. Therefore the curve $\mathbf{x}(u)$ lies in a fixed 2-plane.

Since $\mathbf{x}$ is substantial the vector $(H_1{}^2, \dots, H_1{}^n) \neq 0$ for all $u \in I$, so we may impose the condition that $(H_1{}^2, H_1{}^3, \dots, H_1{}^n) = (1, 0, \dots, 0)$. Equation (3.6) becomes $0 = (dh_1 - 2h_1\,\omega_1{}^1 + \omega_2{}^0) \wedge \omega_0{}^1$, and we see that we may impose the condition $h_1 = 0$. Equations (3.6) and (3.7) reduce to

$$
\begin{aligned}
0 &= \omega_2{}^0 \wedge \omega_0{}^1 \\
0 &= (\omega_2{}^2 - 2\omega_1{}^1) \wedge \omega_0{}^1 \\
0 &= \omega_2{}^j \wedge \omega_0{}^1 \qquad (3 \le j \le n).
\end{aligned}
$$

This introduces n functions, $h_2, \ell_2{}^2, H_2{}^3, \dots, H_2{}^n$, defined by

$$
\omega_2{}^0 = h_2\,\omega_0{}^1,\ \omega_2{}^2 - 2\omega_1{}^1 = \ell_2{}^2\,\omega_0{}^1, \text{ and } \omega_2{}^j = H_2{}^j\,\omega_0{}^1 \qquad (3 \le j \le n).
$$

Equation (3.1) now has the form

$$
d\begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix} = \begin{pmatrix} 0 & \omega_0{}^1 & 0 & 0 & \dots & 0 \\ 0 & \omega_1{}^1 & \omega_0{}^1 & 0 & \dots & 0 \\ h_2\,\omega_0{}^1 & \omega_2{}^1 & \omega_2{}^2 & H_2{}^3\,\omega_0{}^1 & \dots & H_2{}^n\,\omega_0{}^1 \\ \omega_3{}^0 & \omega_3{}^1 & \omega_3{}^2 & \omega_3{}^3 & \dots & \omega_3{}^n \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \omega_n{}^0 & \omega_n{}^1 & \omega_n{}^2 & \omega_n{}^3 & \dots & \omega_n{}^n \end{pmatrix} \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \vdots \\ \mathbf{e}_n \end{pmatrix}, \qquad (3.8)
$$

where $\omega_2{}^2 = 2\omega_1{}^1 + \ell_2{}^2\omega_0{}^1$. The functions h_2, $\ell_2{}^2$, and $H_2{}^3, \dots, H_2{}^n$ are all in row two, and every column other than column 1 has one function. The infinitesimal actions on h_2 and $H_2{}^j$ are computed in the same way and result in equations similar to (3.6) and (3.7),

$$0 = (dh_2 - 3h_2\,\omega_1{}^1 + \sum_{j=3}^{n} H_2{}^j\,\omega_j{}^0) \wedge \omega_0{}^1$$
$$0 = (dH_2{}^j - 3H_2{}^j\,\omega_1{}^1 + \sum_{k=3}^{n} H_2{}^k\,\omega_k{}^j) \wedge \omega_0{}^1 \qquad (3 \le j \le n).$$

We compute the infinitesimal action on $\ell_2{}^2$ by differentiating the relation $\ell_2{}^2\omega_0{}^1 = \omega_2{}^2 - 2\omega_1{}^1$. First

$$d(\ell_2{}^2\,\omega_0{}^1) = d\ell_2{}^2 \wedge \omega_0{}^1 + \ell_2{}^2\,\omega_0{}^1 \wedge \omega_1{}^1$$

and next

$$\begin{aligned} d(\omega_2{}^2 - 2\omega_1{}^1) &= d\omega_2{}^2 - 2d\omega_1{}^1 \\ &= \omega_2{}^1 \wedge \omega_0{}^1 + \sum_{j=3}^{n} H_2{}^j\,\omega_0{}^1 \wedge \omega_j{}^2 - 2\omega_0{}^1 \wedge \omega_2{}^1 \\ &= 3\omega_2{}^1 \wedge \omega_0{}^1 + \sum_{j=3}^{n} H_2{}^j\,\omega_0{}^1 \wedge \omega_j{}^2. \end{aligned}$$

Subtracting the first and last equation gives

$$0 = (d\ell_2{}^2 - \ell_2{}^2\omega_1{}^1 - 3\omega_2{}^1 + \sum_{j=3}^{n} H_2{}^j\omega_j{}^2) \wedge \omega_0{}^1.$$

Once again the vector $(H_2{}^3, \dots, H_2{}^n)$ plays the key role. If this vector vanishes for all u, then $d(\mathbf{e}_0 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2) \equiv 0 \pmod{\mathbf{e}_0 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2}$ and the curve $\mathbf{x}(u)$ lies in a 3 dimensional subspace. Since $\mathbf{x}$ is substantial this vector never vanishes. Thus we may use the general linear action to restrict $(H_2{}^3, \dots, H_2{}^n) = (1, 0, \dots, 0)$. We may also restrict h_2 to 0 and $\ell_2{}^2$ to 0.

These restrictions induce n new functions, all in row three, h_3, $\ell_3{}^2$, $\ell_3{}^3$, $H_3{}^4, \dots, H_3{}^n$. The Maurer-Cartan matrix from (3.8) takes the form

$$\begin{pmatrix} 0 & \omega_0{}^1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \omega_1{}^1 & \omega_0{}^1 & 0 & 0 & \dots & 0 \\ 0 & \omega_2{}^1 & 2\omega_1{}^1 & \omega_0{}^1 & 0 & \dots & 0 \\ h_3\,\omega_0{}^1 & \omega_3{}^1 & \omega_3{}^2 & \omega_3{}^3 & H_3{}^4\,\omega_0{}^1 & \dots & H_3{}^n\,\omega_0{}^1 \\ \omega_4{}^0 & \omega_4{}^1 & \omega_4{}^2 & \omega_4{}^3 & \omega_4{}^4 & \dots & \omega_4{}^n \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \omega_n{}^0 & \omega_n{}^1 & \omega_n{}^2 & \omega_n{}^3 & \omega_n{}^4 & \dots & \omega_n{}^n \end{pmatrix},$$

where $\omega_3{}^2 - 3\omega_2{}^1 = \ell_3{}^2\omega_0{}^1$ and $\omega_3{}^3 - 3\omega_1{}^1 = \ell_3{}^3\omega_0{}^1$.

Rather than compute the action on these terms, we will now go to the inductive step of the calculation. The Maurer-Cartan matrix will have a certain structure. There will be n functions all in the same row, say row p (e.g., $p = 3$ in the above matrix). One function will be in column 0, and there will be one function in each of columns 2 through n. Below row p, all the 1-forms will be independent group forms. The 1-forms in column 1, starting with the form in row 1 and continuing to the last row, are group forms. Above row p, each band above the superdiagonal is zero. The remaining bands that are parallel to the diagonal form a "left justified" Pascal's triangle, with the relations linking a 1-form up the band to the 1-form in column one.

For the inductive step, we assume that we have restricted the curve's frames so that for a fixed $p \leq n-1$ the Maurer-Cartan matrix satisfies the conditions in the following list. Our goal will be to show that we can further restrict the choice of frames so that the Maurer-Cartan matrix satisfies the listed conditions with p replaced by $p+1$.

A. For each row m, with $0 \leq m < p$

A.1 $\omega_m{}^0 = 0 = \omega_m{}^{m+2} = \cdots = \omega_m{}^n$ (column 0 is 0 and bands 2 or more above the diagonal are zero).

A.2 $\omega_m{}^{m+1} = \omega_0{}^1$ (the superdiagonal equals $\omega_0{}^1$).

A.3 $\omega_m{}^{j+1} = \binom{m}{j} \omega_{m-j}{}^1$ for $1 \leq j \leq m-1$ (the binomial coefficients from the left justified Pascal's triangle.)

B. For row p, there are n functions $h_p, \ell_p{}^2, \ldots, \ell_p{}^p, H_p{}^{p+1}, \ldots, H_p{}^n$ such that

B.1 $\omega_p{}^0 = h_p\, \omega_0{}^1$ (small h in column 0).

B.2 $\omega_p{}^{j+1} - \binom{p}{j} \omega_{p-j}{}^1 = \ell_p{}^{j+1}\, \omega_0{}^1$ for $1 \leq j \leq p-1$ (Pascal from column 2 to the diagonal.)

B.3 $\omega_p{}^j = H_p{}^j\, \omega_0{}^1$ for $p+1 \leq j \leq n$ (the H vector, running from the superdiagonal to the right edge).

To achieve our goal, we need to show that we can restrict our choice of frames so that the function $H_p{}^{p+1} = 1$ and all the other functions, $h_p, \ell_p{}^2, \ldots, \ell_p{}^p, H_p{}^{p+2}, \ldots, H_p{}^n$, equal 0.

We will compute the infinitesimal action on the functions $H_p{}^j$. Fix j with $p+1 \leq j \leq n$, so that $\omega_p{}^j = H_p{}^j\, \omega_0{}^1$. We have

$$\begin{aligned} d\omega_p{}^j &= \sum_{k=0}^{n} \omega_p{}^k \wedge \omega_k{}^j \\ &= \left(\sum_{m=0}^{p-1} \omega_p{}^m \wedge \omega_m{}^j \right) + \omega_p{}^p \wedge \omega_p{}^j + \left(\sum_{k=p+1}^{n} \omega_p{}^k \wedge \omega_k{}^j \right). \end{aligned} \tag{3.9}$$

In the first term of (3.9), we have that $m \leq p-1$ and $p+1 \leq j$, so by (A.1) $\omega_m{}^j = 0$ for each m and the first term vanishes. In the second term, $\omega_p{}^j = H_p{}^j\, \omega_0{}^1$ and by (B.2) $\omega_p{}^p \wedge \omega_0{}^1 = p\, \omega_1{}^1 \wedge \omega_0{}^1$ (pick $j = p-1$ and wedge the equation in (B.2) by $\omega_0{}^1$). We see that $\omega_p{}^p \wedge \omega_p{}^j = p\, H_p{}^j\, \omega_1{}^1 \wedge \omega_0{}^1$. In the last term, we use (B.3) to replace $\omega_p{}^k$ with $H_p{}^k\, \omega_0{}^1$. This gives the equation

$$d\omega_p{}^j = p\, H_p{}^j\, \omega_1{}^1 \wedge \omega_0{}^1 + \sum_{k=p+1}^{n} H_p{}^k\, \omega_0{}^1 \wedge \omega_k{}^j.$$

We also have that $d\omega_p{}^j = d(H_p{}^j\,\omega_0{}^1) = dH_p{}^j \wedge \omega_0{}^1 + H_p{}^j\omega_0{}^1 \wedge \omega_1{}^1$. Subtracting these two equations we find that

$$0 = \left(dH_p{}^j - (p+1)H_p{}^j\omega_1{}^1 + \sum_{k=p+1}^{n} H_p{}^k\,\omega_k{}^j \right) \wedge \omega_0{}^1, \quad (p+1 \le j \le n)$$

which in matrix form is

$$\begin{aligned} 0 \equiv\, & d(H_p{}^{p+1}, \ldots, H_p{}^n) - (p+1)(H_p{}^{p+1}, \ldots, H_p{}^n)\omega_1{}^1 \\ & + (H_p{}^{p+1}, \ldots, H_p{}^n) \begin{pmatrix} \omega_{p+1}{}^{p+1} & \ldots & \omega_{p+1}{}^n \\ \vdots & & \vdots \\ \omega_n{}^{p+1} & \ldots & \omega_n{}^n \end{pmatrix} \quad (\text{mod } \omega_0{}^1). \end{aligned}$$

We see that the H vector transforms by an arbitrary general linear action and by a multiplication by a $p+1$ power. If the vector $(H_p{}^{p+1}, \ldots, H_p{}^n)$ equals 0 for all $u \in I$, then

$$d(\mathbf{e}_0 \wedge \cdots \wedge \mathbf{e}_p) = \left(\sum_{i=1}^{p} \omega_i{}^i \right) \mathbf{e}_0 \wedge \cdots \wedge \mathbf{e}_p \equiv 0 \qquad (\text{mod } \mathbf{e}_0 \wedge \cdots \wedge \mathbf{e}_p),$$

thus the $(p+1)$-plane spanned by $\{\mathbf{e}_0, \ldots, \mathbf{e}_p\}$ is constant and the curve $\mathbf{x}(u)$ lies in this plane.

Since $\mathbf{x}$ is substantial the vector $(H_p{}^{p+1}, \ldots, H_p{}^n)$ never equals zero. Hence we may restrict our frames so that $(H_p{}^{p+1}, \ldots, H_p{}^n) = (1, 0, \ldots, 0)$, which implies the relations

$$\begin{aligned} \omega_p{}^{p+1} &= \omega_0{}^1 \quad \text{which is (A.2) for } m = p, \\ 0 &= \omega_p{}^{p+2} = \cdots = \omega_p{}^n \quad \text{which is all but 1 equation in (A.1) for } m = p, \\ 0 &= (\omega_{p+1}{}^{p+1} - (p+1)\,\omega_1{}^1) \wedge \omega_0{}^1 \\ 0 &= \omega_{p+1}{}^j \wedge \omega_0{}^1 \qquad (p+2 \le j \le n). \end{aligned}$$

Therefore, we must have functions $\ell_{p+1}{}^{p+1}$ and $H_{p+1}{}^{p+2}, \ldots, H_{p+1}{}^n$ defined by

$$\begin{aligned} \omega_{p+1}{}^{p+1} - (p+1)\,\omega_1{}^1 &= \ell_{p+1}{}^{p+1}\,\omega_0{}^1 \\ \text{and} \quad \omega_{p+1}{}^j &= H_{p+1}{}^j\,\omega_0{}^1 \qquad (p+2 \le j \le n). \end{aligned}$$

This shows that (B.3) is true for $p+1$ and that (B.2) is true for $j = (p+1) - 1$.

By an almost identical calculation, we arrive at the equation

$$\begin{aligned} 0 &= \left(dh_p - (p+1)\,h_p\,\omega_1{}^1 + \sum_{k=p+1}^{n} H_p{}^k\,\omega_k{}^0 \right) \wedge \omega_0{}^1 \\ &= \left(dh_p - (p+1)\,h_p\,\omega_1{}^1 + \omega_{p+1}{}^0 \right) \wedge \omega_0{}^1. \end{aligned} \tag{3.10}$$

The function h_p transforms by a multiplication by a $p+1$ power and by a translation. We may therefore further restrict the frames so that $h_p = 0$, which implies the relations ${\omega_p}^0 = 0$ and ${\omega_{p+1}}^0 \wedge {\omega_0}^1 = 0$. Thus there is a function h_{p+1} such that ${\omega_{p+1}}^0 = h_{p+1}\,{\omega_0}^1$ and we see that (A.1), (A.2), (B.1) and (B.3) all hold for row $p+1$, as well as (B.2) for the case $j = p$. All that remains is to verify (A.3) for $1 \le j \le p$ and (B.2) for $1 \le j \le p-1$.

To do this, we need to compute the infinitesimal action on the functions ${\ell_p}^{j+1}$, $(1 \le j \le p-1)$. Thus, we must differentiate the relation

$$ {\ell_p}^{j+1}\,{\omega_0}^1 = {\omega_p}^{j+1} - \binom{p}{j}{\omega_{p-j}}^1 \qquad (1 \le j \le p-1) \tag{3.11} $$

to find the infinitesimal action. There is one important observation we can make. Since $d{\omega_0}^1 = {\omega_0}^1 \wedge {\omega_1}^1$, the exterior derivative

$$ d({\ell_p}^{j+1}\,{\omega_0}^1) = d{\ell_p}^{j+1} \wedge {\omega_0}^1 + {\ell_p}^{j+1}\,{\omega_0}^1 \wedge {\omega_1}^1 \tag{3.12} $$

must be linear in ${\omega_0}^1$. Therefore, when we differentiate the right hand side of (3.11) and reduce our equations as much as possible using (A.1), (A.2), (A.3), (B.1), (B.2)=(3.11) and (B.3) we know that all of the terms that are independent of ${\omega_0}^1$ must cancel out one another. Thus we only need to keep track of the terms that introduce a factor of ${\omega_0}^1$.

We will begin by differentiating the first term on the right hand side.

$$ \begin{aligned} d{\omega_p}^{j+1} &= \sum_{k=0}^{n} {\omega_p}^k \wedge {\omega_k}^{j+1} \\ &= \left(\sum_{k=0}^{j-1} {\omega_p}^k \wedge {\omega_k}^{j+1}\right) + {\omega_p}^j \wedge {\omega_j}^{j+1} + {\omega_p}^{j+1} \wedge {\omega_{j+1}}^{j+1} + \left(\sum_{k=j+2}^{p-1} {\omega_p}^k \wedge {\omega_k}^{j+1}\right) \\ &\qquad + {\omega_p}^p \wedge {\omega_p}^{j+1} + \left(\sum_{k=p+1}^{n} {\omega_p}^k \wedge {\omega_k}^{j+1}\right) \end{aligned} $$

In the first term, all the ${\omega_k}^{j+1} = 0$. In the second term,

$$ {\omega_j}^{j+1} = {\omega_0}^1 \text{ and } {\omega_p}^j = \binom{p}{j-1}{\omega_{p-j+1}}^1 + {\ell_p}^j\,{\omega_0}^1. $$

The third and fifth terms have a common factor of ${\omega_p}^{j+1}$ and can be combined into a a single term $({\omega_p}^p - {\omega_{j+1}}^{j+1}) \wedge {\omega_p}^{j+1}$. We can express this combined term as

$$ \left[(p-j-1){\omega_1}^1 + {\ell_p}^p{\omega_0}^1\right] \wedge \left[\binom{p}{j}{\omega_{p-j}}^1 + {\ell_p}^{j+1}\,{\omega_0}^1\right]. $$

In the fourth term we have ${\omega_p}^k = \binom{p}{k-1}{\omega_{p-k+1}}^1 + {\ell_p}^k\,{\omega_0}^1$ and ${\omega_k}^{j+1} = \binom{k}{j}{\omega_{k-j}}^1$. In the sixth term we have ${\omega_p}^k = {H_p}^k\,{\omega_0}^1$. Recall that with the current choice of frames ${H_p}^k$ equals 1 for $k = p+1$ and equals 0 otherwise. Combining these relations we get

$$ \begin{aligned} d{\omega_p}^{j+1} &= \binom{p}{j-1}{\omega_{p-j+1}}^1 \wedge {\omega_0}^1 + (p-j-1){\ell_p}^{j+1}\,{\omega_1}^1 \wedge {\omega_0}^1 + \binom{p}{j}{\ell_p}^p\,{\omega_0}^1 \wedge {\omega_{p-j}}^1 \\ &\quad + \sum_{k=j+2}^{p-1}\binom{k}{j}{\ell_p}^k\,{\omega_0}^1 \wedge {\omega_{k-j}}^1 + {\omega_0}^1 \wedge {\omega_{p+1}}^{j+1} + \text{terms independent of } {\omega_0}^1. \end{aligned} \tag{3.13} $$

For the second term on the right hand side, we compute

$$\begin{aligned} d\omega_{p-j}{}^1 &= \sum_{k=0}^{n} \omega_{p-j}{}^k \wedge \omega_k{}^1 \\ &= \omega_{p-j}{}^0 \wedge \omega_0{}^1 + \left(\sum_{k=1}^{p-j} \omega_{p-j}{}^k \wedge \omega_k{}^1 \right) + \omega_{p-j}{}^{p-j+1} \wedge \omega_{p-j+1}{}^1 \\ &\qquad + \left(\sum_{k=p-j+2}^{n} \omega_{p-j}{}^k \wedge \omega_k{}^1 \right). \end{aligned}$$

In the first term, $\omega_{p-j}{}^0 = 0$. In the second term $\omega_{p-j}{}^k = \binom{p-j}{k-1}\omega_{p-j-k+1}{}^1$. In the third term $\omega_{p-j}{}^{p-j+1} = \omega_0{}^1$ and in the fourth term each $\omega_{p-j}{}^k = 0$. Combining these relations gives

$$\begin{aligned} d\omega_{p-j}{}^1 &= \sum_{k=1}^{p-j} \binom{p-j}{k-1} \omega_{p-j-k+1}{}^1 \wedge \omega_k{}^1 + \omega_0{}^1 \wedge \omega_{p-j+1}{}^1 \\ &= \omega_0{}^1 \wedge \omega_{p-j+1}{}^1 + \text{terms independent of } \omega_0{}^1. \end{aligned} \tag{3.14}$$

We see from (3.11) that $0 = d(\ell_p{}^{j+1}\,\omega_0{}^1) - d\omega_p{}^{j+1} + \binom{p}{j} d\omega_{p-j}{}^1$. If we substitute (3.12), (3.13) and (3.14) into this equation and collect some terms we have for each $1 \le j \le p-1$

$$\begin{aligned} 0 = \Bigg(d\ell_p{}^{j+1} - (p-j)\ell_p{}^{j+1}\,\omega_1{}^1 + \sum_{k=j+2}^{p} \binom{k}{j} \ell_p{}^k\,\omega_{k-j}{}^1 \\ + \omega_{p+1}{}^{j+1} - \binom{p+1}{j}\omega_{p+1-j}{}^1 \Bigg) \wedge \omega_0{}^1. \end{aligned} \tag{3.15}$$

From this equation we observe that each $\ell_p{}^{j+1}$ is translated by the 1-form positioned in the same column and in the row directly below it, $\omega_{p+1}{}^{j+1}$. We also observe that $\ell_p{}^{j+1}$ depends only on $\ell_p{}^{j+1}, \ldots, \ell_p{}^p$. Working from equation $p-1$ down to equation 1 we see that each $\ell_p{}^{j+1}$ can be set to 0. After restricting the frames so that all the $\ell_p{}^{j+1} = 0$, the equations from (B.2) reduce to

$$\omega_p{}^{j+1} = \binom{p}{j}\omega_{p-j}{}^1, \qquad (1 \le j \le p-1),$$

and equations (3.15) simplify to

$$0 = \left(\omega_{p+1}{}^{j+1} - \binom{p+1}{j}\omega_{p+1-j}{}^1 \right) \wedge \omega_0{}^1, \qquad (1 \le j \le p-1).$$

This shows that there are $p-1$ functions, $\ell_{p+1}{}^2, \ldots, \ell_{p+1}{}^p$, defined by

$$\omega_{p+1}{}^{j+1} - \binom{p+1}{j}\omega_{p+1-j}{}^1 = \ell_{p+1}{}^{j+1}\,\omega_0{}^1, \qquad (1 \le j \le p-1).$$

Together, these equations verify (A.3) and the rest of (B.2).

Having completed the induction, we see that when p takes on its maximum value n, the Maurer-Cartan matrix will have the following form:

$$\begin{pmatrix} 0 & {\omega_0}^1 & 0 & \multicolumn{5}{c}{\dotfill} \\ 0 & {\omega_1}^1 & {\omega_0}^1 & 0 & \multicolumn{4}{c}{\dotfill} \\ 0 & {\omega_2}^1 & 2{\omega_1}^1 & {\omega_0}^1 & 0 & \multicolumn{3}{c}{\dotfill} \\ 0 & {\omega_3}^1 & 3{\omega_2}^1 & 3{\omega_1}^1 & {\omega_0}^1 & 0 & \multicolumn{2}{c}{\dotfill} \\ 0 & {\omega_4}^1 & 4{\omega_3}^1 & 6{\omega_2}^1 & 4{\omega_1}^1 & {\omega_0}^1 & 0 & \dots \\ \multicolumn{8}{c}{\dotfill} \\ 0 & {\omega_{n-1}}^1 & (n-1){\omega_{n-2}}^1 & \multicolumn{3}{c}{\dotfill} & (n-1){\omega_1}^1 & {\omega_0}^1 \\ h_n\,{\omega_0}^1 & {\omega_n}^1 & {\omega_n}^2 & \multicolumn{4}{c}{\dotfill} & {\omega_n}^n \end{pmatrix}$$

In this matrix, the 1-forms ${\omega_1}^1, \dots, {\omega_n}^1$ are independent group forms, and the last row determines the n functions $h_n, {\ell_n}^2, \dots, {\ell_n}^n$ where ${\ell_n}^{j+1}$ is defined by the relation

$$ {\omega_n}^{j+1} = \binom{n}{j}\,{\omega_{n-j}}^1 + {\ell_n}^{j+1}\,{\omega_0}^1 \qquad (1 \le j \le n-1). $$

This completes the proof of lemma 3. ∎

The Maurer-Cartan matrix satisfies all of conditions (A) and (B), with the exception of (B.3) since there are no functions ${H_n}^j$. We easily see that the previous calculations of infinitesimal actions are still valid, and thus we can use these calculations simply by setting all of the H's equal to zero. For instance, from equation (3.10) we see that

$$ dh_n - (n+1)h_n\,{\omega_1}^1 \equiv 0 \qquad (\mathrm{mod}\ {\omega_0}^1) $$

which shows that h_n is multiplied by an $n+1$st power. Suppose first of all that h_n is identically zero. Then from the above matrix we compute that

$$ d(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_n) = \Big(\sum_{i=1}^{n} {\omega_i}^i\Big)\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_n \equiv 0 \qquad (\mathrm{mod}\ \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_n) $$

This shows that the hyperplane spanned by $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ is constant. Since this hyperplane is the osculating n-plane of the curve $\mathbf{x}(u) = \mathbf{e}_0$, a simple calculation shows that the curve must lie in a fixed affine hyperplane parallel to the span of $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$. The problem reduces to an n-dimensional one in general affine geometry.

For a substantial curve $\mathbf{x}$, h_n never vanishes. In this case, we can restrict our frames so that $h_n = \pm 1$, and if $n+1$ is odd then we can always choose positive 1. This reduction implies that there is a function κ_1 defined by

$$ {\omega_1}^1 = \kappa_1\,{\omega_0}^1, $$

and this new relation further implies that $d{\omega_0}^1 = {\omega_0}^1 \wedge {\omega_1}^1 = 0$. This means that ${\omega_0}^1$ must be the differential of some function $s(u)$ on the curve. Any two such functions differ by a constant, and we will call any one of them a *centro-affine arclength for* $\mathbf{x}$. We will say a little more about this arclength function below. For now, we will continue with the infinitesimal actions.

A quick review of the derivation of equation (3.15) shows that only the ${\omega_{p+1}}^{j+1}$ term involved the H functions, so that term will not appear in $d{\ell_n}^{j+1}$. Since we also now have ${\omega_1}^1 \equiv 0 \pmod{{\omega_0}^1}$, equation (3.15) reduces to

$$d{\ell_n}^{j+1} + \sum_{k=j+2}^{n} \binom{k}{j} {\ell_n}^k \, {\omega_{k-j}}^1 - \binom{n+1}{j} {\omega_{n+1-j}}^1 \equiv 0 \qquad \pmod{{\omega_0}^1} \quad (1 \le j \le n-1).$$

Working as before from equation $n-1$ down to equation 1, we see that each ${\ell_n}^{j+1}$ may be translated to 0 by an independent group form. For example, ${\ell_n}^n$ is translated by ${\omega_2}^1$, ${\ell_n}^{n-1}$ by ${\omega_3}^1$, etc., and finally ${\ell_n}^2$ by ${\omega_n}^1$.

Making these final restrictions on the adapted frames uses all of the remaining freedom in the group. Every entry in the Maurer-Cartan matrix is a multiple of the centro-affine arclength ${\omega_0}^1 = ds$, which shows that the centro-affine frames are uniquely determined. The matrix has the form

$$\begin{pmatrix} 0 & 1 & 0 & \multicolumn{5}{c}{\dotfill} \\ 0 & \kappa_1 & 1 & 0 & \multicolumn{4}{c}{\dotfill} \\ 0 & \kappa_2 & 2\kappa_1 & 1 & 0 & \multicolumn{3}{c}{\dotfill} \\ 0 & \kappa_3 & 3\kappa_2 & 3\kappa_1 & 1 & 0 & \multicolumn{2}{c}{\dotfill} \\ 0 & \kappa_4 & 4\kappa_3 & 6\kappa_2 & 4\kappa_1 & 1 & 0 & \dots \\ \multicolumn{8}{c}{\dotfill} \\ \pm 1 & \kappa_n & n\,\kappa_{n-1} & \binom{n}{2}\kappa_{n-2} & \dots & \binom{n}{n-2}\kappa_2 & & n\,\kappa_1 \end{pmatrix} ds.$$

The binomial coefficients of the "left justified" Pascal's triangle appear clearly in the above matrix when we imagine all of the κ_i's are set equal to 1. The last row is a little different. The final binomial coefficient $\binom{n}{n}$ appears in column 0 and may have a minus sign. The above matrix also defines the centro-affine "Frenet equations" of the curve,

$$\frac{d}{ds}\begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \\ \dots \\ \mathbf{e}_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \multicolumn{5}{c}{\dotfill} \\ 0 & \kappa_1 & 1 & 0 & \multicolumn{4}{c}{\dotfill} \\ 0 & \kappa_2 & 2\kappa_1 & 1 & 0 & \multicolumn{3}{c}{\dotfill} \\ 0 & \kappa_3 & 3\kappa_2 & 3\kappa_1 & 1 & 0 & \multicolumn{2}{c}{\dotfill} \\ 0 & \kappa_4 & 4\kappa_3 & 6\kappa_2 & 4\kappa_1 & 1 & 0 & \dots \\ \multicolumn{8}{c}{\dotfill} \\ \pm 1 & \kappa_n & n\,\kappa_{n-1} & \binom{n}{2}\kappa_{n-2} & \dots & \binom{n}{n-2}\kappa_2 & & n\,\kappa_1 \end{pmatrix} \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \\ \dots \\ \mathbf{e}_n \end{pmatrix}. \tag{3.16}$$

We can use the centro-affine Frenet equations to get a formula for the element of arclength, ds. As a notational convenience, we will let $\nu = ds/du$. Then we clearly have that $ds = \nu\, du$ and $\mathbf{e}_1 = \nu^{-1}\,\mathbf{x}'(u)$. From the Frenet equations we get that $d\mathbf{e}_1/ds \equiv \mathbf{e}_2 \pmod{\mathbf{e}_1}$, but we can also calculate that

$$\frac{d\mathbf{e}_1}{ds} \equiv \nu^{-1}\,\frac{d\mathbf{x}'(u)}{ds} \equiv \nu^{-2}\,\mathbf{x}''(u) \qquad \pmod{\mathbf{e}_1},$$

which shows that $\mathbf{e}_2 \equiv \nu^{-2}\,\mathbf{x}''(u) \pmod{\mathbf{e}_1}$. A direct generalization of this calculation shows that in addition to the equations

$$\begin{aligned} \mathbf{e}_0 &= \mathbf{x}(u) \\ \mathbf{e}_1 &= \nu^{-1}\,\mathbf{x}'(u) \end{aligned} \tag{3.17}$$

we also have for each $2 \le p \le n$

$$\mathbf{e}_p \equiv \nu^{-p}\, \mathbf{x}^{(p)}(u) \qquad (\mathrm{mod}\ \mathbf{e}_1, \ldots, \mathbf{e}_{p-1}). \tag{3.18}$$

From this equation with $p = n$ and the Frenet equations we compute that

$$\begin{aligned} \frac{d\mathbf{e}_n}{ds} &\equiv \nu^{-(n+1)}\, \mathbf{x}^{(n+1)}(u) \quad (\mathrm{mod}\ \mathbf{e}_1, \ldots, \mathbf{e}_n) \\ &\equiv \pm \mathbf{e}_0 \quad (\mathrm{mod}\ \mathbf{e}_1, \ldots, \mathbf{e}_n) \text{ by the Frenet equations.} \end{aligned} \tag{3.19}$$

From (3.19) we get

$$\nu^{-(n+1)} \det(\mathbf{x}^{(n+1)}(u), \mathbf{e}_1, \ldots, \mathbf{e}_n) = \det(\frac{d\mathbf{e}_n}{ds}, \mathbf{e}_1, \ldots, \mathbf{e}_n) = \pm \det(\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n)$$

which together with (3.17) and (3.18) implies

$$\left(\frac{ds}{du}\right)^{n+1} = \nu^{n+1} = \pm \frac{\det(\mathbf{x}^{(n+1)}(u), \mathbf{e}_1, \ldots, \mathbf{e}_n)}{\det(\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n)} = \pm \frac{\det(\mathbf{x}^{(n+1)}(u), \mathbf{x}'(u), \ldots, \mathbf{x}^{(n)}(u))}{\det(\mathbf{x}(u), \mathbf{x}'(u), \ldots, \mathbf{x}^{(n)}(u))}. \tag{3.20}$$

We now have an explicit formula for computing a centro-affine arclength in terms of an arbitrary parameter. An orientation preserving arclength is given by

$$s = \int \left| \frac{\det(\mathbf{x}^{(n+1)}(u), \mathbf{x}'(u), \ldots, \mathbf{x}^{(n)}(u))}{\det(\mathbf{x}(u), \mathbf{x}'(u), \ldots, \mathbf{x}^{(n)}(u))} \right|^{1/(n+1)} du. \tag{3.21}$$

Notice that the 1-form under the integral is $\mathrm{GL}(n+1, \mathbf{R})$-invariant and invariant under orientation preserving reparametrization. These two observations confirm that the centro-affine arclength s has the desired invariance. We also see that the plus or minus sign in (3.20) is determined by the equation

$$\mathbf{x}^{(n+1)}(s) \equiv \pm \mathbf{x}(s) \quad (\mathrm{mod}\ \mathbf{x}'(s), \ldots, \mathbf{x}^{(n)}(s)). \tag{3.22}$$

The Frenet equations also give an interesting interpretation of the first curvature, κ_1. If we differentiate the volume element $\mathbf{e}_0 \wedge \cdots \wedge \mathbf{e}_n$ with respect to arclength, we see

$$\frac{d}{ds}(\mathbf{e}_0 \wedge \cdots \wedge \mathbf{e}_n) = (1 + 2 + \cdots + n)\kappa_1\, \mathbf{e}_0 \wedge \cdots \wedge \mathbf{e}_n = \binom{n+1}{2} \kappa_1\, \mathbf{e}_0 \wedge \cdots \wedge \mathbf{e}_n,$$

and if we observe that $\mathbf{e}_0 \wedge \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n = \mathbf{x}(s) \wedge \mathbf{x}'(s) \wedge \cdots \wedge \mathbf{x}^{(n)}(s)$ we see that

$$\binom{n+1}{2} \kappa_1 = \frac{d}{ds} \ln |\det(\mathbf{x}(s), \mathbf{x}'(s), \ldots, \mathbf{x}^{(n)}(s))|.$$

Formula (3.21) shows that s is well defined for substantial curves. We can now state the fundamental theorem for centro-affine curves.

Theorem 3. *Let $I \subset \mathbf{R}$ be an open interval, $\epsilon = \pm 1$, and let $\kappa_1(s), \ldots, \kappa_n(s)$ be smooth functions on I. Then there is a smooth substantial immersion $s \mapsto \mathbf{x}(s)$ from I to $\mathbf{R}_0^{n+1}$ such that s is a centro-affine arclength, $\kappa_1(s), \ldots, \kappa_n(s)$ are the centro-affine curvatures, and*

$$\mathbf{x}^{(n+1)}(s) \equiv \epsilon\, \mathbf{x}(s) \quad (\mathrm{mod}\ \mathbf{x}'(s), \ldots, \mathbf{x}^{(n)}(s)).$$

Moreover, $\mathbf{x}$ is uniquely determined up to a centro-affine motion of $\mathbf{R}_0^{n+1}$.

The proof of this theorem is standard. Use $\kappa_1(s), \dots, \kappa_n(s)$ and $\epsilon = \pm 1$ to find a solution of the Frenet equations (3.16) in $\mathrm{GL}(n+1, \mathbf{R})$, then let $\mathbf{x}(s) = \mathbf{e}_0(s)$. The remainder of the proof is a standard verification using the Frenet equations.

Equation (3.20) provides a reasonable way to compute ds/du, which therefore allows one to compute derivatives with respect to arclength. We will use this fact, along with the structure equations, to provide a method for constructing the centro-affine frame and the centro-affine curvatures.

Assume that $\mathbf{x}$ is a substantial curve and s is an arclength parameter for $\mathbf{x}$. By condition (S1) and equation (3.22) there must exist unique functions $c_1(s), \dots, c_n(s)$ satisfying

$$\mathbf{x}^{(n+1)}(s) = \pm\mathbf{x}(s) + c_n(s)\,\mathbf{x}'(s) + \cdots + c_1(s)\,\mathbf{x}^{(n)}(s). \tag{3.23}$$

Using this equation, we can write

$$\frac{d}{ds}\begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \\ \dots \\ \mathbf{x}^{(n-1)} \\ \mathbf{x}^{(n)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ \pm 1 & c_n & c_{n-1} & c_{n-2} & \dots & c_1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \\ \dots \\ \mathbf{x}^{(n-1)} \\ \mathbf{x}^{(n)} \end{pmatrix}$$

If we let $W(s)$ be the matrix with rows $\big(\mathbf{x}(s), \mathbf{x}'(s), \dots, \mathbf{x}^{(n)}(s)\big)$, then we can write the above equation as

$$W'(s) = C(s)\,W(s)$$

where $C(s)$ is defined by the above equation. Since we are differentiating with respect to arclength, $\nu = 1$ in equations (3.17) and (3.18), which shows that we can write the frame in terms of $\mathbf{x}, \mathbf{x}', \dots, \mathbf{x}^{(n)}$ as

$$\begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \dots \\ \mathbf{e}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & m_{21} & 1 & 0 & \dots & 0 \\ 0 & m_{31} & m_{32} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & m_{n1} & m_{n2} & m_{n3} & \dots & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \\ \mathbf{x}'' \\ \mathbf{x}''' \\ \dots \\ \mathbf{x}^{(n)} \end{pmatrix},$$

or as

$$E(s) = M(s)\,W(s),$$

where $E(s)$ is the matrix representing the frames and $M(s)$ is the lower triangular matrix in the above equation. Differentiating this last equation and comparing with the Frenet equations (3.16) we have

$$E'(s) = \Big(M'\,M^{-1} + M\,C\,M^{-1}\Big)E(s),$$

and we see that $M'\,M^{-1} + M\,C\,M^{-1}$ must be the matrix of centro-affine curvatures in (3.16). We can use the relations among the entries of the curvature matrix to uniquely solve for the m_{ij}'s in terms of the $c_1, \dots, c_n$. This will determine the matrix $M(s)$ and therefore $E(s)$ in terms of $c_1, \dots, c_n$. The centro-affine curvatures will be expressed in terms of $c_1, \dots, c_n$ and their derivatives with respect to arclength.

Suppose that we are working with a substantial curve in $\mathbf{R}^4$ with $\mathbf{x}^{(4)}(s) = \mathbf{x}(s) + c_3(s)\mathbf{x}'(s) + c_2(s)\,\mathbf{x}''(s) + c_1(s)\mathbf{x}'''(s)$. Then we can write

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & m_1 & 1 & 0 \\ 0 & m_3 & m_2 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & c_3 & c_2 & c_1 \end{pmatrix} \text{ and } K = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & \kappa_1 & 1 & 0 \\ 0 & \kappa_2 & 2\kappa_1 & 1 \\ 1 & \kappa_3 & 3\kappa_2 & 3\kappa_1 \end{pmatrix},$$

where K is the matrix of centro-affine curvatures. We know that

$$K = M'\,M^{-1} + M\,C\,M^{-1}. \tag{3.24}$$

The diagonal entries of this equation are

$$\begin{aligned} \kappa_1 &= -m_1 \\ 2\kappa_1 &= m_1 - m_2 \\ 3\kappa_1 &= m_2 + c_1, \end{aligned}$$

which allows one to eliminate κ_1 from the equations and solve for m_1 and m_2 in terms of c_1, giving

$$m_1 = -\frac{c_1}{6} \quad \text{and} \quad m_2 = -\frac{c_1}{2}.$$

If we substitute these equations into (3.24) and consider the equations on the subdiagonal, we have

$$\begin{aligned} \kappa_2 &= -m_3 - \frac{c_1'}{6} + \frac{c_1^2}{18} \\ 3\kappa_2 &= m_3 - \frac{c_1'}{2} + \frac{c_1^2}{4} + c_2, \end{aligned}$$

and we can eliminate κ_2 from these equations and solve for

$$m_3 = -\frac{c_1^2}{48} - \frac{c_2}{4}.$$

Thus we have solved for the entries of M in terms of c_1, c_2 and c_3. We get the centro-affine curvatures in terms of the c_i's and their first derivatives by substituting the values of m_1, m_2 and m_3 into (3.24).

Notice that when we solved for m_3, the terms involving c_1' cancel. This will happen in general, as long as one solves the equations along the diagonal first, and then sequentially along each subdiagonal. From equation (3.21) we see that centro-affine arclength depends on derivatives of order $n+1$, thus the functions $c_1, \ldots, c_n$ also depend on derivatives of order $n+1$. Since we can solve for the matrix M in terms of the functions $c_1, \ldots, c_n$, we see that the centro-affine frame depends on derivatives of order $n+1$. Since the curvatures depend on the functions $c_1, \ldots, c_n$ and their first derivatives, we see that the curvatures are of order $n+2$.

4 Closing Remarks

The plane version of theorem 3 goes back to at least 1933 [MM]. A discussion of centro-affine plane curves, as well as a very brief discussion of centro-affine space curves, can be found in [SS, §12 and §16]. Their invariants for curves in $\mathbf{R}^3$ (see page 89) are exactly the coefficients in equation (3.23) for $n = 2$. A very detailed discussion of the centro-affine plane theory of curves ($n = 1$) can be found in Laugwitz [La]. An application of centro-affine curve theory in the plane to differential equations appears in Borůvka's book [Bo]. We are able to apply a similar sort of analysis to the lowest dimensional problem in control theory, and intend to generalize to higher dimensional problems.

Various authors have studied other affine invariants of curves and surfaces. We note Paukowitsch [Pa] developed an equi-affine invariant moving frame for curves in arbitrary dimension. Pabel [Pl] relates the equi-affine curve theory to a structure from classical analysis.

Laugwitz also discusses the case of centro-affine hypersurfaces. He derives the same invariants, but without using moving frames. His application to Finsler geometry is very much in the spirit of our applications to control theory.

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