Measures with finite semi-variation

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Abstract

The purpose of this paper is to characterize the Banach spaces and the locally convex spaces E for which bounded additive measures or bounded σ-additive measures with values in $\mathcal{L}(E, F)$, the space of continuous linear maps from E into F, are of bounded semi-variation for any Banach space or locally convex space F.

This paper gives an answer to a problem posed by D.H. Tucker in [6].

1 Introduction and Definitions

Among the more interesting and useful properties of operators which are representable by integrals of scalar valued functions with respect to scalar valued measures is that such integers (operators) have certain weakened forms of continuity with respect to the integrals. The main examples are the dominated convergence theorem, the monotone convergence theorem and the bounded convergence theorem.

When one moves afield from the scalar valued cases, these results become at best questionable. The relationships which exist in the scalar case relating convergence in measure and convergence pointwise almost everywhere no longer obtain. Pathological examples abound. In [7], examples are given which show that neither type of convergence implies the other. Indeed an example is given of a scalar valued sequence which converges point-wise everywhere to one and in measure to zero. In [8] an example is given of a vector valued sequence on $[0, 1]$, the measure being ordinary Lebesgue measure, in which the functions converge in measure, but no subsequence converges almost everywhere. The pathology in the first example was due to the nature of the measure, that in the second was due to the nature of the range space for the functions involved.

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In the paper [8], the weakened continuity properties were investigated for the case of representing measures, that is, measures which are obtained as a representing measure for a linear operator on a space of continuous vector valued functions, the operator being continuous with respect to the topology of uniform convergence on the function space. Such measures are additive, (but not necessarily countably additive), and are of bounded semi-variation. It is shown there that for such a measure, countable additivity (or continuity in a given topology) implies both a bounded convergence theorem and a dominated convergence theorem relative to pointwise convergence almost everywhere in a similar topology. A converse theorem also holds with a change of topologies and equivalence holds in the case of Banach spaces. As it happens, each of these is also equivalent to a type of weak compactness for the operator represented by the measure.

In [5], a representation theorem is presented for continuous linear mappings from the space C into a linear normed space Y, where C denotes the space of continuous functions from the interval $[0, 1]$ into a normed space X. It is shown that those linear mappings can be represented by a function from $[0, 1]$ to the space $B(X, Y)$ of bounded linear transformations from X to Y which are of bounded semi-variation. The correspondance between the linear mapping φ and the function K of bounded semi-variation is given by $\varphi(f) = \int_0^1 f \; dK(t)$ where the integral is a Stieltjes integral.

The fact that the function K is of bounded semi-variation is important in showing the uniqueness of K in the representation theorem.

If the function K is weakly of bounded semi-variation (i.e. if $y'K$ is of bounded semi-variation for each y' in the dual Y' of Y) then D.H. Tucker gets a representation theorem with Y replaced by the weak sequential completion of Y denoted by Y^+.

In [6], an example is given of a function which is weakly of bounded semi-variation but is not of bounded semi-variation. Denoting by wbv the space of functions which are weakly of bounded semi-variation and by G the space of functions which are of bounded semi-variation, the following is pointed out : "*It would be of interest to know a characterization of those spaces X such that wbv is G for all Y.*"

The purpose of this paper is to provide an answer to that question.

The setting for this investigation is as follows. Suppose E and F are Banach spaces, $\mathcal{L}(E, F)$ is the space of continuous linear operators from E into F, T is a set and $\mathcal{A}$ is a ring (σ-ring) of subsets of T. Under what conditions on E (F, T and $\mathcal{A}$ are arbitrary) does each bounded additive measure on $\mathcal{A}$ (countably additive measure on $\mathcal{A}$) have finite semivariation, i.e., every potential representing measure has the weakened continuity properties.

Definitions and notations.

We will say that a Banach space E has property (P) (resp. (P_σ)) if for each set T, each ring (resp. σ-ring) $\mathcal{A}$ of subsets of T and each Banach space F, each bounded additive (resp. countably additive) set function from $\mathcal{A}$ to $\mathcal{L}(E, F)$ has finite semi-variation.

It is well known that finite dimensional spaces have property (P).

The main result of this paper is the following : only finite dimensional Banach spaces have property (P) and only finite dimensional Banach spaces have property (P_σ).

If m is a set function defined on a ring of subsets of a set T with values in $\mathcal{L}(E,F)$, $\tilde{m}$ will denote the semi-variation of m and $S(X,\mathcal{A},E)$ will denote the vector space of $\mathcal{A}$-step functions with values in E.

Let us also recall that a set function m is of finite semi-variation if and only if the integral mapping defined on the space of E-valued step functions with values in F is continuous.

2 Preliminary result

Proposition 1 *The following are equivalent :*

(1) E has property (P_σ).

(2) For each set T, each σ-ring $\mathcal{A}$ of subsets of T and each family $(F_\alpha)_{\alpha\in\Lambda}$ of Banach spaces, if $(m_\alpha)_{\alpha\in\Lambda}$ is a bounded family of uniformly σ-additive set functions from $\mathcal{A}$ to $\mathcal{L}(E,F_\alpha)$ then there exists $K>0$ such that $\widetilde{m_\alpha}(T)\le K$ for each α in Λ.

(3) For each set T, each σ-ring $\mathcal{A}$ of subsets of T, if $(m_\alpha)_{\alpha\in\Lambda}$ is a bounded family of uniformly σ-additive set functions from $\mathcal{A}$ to E' then there exists $K>0$ such that $\widetilde{m_\alpha}(T)\le K$ for each α in Λ.

Proof.

(1) $\Rightarrow$ (2).

Let $(m_\alpha)_{\alpha\in\Lambda}$ a bounded family of countably additive set functions from $\mathcal{A}$ to $\mathcal{L}(E,F_\alpha)$ and let

$$F=\{\,x\mid x=(x_\alpha)_{\alpha\in\Lambda}, x_\alpha\in F_\alpha, \exists C>0:\|x_\alpha\|\le C\quad\forall\alpha\in\Lambda\}.$$

Equipped with the norm defined by $\|(x_\alpha)_{\alpha\in\Lambda}\|=\sup_{\alpha\in\Lambda}\|x_\alpha\|$, F is a Banach space and the set function m defined on $\mathcal{A}$ by $m(A)x=(m_\alpha(A)x)_{\alpha\in\Lambda}$ is a countably additive set function.

By our assumption m has finite semi-variation and the result follows from the fact that $\widetilde{m_\alpha}(A)\le\widetilde{m}(A)$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1).

Let $m:\mathcal{A}\longrightarrow\mathcal{L}(E,F)$ a countably additive set function. The family $\{\,y'\circ m\mid y'\in F',\|y'\|\le 1\}$ is a bounded family of countably additive set functions from $\mathcal{A}$ to E'.

By our assumption, there exists $K>0$ such that $\widetilde{y'\circ m}(T)\le K\quad\forall y'\in F'$.

As $\widetilde{y'\circ m}(A)=\overline{y'\circ m}(A)$ and $\widetilde{m}(A)=\sup_{y'\in F'\|y'\|\le 1}\overline{y'\circ m}(A)$, the result follows immediately. ∎

Remark : We have a similar result for property (P) with the same proof.

3 Main Result

We now state and prove the main result of this paper.

Theorem 2 *The following are equivalent :*

(1) E has property (P).

(2) E has property (P_σ).

(3) E is finite dimensional.

Proof.

We only have to prove that (2) implies (3).

Let us denote by $\mathcal{P}_f(\mathbb{N})$ the set of finite subsets of $\mathbb{N}$ and by $F = \mathcal{C}_0(\mathbb{N})\widehat{\otimes}E$ the projective tensor product of $\mathcal{C}_0(\mathbb{N})$, the space of (continuous) functions from the locally compact set $\mathbb{N}$ into $\mathbb{R}$ tending to 0 at infinity, with E.

We begin by proving that if the additive set function m defined from $\mathcal{P}_f(\mathbb{N})$ to $\mathcal{L}(E,F)$ by $m(A)(x) = \varphi_A \otimes x$ has finite semi-variation, then E is finite dimensional.

In that case, the canonical integration mapping $\int . \, dm$ from $S(\mathbb{N}, \mathcal{P}_f(\mathbb{N}), E)$ into F is continuous if $S(\mathbb{N}, \mathcal{P}_f(\mathbb{N}), E)$, the space of $\mathcal{P}_f(\mathbb{N})$-step functions from $\mathbb{N}$ to E, is equipped with the uniform convergence topology.

As $\int \sum_{i=1}^n \varphi_{A_i} x_i \, dm = \sum_{i=1}^n m(A_i)x_i = \sum_{i=1}^n m(A_i) \otimes x_i$, this canonical mapping is the canonical embedding of $S(\mathcal{P}_f(\mathbb{N}), \mathbb{N}, E)$ into $\mathcal{C}_0(\mathbb{N})\widehat{\otimes}E$.

The space $S(\mathcal{P}_f(\mathbb{N}), \mathbb{N}, E)$ being dense in $\mathcal{C}_0(\mathbb{N}, E)$, this canonical mapping may be continuously extended to $\mathcal{C}_0(\mathbb{N}, E)$.

As $\mathcal{C}_0(\mathbb{N}, E)$ may be identified with $\mathcal{C}_0(\mathbb{N})\hat{\hat{\otimes}}E$, the topology of $\mathcal{C}_0(\mathbb{N})\hat{\hat{\otimes}}E$ if finer than the topology of $\mathcal{C}_0(\mathbb{N})\widehat{\otimes}E$ and the two topologies coincide on the tensor products. It follows from prop. 33 p. 152 of chap.I of [3] that E is finite dimensional.

We now prove that (2) implies (3).

Let us suppose that E is infinite dimensional and choose for T the set of the integers, $\mathcal{A}$ the set of subsets of T and $F = \mathcal{C}_0(\mathbb{N})\widehat{\otimes}E$.

We will build a bounded family $(m_n)_{n\in\mathbb{N}}$ of uniformly σ-additive set functions defined on $\mathcal{A}$ with values in $\mathcal{L}(E,F)$ which doesn't have uniformly bounded semi-variation.

According to proposition 1, E will not have property (P_σ).

If E is infinite dimensional, the additive set function m defined on $\mathcal{P}_f(\mathbb{N})$ into $\mathcal{L}(E,F)$ by $m(A)(x) = \varphi_A \otimes x$ doesn't have finite semi-variation.

It follows that for each integer n, there exists a finite number p_n of finite subsets of $\mathbb{N}$, $A_{1,n}, A_{2,n}, \cdots, A_{p_n,n}$ which are mutually disjoint and vectors $x_{1,n}, x_{2,n}, \cdots, x_{p_n,n}$ in the unit ball of E such that

$$\| \sum_{i=1}^{p_n} m(A_{i,n})(x_{i,n}) \| \geq n.$$

Let us denote by B_n the union $\cup_{i=1}^{p_n} A_{i,n}$ which is a subset in $\mathcal{P}_f(\mathbb{N})$.

For $n \geq 1$, we define $m_n : \mathcal{A} \longrightarrow \mathcal{L}(E,F)$ by

$$m_n(A) = \frac{1}{\sqrt{n}} m(A \cap B_n).$$

As each m_n is σ-additive and $\|m_n(A)\| \leq \frac{1}{\sqrt{n}}$, the family $(m_n)_{n\geq 1}$ is uniformly bounded and uniformly σ-additive.

Nevertheless, the family $(m_n)_{n\geq 1}$ doesn't have uniformly finite semi-variation as $\| \sum_{n=1}^{p_n} m_n(A_{i,n})(x_{i,n})\| \geq \sqrt{n}$. ■

4 The locally convex case

We recall that if E and F are locally convex spaces and m is a set function from $\mathcal{A}$ to $\mathcal{L}(E,F)$, then m has finite semi-variation if for any continuous semi-norm p on F, there exists a continuous semi-norm q on E such that

$$\sup\{q(\sum_{i=1}^{n} m(A_i)(x_i)) \mid n \in \mathbb{N}^*, A_i \text{ mutually disjoint in } \mathcal{A}, x_i \in E : p(x_i) \leq 1\} < \infty.$$

It follows that m is of finite semi-variation if for any continuous semi-norm p on F there exists a continuous semi-norm q on E such that the set function $m_{p,q}$ obtained by composing m with the canonical injection from the space $\mathcal{L}(E,F)$ to $\mathcal{L}(E_q,F_p)$ is of finite semi-variation where E_q (resp. F_p) denotes the completion of the quotient of E (resp. F) by the kernel of the semi-norm p (resp. q).

As in the Banach case, m is of finite semi-variation if and only if the integral mapping from the space $S(X,\mathcal{A},E)$ into F is continuous.

Following the Banach case, we will say that a locally convex space E has property (P) if for each set T, each ring $\mathcal{A}$ of subsets of T and each locally convex space F, each bounded additive set function from $\mathcal{A}$ to $\mathcal{L}(E,F)$ has finite semi-variation and that a locally convex space E has property (P_σ) if for each set T, each σ-ring $\mathcal{A}$ of subsets of T and each locally convex space F, each countably additive set function from $\mathcal{A}$ to $\mathcal{L}(E,F)$ has finite semi-variation.

The proof of the following theorem is very easy. It only uses the fact that a locally convex space E is nuclear if and only if the Banach space obtained by completion of the quotient of E by the kernel of a continuous semi-norm is finite dimensional.

Theorem 3 *The following are equivalent :*
(1) The locally convex space E has property (P).
(2) The locally convex space E has property (P_σ).
(3) The locally convex space E is nuclear.

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