Spectral measures and automatic continuity

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Let X be a locally convex Hausdorff space (briefly, lcHs) and $L(X)$ denote the space of all continuous linear operators of X into itself. The space $L(X)$ is denoted by $L_s(X)$ when it is equipped with the topology of pointwise convergence in X (i.e. the strong operator topology). By a *spectral measure* in X is meant a σ-additive map $P : \Sigma \to L_s(X)$, defined on a σ-algebra Σ of subsets of some set Ω, which is multiplicative (i.e. $P(E \cap F) = P(E)P(F)$ for $E, F \in \Sigma$) and satisfies $P(\Omega) = I$, the identity operator in X. This concept is a natural extension to Banach and more general lc-spaces X of the notion of the resolution of the identity for normal operators in Hilbert spaces, [7,11,19,21].

Since a spectral measure $P : \Sigma \to L_s(X)$ is, in particular, a vector measure (in the usual sense, [9]) it has an associated space $\mathcal{L}^1(P)$ of P-integrable functions. For each $x \in X$, there is an induced X-valued vector measure $Px : \Sigma \to X$ defined by $Px : E \mapsto P(E)x$, for $E \in \Sigma$, and its associated space $\mathcal{L}^1(Px)$ of Px-integrable functions. It is routine to check that every ($\mathbb{C}$-valued) function $f \in \mathcal{L}^1(P)$ necessarily belongs to $\mathcal{L}^1(Px)$, for each $x \in X$, and that the continuous linear operator $P(f) = \int_\Omega f dP$ in X satisfies $P(f)x = \int_\Omega f d(Px)$, for each $x \in X$.

The topic of this note is the converse question. Namely, suppose that f is a $\mathbb{C}$-valued, Σ-measurable function which belongs to $\mathcal{L}^1(Px)$, for each $x \in X$. Then the map $P_{[f]} : X \to X$ defined by

$$(1) \qquad P_{[f]} : x \mapsto \int_\Omega f d(Px), \qquad x \in X,$$

is linear and everywhere defined. There arises the question of when is $P_{[f]}$ continuous? Besides its intrinsic interest as a question about automatic continuity it is also a question of practical relevance since, in the treatment of concrete examples, it is typically easier to establish integrability with respect to the X-valued measure

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Px (for each $x \in X$) than to establish integrability directly with respect to the $L_s(X)$-valued measure P.

The most general result concerned with this question (for f satisfying $f \in \mathcal{L}^1(Px)$, for each $x \in X$) is Proposition 1.2 of [5], which states that $P_{[f]}$ given by (1) is continuous whenever the following conditions are satisfied:

(2.i) X is quasicomplete,

(2.ii) $L_s(X)$ is sequentially complete, and

(2.iii) $P(\Sigma) = \{P(E); E \in \Sigma\}$ is an equicontinuous subset of $L(X)$.

The proof of this result is based on an elegant automatic continuity result of P.G. Dodds and B. de Pagter, [3; Corollary 5.7], which states (under (2.i) and (2.ii)) that if $\mathcal{M}$ is a strongly equicontinuous Boolean algebra of projections ($\mathcal{M} = P(\Sigma)$ in (2.iii) satisfies this) in $L(X)$, then an everywhere defined linear operator $T : X \to X$ which leaves invariant every closed, $\mathcal{M}$-invariant subspace of X is necessarily continuous.

Despite its generality, Proposition 1.2 of [5] is not applicable in many situations of interest. For instance, there exist quasicomplete (even complete) lcH-spaces X such that $L_s(X)$ fails to be sequentially complete; see [15; §3] and [17], for example. Similarly, there exist large classes of lcH-spaces X which are sequentially complete but not quasicomplete and, of course, spaces which are not even sequentially complete. Moreover, many spectral measures of interest fail to be equicontinuous; any non-trivial spectral measure in a Banach space with its weak topology or in a dual Banach space with its weak-star topology fails to be equicontinuous, [12; Proposition 4]. Accordingly, it would be useful to have available an analogue of Proposition 1.2 in [5] without such stringent hypotheses. We now present such an extension in which the assumptions (2.i)–(2.iii) are significantly relaxed. This is possible because the proof is no longer based on the automatic continuity result of Dodds and de Pagter, [3; Corollary 5.7], which relies on the theory of order and lc-Riesz spaces, but is based directly on the theory of integration.

In order to formulate this result recall that the *sequential closure* $[\Lambda]_s$ of a subset Λ of a topological space Z is the smallest subset W of Z which contains Λ and has the property that $z \in W$ whenever $z \in Z$ is the limit of a sequence from W. The space $[\Lambda]_s$ is equipped with the relative topology from Z. If Z is a lcHs and $m : \Sigma \to Z$ is a vector measure, then the sequential closure of $\Lambda = \operatorname{span}(m(\Sigma))$ is denoted simply by $[Z]_m$.

A spectral measure P is called *equicontinuous* if (2.iii) is satisfied. A lcHs X is said to have the *closed graph property* if every closed, linear map of X into itself is necessarily continuous. The vector space of all closed, linear maps of (all of) X into itself is denoted by $\mathcal{C}(X)$. In the following conditions X is a lcHs and P is a spectral measure in X :

(H1) X is barrelled.

(H2) X has the closed graph property and $\{P_{[f]}; f \in \cap_{x\in X}\mathcal{L}^1(Px)\} \subseteq \mathcal{C}(X)$.

(H3) $[L_s(X)]_P$ is sequentially complete.

Theorem 1. *Let X be a lcHs and P be a spectral measure in X. Suppose that any one of* (H1), (H2), (H3) *is satisfied. Then the linear operator $P_{[f]}$ given by* (1) *is continuous, for every $f \in \cap_{x\in X}\mathcal{L}^1(Px)$. In particular, $\mathcal{L}^1(P) = \bigcap_{x\in X} \mathcal{L}^1(Px)$.*

We remark that if $L_s(X)$ is sequentially complete (i.e. (2.ii) holds), then $[L_s(X)]_P$ is also sequentially complete. Examples 12 & 16 show that the sequential completeness of $[L_s(X)]_P$ is weaker than (2.ii). Moreover, (2.ii) implies that X itself must be sequentially complete; this is not the case for $[L_s(X)]_P$; i.e. $[L_s(X)]_P$ can be sequentially complete without X being sequentially complete (cf. Examples 12 & 16). In none of (H1)–(H3) is it necessary to assume condition (2.i). Indeed, no completeness properties what-so-ever are required of X. In (H1) the equicontinuity of P follows from the barrelledness of X (i.e. (2.iii) is present), but in (H2) and (H3) there is no requirement of P being equicontinuous. So, Theorem 1 is a genuine and non-trivial extension of Proposition 1.2 in [5].

1 Preliminaries

Let Σ be a σ-algebra of subsets of a non-empty set Ω and Z be a lcHs. A function $m : \Sigma \to Z$ is a *vector measure* if it is σ-additive. Given $z' \in Z'$ (the continuous dual space of Z), let $\langle z', m\rangle$ denote the complex measure $E \mapsto \langle z', m(E)\rangle$ for $E \in \Sigma$. Its total variation measure is denoted by $|\langle z', m\rangle|$. The space of all $\mathbb{C}$-valued, Σ-simple functions is denoted by $\mathrm{sim}(\Sigma)$.

A Σ-measurable function $f : \Omega \to \mathbb{C}$ is called *m-integrable* if it is $\langle z', m\rangle$-integrable for every $z' \in Z'$ and if, for each $E \in \Sigma$, there exists an element $\int_E f dm$ in Z such that $\langle z', \int_E f dm\rangle = \int_E f d\langle z', m\rangle$, for $z' \in Z'$. The linear space of all m-integrable functions is denoted by $\mathcal{L}^1(m)$; it contains $\mathrm{sim}(\Sigma)$.

A set $E \in \Sigma$ is called *m-null* if $m(F) = 0$ for every $F \in \Sigma$ such that $F \subseteq E$. A $\mathbb{C}$-valued, Σ-measurable function on Ω is said to be *m-essentially bounded* if it is bounded off an m-null set. The space of all m-essentially bounded functions is denoted by $\mathcal{L}^\infty(m)$. If $[Z]_m$ is sequentially complete, then

$$\mathcal{L}^\infty(m) \subseteq \mathcal{L}^1(m); \tag{3}$$

see [13, 18]. The inclusion (3) is not always valid; see [13], for example.

Let X be a lcHs and $P : \Sigma \to L_s(X)$ be a spectral measure. The multiplicativity of P implies that $E \in \Sigma$ is P-null iff $P(E) = 0$. Integrability with respect to P is simpler than for general vector measures. Given $f \in \mathcal{L}^1(P)$, the continuous operator $\int_\Omega f dP$ is also denoted by $P(f)$.

Lemma 2 ([15; Lemma 1.2]). *Let $P : \Sigma \to L_s(X)$ be a spectral measure. The following statements for a Σ-measurable function $f : \Omega \to \mathbb{C}$ are equivalent.*

(i) *The function f is P-integrable.*

(ii) *The function f is Px-integrable for each $x \in X$, and there is $T_1 \in L(X)$ such that $T_1 x = \int_\Omega f d(Px)$, for $x \in X$.*

(iii) *There exist functions $s_n \in \mathrm{sim}(\Sigma)$, for $n \in \mathbb{N}$, converging pointwise to f, such that $\{P(s_n)\}_{n=1}^\infty$ converges in $L_s(X)$ to some $T_2 \in L(X)$.*

(iv) *There exist functions $f_n \in \mathcal{L}^1(P)$, for $n \in \mathbb{N}$, converging pointwise to f, such that $\{P(f_n)\}_{n=1}^\infty$ converges in $L_s(X)$ to some $T_3 \in L(X)$.*

In this case $T_j = P(f)$, for each $j = 1, 2, 3$ and

$$\int_E f dP = P(f)P(E) = P(E)P(f), \qquad E \in \Sigma. \tag{4}$$

As a simple consequence we have the following useful result which is well known under conditions (2.i) and (2.ii); see [4; Lemma 1.3]. However, the proof given there is not directly applicable in our general setting since the dominated convergence theorem for vector measures may not be available and (3) may fail to hold, [13].

Corollary 2.1. *Let $P : \Sigma \to L_s(X)$ be a spectral measure and let $f, g \in \mathcal{L}^1(P)$. Then also $fg \in \mathcal{L}^1(P)$ and*

$$\int_E fg dP = P(f)P(g)P(E) = P(g)P(f)P(E), \qquad E \in \Sigma. \tag{5}$$

Proof. By Lemma 2, choose $s_n \in \operatorname{sim}(\Sigma)$, for $n \in \mathbb{N}$, such that $s_n \to g$ pointwise on Ω and $\int_E s_n dP \to \int_E g dP$ in $L_s(X)$, for each $E \in \Sigma$. Then (4) implies that $\int_E f s_n dP = (\int_E f dP) \cdot (\int_E s_n dP)$ for $E \in \Sigma,\ n \in \mathbb{N}$. Accordingly,

$$\lim_{n\to\infty} \int_E f s_n dP = \lim_{n\to\infty} \Big(\int_E f dP\Big) \cdot \Big(\int_E s_n dP\Big) = \Big(\int_E f dP\Big) \cdot \Big(\int_E g dP\Big)$$

in $L_s(X)$, for each $E \in \Sigma$. Since $s_n f \to gf$ pointwise on Ω, Lemma 2 implies that $fg \in \mathcal{L}^1(P)$ and (5) holds. ■

A spectral measure P in X has the *bounded-pointwise intersection property* if

$$\mathcal{L}^\infty(P) \cap \Big(\bigcap_{x\in X} \mathcal{L}^1(Px)\Big) \subseteq \mathcal{L}^1(P). \tag{6}$$

Example 3. Not every spectral measure has the bounded-pointwise intersection property. Let λ be Lebesgue measure on the Borel subsets Σ of $\Omega = [0, 1]$. Let $X = \mathcal{L}^1(\lambda)$, equipped with the lcH-topology $\sigma(\mathcal{L}^1(\lambda), \operatorname{sim}(\Sigma))$. The duality between $\mathcal{L}^1(\lambda)$ and $\operatorname{sim}(\Sigma)$ is the usual $\langle \mathcal{L}^1(\lambda), \mathcal{L}^\infty(\lambda)\rangle$ duality $\langle f, g\rangle = \int_\Omega fg d\lambda$, where $\operatorname{sim}(\Sigma)$ is interpreted as a linear subspace of $\mathcal{L}^\infty(\lambda)$. It is a consequence of the Vitali-Hahn-Saks and Radon-Nikodym theorems that X is sequentially complete. Let $P : \Sigma \to L_s(X)$ be the spectral measure given by $P(E) : h \mapsto \chi_E h$ for $h \in X$ and $E \in \Sigma$. It is routine to check, for each $h \in X$, that $\mathcal{L}^1(Ph)$ is the space of all Σ-measurable functions $\varphi : \Omega \to \mathbb{C}$ such that $\varphi h \in X$, with integrals given by $\int_E \varphi d(Ph) = \chi_E \varphi h$ for $E \in \Sigma$. It follows that the left-hand-side of (6) is equal to $\mathcal{L}^\infty(\lambda)$. However, it is also routine to check that $\mathcal{L}^1(P) = \operatorname{sim}(\Sigma)$ with $P(\varphi)$ being the operator in X of multiplication of φ, for each $\varphi \in \operatorname{sim}(\Sigma)$. Hence, (6) fails to hold. ■

Lemma 4. *Every equicontinuous spectral measure has the bounded-pointwise intersection property.*

Proof. Let $P : \Sigma \to L_s(X)$ be an equicontinuous spectral measure. Fix $f \in \mathcal{L}^\infty(P) \cap (\cap_{x\in X} \mathcal{L}^1(Px))$. Choose $s_n \in \operatorname{sim}(\Sigma)$, for $n \in \mathbb{N}$, satisfying $|s_n| \le |f|$ and such that $s_n \to f$ uniformly on Ω. Fix $x \in X$. By the dominated convergent theorem applied to the vector measure Px, interpreted as taking its values in the completion $\overline{X}$ of X, [9; II Theorem 4.2], it follows that $\int_\Omega s_n d(Px) \to \int_\Omega f d(Px)$ in $\overline{X}$. But, since $f \in \mathcal{L}^1(Px)$ the integral $\int_\Omega f d(Px)$ is actually an element of X rather than $\overline{X}$. This shows that $P(s_n)x \to P_{[f]}x$, i.e. $\{P(s_n)\}_{n=1}^\infty$ converges pointwise on X to the operator $P_{[f]}$.

Let q be a continuous seminorm in X, in which case $q(x) = \sup\{|\langle x', x\rangle|; x' \in W\}$, for $x \in X$, for some equicontinuous set $W \subseteq X'$. Given any $g \in \mathcal{L}^1(P)$ satisfying $|g| \leq |f|$ we have

$$q(P(g)x) = \sup_{x' \in W} |\int_\Omega g d\langle x', Px\rangle| \leq \sup_{x' \in W} \int_\Omega |f| d|\langle x', Px\rangle| \leq 4\|f\|_\infty \tilde{q}(x),$$

for each $x \in X$, where $\tilde{q}$ is the continuous seminorm in X corresponding to the equicontinuous subset $\tilde{W} = \{P(E)'x'; E \in \Sigma, x' \in W\}$ of X'. This shows that $H = \{P(g); g \in \mathcal{L}^1(P),\ |g| \leq |f|\}$ is an equicontinuous subset of $L(X)$. Since $\{P(s_n)\}_{n=1}^\infty \subseteq H$ it follows that the pointwise limit operator $P_{[f]}$ is actually continuous, i.e. $P_{[f]} \in L(X)$. Lemma 2 implies that $f \in \mathcal{L}^1(P)$. ∎

Lemma 4 shows that the spectral measure in Example 3 cannot be equicontinuous.

Example 5. Let $X = \ell^2$ equipped with its weak topology $\sigma(\ell^2, \ell^2)$, in which case X is a quasicomplete lcHs. Let $\Sigma = 2^{\mathbb{N}}$ and $P : \Sigma \to L_s(X)$ be the spectral measure given by $P(E) : x \mapsto \chi_E x$ (co-ordinatewise multiplication), for $x \in X, E \in \Sigma$. Then P is surely not equicontinuous, [12; Proposition 4]. However, it is routine to check that $\mathcal{L}^1(P) = \mathcal{L}^\infty(P) = \ell^\infty$ (cf. [15; Example 3.8], for example) and so P has the bounded-pointwise intersection property. This shows that equicontinuity is not necessary for the bounded-pointwise intersection property. ∎

Given a spectral measure $P : \Sigma \to L_s(X)$ and a Σ-measurable function $\varphi : \Omega \to \mathbb{C}$, let $\varphi^{[n]} \in \mathcal{L}^\infty(P)$ denote the function $\varphi\chi_{E(n)}$ where $E(n) = \{\omega \in \Omega; |\varphi(\omega)| \leq n\}$ for each $n \in \mathbb{N}$. The *integration map* $I_P : \mathcal{L}^1(P) \to L(X)$ is defined by $I_P : f \mapsto \int_\Omega f dP$, for $f \in \mathcal{L}^1(P)$.

Lemma 6. *Let $P : \Sigma \to L_s(X)$ be a spectral measure with the bounded-pointwise intersection property. Let $f \in \cap_{x \in X} \mathcal{L}^1(Px)$. Then $\{P(f^{[n]})\}_{n=1}^\infty$ is a Cauchy sequence in $L_s(X)$ with $f^{[n]} \in \mathcal{L}^1(P)$, for each $n \in \mathbb{N}$, and*

$$P(f^{[n]}) \in [L_s(X)]_P, \qquad n \in \mathbb{N}. \tag{7}$$

Moreover, $P(f^{[n]}) \to P_{[f]}$ pointwise on X.

Proof. By hypothesis $f^{[n]} \in \mathcal{L}^1(P)$ for $n \in \mathbb{N}$. Accordingly, $P(f^{[n]}) \in I_P(\mathcal{L}^1(P))$ for each $n \in \mathbb{N}$. Since $[L_s(X)]_P$ is the sequential closure of $I_P(\mathcal{L}^1(P))$ in the lcHs $L_s(X)$, [13; p.347], it is clear that (7) holds.

Fix $x \in X$. Since $|f^{[n]}| \leq |f|$, for $n \in \mathbb{N}$, and $f^{[n]} \to f$ pointwise on Ω it follows from the dominated convergence theorem applied to Px, interpreted as taking its values in the completion $\overline{X}$ of X, that $\int_\Omega f^{[n]} d(Px) \to \int_\Omega f d(Px)$ in $\overline{X}$. Since $f \in \mathcal{L}^1(Px)$ the integral $\int_\Omega f d(Px)$ is actually an element of X. Moreover, since $f^{[n]} \in \mathcal{L}^1(P)$ we have $\int_\Omega f^{[n]} d(Px) = P(f^{[n]})x$ for $n \in \mathbb{N}$. Accordingly, $P(f^{[n]})x \to \int_\Omega f d(Px) = P_{[f]}x$, for each $x \in X$. In particular, $\{P(f^{[n]})\}_{n=1}^\infty$ is Cauchy in $L_s(X)$. ∎

2 Proof of Theorem 1

Assume that (H1) is satisfied. The σ-additivity of P implies that $P(\Sigma)$ is a bounded subset of $L_s(X)$. So, the barelledness of X ensures that P is equicontinuous. Then P has the bounded-pointwise intersection property by Lemma 4.

Let $f \in \cap_{x\in X}\mathcal{L}^1(Px)$. Lemma 6 implies that $P(f^{[n]}) \to P_{[f]}$ pointwise on X and that $P(f^{[n]}) \in L(X)$ for each $n \in \mathbb{N}$. Accordingly, the Banach-Steinhaus theorem for barrelled spaces ensures that $P_{[f]} \in L(X)$. Lemma 2 guarantees that $f \in \mathcal{L}^1(P)$.

Suppose now that (H2) is satisfied. If $f \in \cap_{x\in X}\mathcal{L}^1(Px)$, then (H2) implies that $P_{[f]} \in \mathcal{C}(X)$ and hence, the closed graph property of X ensures that $P_{[f]} \in L(X)$. Again Lemma 2 guarantees that $f \in \mathcal{L}^1(P)$.

Finally assume that (H3) is given. It has been noted (cf. (3)) that the sequential completeness of $[L_s(X)]_P$ guarantees that $\mathcal{L}^\infty(P) \subseteq \mathcal{L}^1(P)$ and hence, P has the bounded-pointwise intersection property. Let $f \in \cap_{x\in X}\mathcal{L}^1(Px)$. Lemma 6 implies that $\{P(f^{[n]})\}_{n=1}^\infty$ is a Cauchy sequence in $[L_s(X)]_P$ and hence, there is $T \in L(X)$ such that $P(f^{[n]}) \to T$ in $L_s(X)$. Since also $P(f^{[n]}) \to P_{[f]}$ pointwise on X, by Lemma 6, it follows that $T = P_{[f]}$ and so $P_{[f]} \in L(X)$. Lemma 2 shows that $f \in \mathcal{L}^1(P)$. ∎

3 Conditions (H1)–(H3) and examples

In this section we exhibit various criteria which show that conditions (H1)–(H3) are indeed quite general. We also present a series of relevant examples which illustrate various related phenomena.

In the proof of Theorem 1, under assumption (H1), the barrelledness of X played two distinct roles. Firstly to ensure that P is equicontinuous and secondly to guarantee that the pointwise limit operator $P_{[f]}$ is actually continuous. The equicontinuity of P actually follows from the weaker hypothesis that X is merely quasibarrelled, [15; Lemma 1.3]. This is a substantial improvement since all metrizable lcH-spaces (in particular, all normed spaces) are quasibarrelled. However, to exhibit interesting examples of (non-complete) metrizable spaces which are actually barrelled is not so straightforward (cf. Example 12), although non-complete barrelled spaces exist in every infinite dimensional Banach space, [1; p.3, Exercise 6]. Unfortunately, the following example shows that it is not possible in (H1) to relax the barrelledness of X to quasibarrelledness.

Example 7. Let $X = c_{00}$ denote the dense subspace of the Banach space ℓ^1 consisting of those elements of ℓ^1 with only finitely many non-zero co-ordinates. As noted above X is quasibarrelled. Define a spectral measure P on $\Sigma = 2^{\mathbb{N}}$ by $P(E)x = \chi_E x$ (co-ordinatewise multiplication), for $E \in \Sigma$ and $x \in X$. For each $x \in X$, the space $\mathcal{L}^1(Px) = \mathbb{C}^{\mathbb{N}}$ consists of all functions $\varphi : \mathbb{N} \to \mathbb{C}$; the integrals are given by $\int_E \varphi d(Px) = \chi_E \varphi x$ for $E \in \Sigma$. Let $f(n) = n$ for each $n \in \mathbb{N}$, in which case $f \in \cap_{x\in X}\mathcal{L}^1(Px)$. However, $P_{[f]} : X \to X$ is the linear operator given by $P_{[f]} : x \mapsto f \cdot x$ for each $x \in X$, which is surely not continuous. ∎

Concerning (H2), it is known that all Fréchet lcH-spaces (and many others) have the closed graph property; such spaces are also barrelled. However, there also exist large classes of spaces with the closed graph property which are not barrelled. This is the case for every infinite dimensional Banach space equipped with its weak topology. For reflexive Banach spaces this shows there even exist quasicomplete spaces with the closed graph property which are not barrelled (as they are not Mackey spaces). It would be interesting to know whether or not all barrelled lc-spaces have the closed graph property. The inclusion

$$\{P_{[f]}; f \in \bigcap_{x\in X} \mathcal{L}^1(Px)\} \subseteq \mathcal{C}(X), \tag{8}$$

which is a requirement of (H2), is not always easy to check in practice. So, the following criterion is of some interest; it is known to hold under the additional hypotheses (2.i) and (2.ii), [4; Proposition 1.8], which are *not* assumed here.

Proposition 8. *Every equicontinuous spectral measure satisfies* (8).

Proof. Let $P : \Sigma \to L_s(X)$ be an equicontinuous spectral measure and $f \in \cap_{x\in X}\mathcal{L}^1(Px)$. Let $D_f = \{x \in X; \lim_{n\to\infty} \int_\Omega f^{[n]} d(Px)$ exists in $X\}$. Fix $x \in X$. The dominated convergence theorem applied to Px (considered as taking its values in the completion of X), the fact that $f \in \mathcal{L}^1(Px)$ and the fact that $f^{[n]} \to f$ pointwise on Ω, together imply that $x \in D_f$ as the limit

$$P_{[f]}x = \lim_{n\to\infty} \int_\Omega f^{[n]} d(Px) = \int_\Omega f d(Px)$$

actually exists in X. Hence, $D_f = X$ and $P_{[f]}$ is precisely the operator "$P(f)$" with domain $D(P(f))$ as defined on p.146 of [4]. An examination of the proof of Proposition 1.8 in [4] shows that the hypotheses (2.i) and (2.ii) which are assumed there are only used to ensure that all essentially bounded functions are P-integrable and Px-integrable, for each $x \in X$. However, once f is specified (in our case from $\cap_{x\in X}\mathcal{L}^1(Px)$)) the only bounded Σ-measurable functions which are actually considered in the proof of [4; Proposition 1.8] are the truncated functions $f^{[n]}$, for $n \in \mathbb{N}$. But, we can apply Lemma 4 to establish that each $f^{[n]} \in \mathcal{L}^1(P)$, for $n \in \mathbb{N}$. Equipped with this fact the proof of Proposition 1.8 in [4] carries over easily to the present setting to show that $P_{[f]} \in \mathcal{C}(X)$. ∎

The next example shows that the equicontinuity requirement of Proposition 8 is sufficient but not necessary for (8) to hold.

Example 9. Let $X = c_{00}$ (cf. Example 7 for the notation) equipped with the lcH-topology $\sigma(c_{00}, \ell^\infty)$. Define a spectral measure P on $\Sigma = 2^{\mathbb{N}}$ as in Example 7. By the same reasoning as in Example 7 it is routine to check that $\cap_{x\in X}\mathcal{L}^1(Px) = \mathbb{C}^{\mathbb{N}}$. Moreover, for each $f \in \cap_{x\in X}\mathcal{L}^1(Px)$, the linear operator $P_{[f]} : X \to X$ is given by $x \mapsto f \cdot x$ (co-ordinatewise multiplication), for $x \in X$. It is straightforward to check that $P_{[f]} \in \mathcal{C}(X)$ and hence, (8) is satisfied. However, P is not equicontinuous, [15; Example 3.8]. ∎

We recall an alternative description of the sequential closure of a subset Λ of a topological space Z. Let $[\Lambda]$ denote the set of all elements in Z which are the limit of some sequence of points from Λ. Define $\Lambda_0 = \Lambda$. Let Ω_1 be the smallest uncountable ordinal. Suppose that $0 < \alpha < \Omega_1$ and that Λ_β has been defined for all ordinals $\beta \in [0, \alpha)$. Define $\Lambda_\alpha = [\cup_{0\le\beta<\alpha}\Lambda_\beta]$. Then $\cup_{0\le\alpha<\Omega_1}\Lambda_\alpha$ is the sequential closure of Λ in Z.

The following result shows that (H1) together with some mild completeness requirement of the *(sequential) cyclic* spaces $[X]_{Px}$, for each $x \in X$, ensures that (H3) is satisfied.

Proposition 10. *Let X be a barrelled lcHs and $P : \Sigma \to L_s(X)$ be a spectral measure with $[X]_{Px}$ sequentially complete for each $x \in X$. Then $[L_s(X)]_P$ is sequentially complete.*

Proof. Let $\{T_n\}_{n=1}^\infty$ be a Cauchy sequence in $[L_s(X)]_P$. If $Y = \mathrm{span}\{P(\Sigma)\}$, then $[L_s(X)]_P = [Y]_P$ by definition and so there exists an ordinal number $\alpha \in [0, \Omega_1)$ such that $\{T_n\}_{n=1}^\infty \subseteq Y_\alpha$. Fix $x \in X$ and let $Z(x) = \mathrm{span}\{Px(\Sigma)\}$. Now, $\{T_n x\}_{n=1}^\infty$ is Cauchy in X. But, it is easy to see that actually $\{T_n x\}_{n=1}^\infty \subseteq (Z(x))_\alpha \subseteq [X]_{Px}$ and hence, there exists $Tx \in [X]_{Px}$ such that $T_n x \to Tx$. Since X is barrelled, the Banach-Steinhaus theorem ensures that the everywhere defined linear operator $T : X \to X$ specified by $Tx = \lim_{n\to\infty} T_n x$, for $x \in X$, is continuous. By definition $T \in [Y_\alpha] \subseteq [L_s(X)]_P$. ∎

Remark 11. (i) In general, it is not possible to omit the barrelledness assumption in Proposition 10 or to replace it with the weaker condition of quasibarrelledness. For, let X and P be as in Example 7. Then X is quasibarrelled (being a normed space) and, being finite dimensional, $[X]_{Px}$ is complete for each $x \in X$. However, $[L_s(X)]_P$ is not sequentially complete. To see this define operators $T_n \in \mathrm{span}\{P(\Sigma)\}$ by $T_n = \Sigma_{k=1}^n kP(\{k\})$, for each $n \in \mathbb{N}$. Then $\{T_n\}_{n=1}^\infty$ is a Cauchy sequence in $[L_s(X)]_P$ with no limit in $L_s(X)$.

(ii) The lcHs $[L_s(X)]_P$ can be sequentially complete without the spaces $[X]_{Px}$, for $x \in X$, being sequentially complete. Indeed, let $X = c_0$ equipped with its weak topology $\sigma(c_0, \ell^1)$. Define a spectral measure P on $\Sigma = 2^{\mathbb{N}}$ by $P(E)x = \chi_E x$ (coordinatewise multiplication) for $E \in \Sigma$ and $x \in X$. Then the element $x_0 = \left(\frac{1}{n}\right)_{n=1}^\infty$ of X has the property that $[X]_{Px_0} = X$ is not sequentially complete.

Let $\{e_n\}_{n=1}^\infty$ and $\{h_n\}_{n=1}^\infty$ be the standard bases in c_0 and ℓ^1, respectively. To show that $[L_s(X)]_P$ is sequentially complete it suffices to show that $I_P(\mathcal{L}^1(P))$ is sequentially complete in $L_s(X)$; see Proposition 13. It is straightforward to verify that $\mathcal{L}^1(P) = \ell^\infty$ and, for each $f \in \mathcal{L}^1(P)$, the operator $P(f)$ is (co-ordinatewise) multiplication (in X) by f. Let $\{f_n\}_{n=1}^\infty \subseteq \ell^\infty$ be functions such that $\{P(f_n)\}_{n=1}^\infty$ is Cauchy in $L_s(X)$. Then, given $m \in \mathbb{N}$, the sequence $\langle h_m, P(f_n)e_m\rangle = f_n(m)$, for $n \in \mathbb{N}$, is Cauchy in $\mathbb{C}$ and hence, has a limit, say $f(m)$. Moreover, for each $x \in X$, the Cauchy sequence $\{P(f_n)x\}_{n=1}^\infty$ is bounded in X and hence, is norm bounded in c_0. Since each operator $P(f_n)$, for $n \in \mathbb{N}$, is also continuous for the norm topology for c_0 it follows from the principle of uniform boundedness that $\sup_n \|f_n\|_\infty = \sup_n \|P(f_n)\| < \infty$. Accordingly, $f \in \ell^\infty$. To establish that $P(f_n) \to P(f)$ in $L_s(X)$ it suffices to show that $\langle h, P(f_n)x\rangle \to \langle h, P(f)x\rangle$, for each $h \in \ell^1$ and $x \in c_0$. Since $\langle h, P(g)x\rangle = \int_{\mathbb{N}} g d\langle h, Px\rangle$, for each $g \in \mathcal{L}^1(P)$, this follows from the dominated convergence theorem applied to the complex measure $\langle h, Px\rangle$. ∎

We present some non-trivial applications of Proposition 10.

Example 12. (A) An increasing sequence $\{k(n)\}_{n=1}^\infty \subseteq \mathbb{N}$ is said to have *density zero* if $\lim_{n\to\infty} \dfrac{n}{k(n)} = 0$. For $\xi \in \ell^1$, let $\mathrm{supp}(\xi) = \{n \in \mathbb{N};\ \xi_n \neq 0\}$. Let X denote the dense subspace of ℓ^1 consisting of all elements ξ for which $\mathrm{supp}(\xi)$ has density zero.

The normed space X is barrelled, [10; p.369], and clearly non-complete. Define a spectral measure P on $\Sigma = 2^{\mathbb{N}}$ by $P(E)\xi = \chi_E \xi$ for $E \in \Sigma$ and $\xi \in X$. Fix $\xi \in X$. It is routine to check that $[X]_{P\xi} = \{\eta \in \ell^1;\ \mathrm{supp}(\eta) \subseteq \mathrm{supp}(\xi)\}$ which is complete, being isomorphic to ℓ^1 if $\mathrm{supp}(\xi)$ is an infinite set and finite dimensional otherwise. Proposition 10 implies that $[L_s(X)]_P$ is sequentially complete (i.e. (H3) is satisfied), whereas neither of (2.i) nor (2.ii) is satisfied.

(B) Let Y be an infinite dimensional Banach space and $(\Omega, \Sigma, \lambda)$ be a finite, positive measure space. Let $X = \mathcal{P}(\lambda, Y)$ denote the space of all Y-valued, Pettis λ-integrable functions equipped with the norm $\|f\| = \sup_{\|y'\| \le 1} \int_\Omega |\langle y', f(w)\rangle| d\lambda(w)$, for $f \in X$; see [2; p.224]. The normed space X is always barrelled, [6], and is non-complete whenever λ is non-atomic, [20; p.131]. For each $\varphi \in \mathcal{L}^\infty(\lambda)$ define $M_\varphi f : w \mapsto \varphi(w) f(w)$, for a.e. $w \in \Omega$ and $f \in X$. Then $M_\varphi \in L(X)$ and $\|M_\varphi\| \le \|\varphi\|_\infty$. For each $E \in \Sigma$ let $P(E) = M_{\chi_E}$. It turns out that $P : \Sigma \to L_s(X)$ is a spectral measure and it can be shown (with some effort!) that $[X]_{Pf}$ is complete for each $f \in X$. Proposition 10 shows that $[L_s(X)]_P$ is sequentially complete.

(C) Let $\mathcal{A}$ denote the set of all functions $a : \mathbb{N} \to \{0,1\}$ such that $n^{-1}\Sigma_{k=1}^n a(k) \to 0$. Let X denote the dense subspace of the Fréchet lcHs $\mathbb{C}^{\mathbb{N}}$ (topology of pointwise convergence on $\mathbb{N}$) consisting of all elements of the form $x = a \cdot z$ (co-ordinatewise multiplication) for some $a \in \mathcal{A}$ and $z \in \mathbb{C}^{\mathbb{N}}$. The non-complete, metrizable lcHs X is barrelled, [8; pp.59–60]. Define a spectral measure P on $\Sigma = 2^{\mathbb{N}}$ by $P(E) = \chi_E x$ for $E \in \Sigma$ and $x \in X$. Fix $x \in X$. Direct calculation shows that $[X]_{Px} = \{z \in \mathbb{C}^{\mathbb{N}};\ \mathrm{supp}(z) \subseteq \mathrm{supp}(x)\}$ which is complete, being isomorphic to $\mathbb{C}^{\mathbb{N}}$ if $\mathrm{supp}(x)$ is an infinite set and finite dimensional otherwise. Again Proposition 10 implies condition (H3). ∎

Proposition 10 gives one criterion for checking condition (H3). But, it is restricted to the class of barrelled spaces. Another criterion, of a different kind is given by the following

Proposition 13. *Let $P : \Sigma \to L_s(X)$ be a spectral measure. If the range $I_P(\mathcal{L}^1(P))$ of the integration map is a sequentially complete subspace of $L_s(X)$, then $[L_s(X)]_P = I_P(\mathcal{L}^1(P))$. In particular, $[L_s(X)]_P$ is sequentially complete.*

Proof. Clearly, if Y is a sequentially complete subspace of a lcHs Z, then the sequential closure of Y in Z coincides with Y itself. But, it was noted in the proof of Lemma 6 that the sequential closure of $I_P(\mathcal{L}^1(P))$ in $L_s(X)$ is precisely $[L_s(X)]_P$. ∎

Propositions 10 and 13 provide criteria which guarantee the sequential completeness of $[L_s(X)]_P$. In the event that these sufficient conditions are not satisfied it seems useful to have available further techniques which can be used to test whether or not (H3) is satisfied. We proceed to formulate such a criterion.

A vector measure $m : \Sigma \to Z$, with Z a lcHs, is called *countably determined*, [14], if there exists a countable set $\{x'_n\}_{n=1}^\infty \subseteq X'$ with the property that a set $E \in \Sigma$ is m-null iff it is $\langle x'_n, m\rangle$-null for each $n \in \mathbb{N}$.

Proposition 14. *Let X be a lcHs and $P : \Sigma \to L_s(X)$ be a spectral measure which is countably determined. Then $[L_s(X)]_P$ coincides with the range $I_P(\mathcal{L}^1(P))$ of the integration map I_P.*

Proof. Suppose that $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{L}^1(P)$ has the property that $\{I_P(f_n)\}_{n=1}^{\infty}$ is convergent in $L_s(X)$, to say $T \in L(X)$. It follows from (4) that $\{I_P(\chi_E f_n)\}_{n=1}^{\infty}$ is also convergent in $L_s(X)$, for $E \in \Sigma$ (to $P(E)T$, of course). Hence, I_P is Σ-*converging* in the sense of [14; §2]. Since P is countably determined the conclusion follows from, [14; Proposition 2.6]. ■

Remark 15. If X is a separable, metrizable lcHs, then every spectral measure in X is countably determined, [14; Corollary 1.8]. The spectral measure P of Remark 11(ii) is countably determined, but X is not metrizable. ■

Proposition 14 shows that if P is countably determined, then deciding about the non-sequential completeness of $[L_s(X)]_P$ which is defined via a long transfinite procedure, reduces to deciding about the non-sequential completeness of the range of the integration map $I_P : \mathcal{L}^1(P) \to L_s(X)$. A useful observation in this regard, at least for equicontinuous spectral measures, is the fact that I_P is then a bicontinuous isomorphism of the lcHs $\mathcal{L}^1(P)$, equipped with the topology of uniform convergence of indefinite integrals, [4; §1], onto its range $I_P(L^1(P))$ equipped with the relative topology from $L_s(X)$. This can be found in [16] after noting that the proof of the isomorphic property of I_P (onto its range) given there does not rely on the assumed completeness hypotheses of X and $L_s(X)$.

The following example shows that (H3) is a different type of condition than (H1).

Example 16. Let X denote the lcHs ℓ^1 equipped with the topology of uniform convergence on the relatively compact subsets of c_0. Then X is complete (with $X' = c_0$) but $L_s(X)$ is not even sequentially complete, [15; §3]. Since every sequentially complete, quasibarrelled space is barrelled it follows that X is not even quasibarrelled (otherwise $L_s(X)$ would be sequentially complete as a consequence of the Banach-Steinhaus theorem). So, (H1) is surely not satisfied.

Let $\Sigma = 2^{\mathbb{N}}$ and, for each $E \in \Sigma$, define $P(E) : X \to X$ by $P(E)\xi = \chi_E \xi$, for $\xi \in X$. Then $P(E)$ is continuous and $P : \Sigma \to L_s(X)$ is a spectral measure. From the fact that $\cup_{E\in\Sigma}\{\chi_E \xi; \xi \in K\}$ is a relatively compact subset of c_0 whenever K is a compact set in c_0 it follows that P is actually equicontinuous.

It is straightforward to check that $\mathcal{L}^1(P) = \ell^\infty$ and, for each $f \in \ell^\infty$, the operator $P(f) = I_P(f)$ is co-ordinatewise multiplication in X by f. Let $\{P(f_n)\}_{n=1}^{\infty}$ be a Cauchy sequence in $L_s(X)$, with $f_n \in \mathcal{L}^1(P)$ for each $n \in \mathbb{N}$. Fix $x \in X$. Since $\{P(f_n)x\}_{n=1}^{\infty}$ is Cauchy in X it is $\sigma(X, X')$-bounded, i.e. it is bounded as a subset of ℓ^1 for the weak-star topology $\sigma(\ell^1, c_0)$. Hence, $\{P(f_n)x\}_{n=1}^{\infty}$ is actually a norm bounded subset of ℓ^1. Since each multiplication operator $P(f_n)$, for $n \in \mathbb{N}$, is also continuous on the Banach space ℓ^1 it follows from the principle of uniform boundedness that $M = \sup_n \|P(f_n)\| < \infty$. Hence,

$$\|f_n\|_\infty = \sup_m |f_n(m)| = \sup_m \|P(f_n)h_m\|_1 \leq \|P(f_n)\| \leq M, \tag{9}$$

for each $n \in \mathbb{N}$.

Let $\{e_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ be as in Remark 11(ii). Since $\{\langle e_m, P(f_n)h_m\rangle\}_{n=1}^{\infty} = \{f_n(m)\}_{n=1}^{\infty}$ is Cauchy for each $m \in \mathbb{N}$ it follows from the completeness of $\mathbb{C}$ and (9) that there exists $f \in \ell^\infty$ such that $f_n \to f$ pointwise on $\mathbb{N}$. Fix $x \in X$. By (9) and the dominated convergence theorem for the vector measure Px in the complete space X it follows that $\int_\Omega f_n d(Px) \to \int_\Omega f d(Px)$ in X, i.e. $P(f_n)x \to P(f)x$. This shows that $P(f_n) \to P(f)$ in $L_s(X)$. Hence $I_P(L^1(P))$ is sequentially complete and so (H3) is satisfied by Proposition 13.

Since X is complete it is clear that $[X]_{Px}$ is sequentially complete, for each $x \in X$. This shows that the barrelledness assumption in Proposition 10 is not a necessary condition although, without it, the result fails in general; see Remark 11. ■

Examples 3 and 7 show that the conclusion of Theorem 1 is not satisfied for arbitrary spectral measures, even equicontinuous ones. The next example shows that, despite its generality, Theorem 1 does not cover "all cases".

Example 17. (A) Let $\mathbb{C}^{\mathbb{N}}$ denote the space of all $\mathbb{C}$-valued functions on $\mathbb{N}$ equipped with the pointwise convergence topology. Then $\mathbb{C}^{\mathbb{N}}$ is a Fréchet lcHs. Let $X = \ell^\infty$, equipped with the relative topology from $\mathbb{C}^{\mathbb{N}}$ in which case X is metrizable and hence, is a Mackey space. Since $X' = c_{00}$ is not quasicomplete for the weak-star topology $\sigma(c_{00}, \ell^\infty)$ it follows that X is not barrelled, [10; p.305]. Let $\Sigma = 2^{\mathbb{N}}$ and define a spectral measure $P : \Sigma \to L_s(X)$ by $P(E)x = \chi_E.x$ for $E \in \Sigma$ and $x \in X$. Since X is quasibarrelled P is necessarily equicontinuous. If $x = \mathbb{1}$ is the function constantly equal to 1 on $\mathbb{N}$, then it is routine to check that $[X]_{P\mathbb{1}} = X$ and so not all (sequential) cyclic spaces $[X]_{Px}$, for $x \in X$, are sequentially complete. It turns out that $\mathcal{L}^1(P) = \ell^\infty = \cap_{x\in X}\mathcal{L}^1(Px)$ and that I_P is an isomorphism of $\mathcal{L}^1(P)$ onto its range. From this it can be shown that $\mathcal{L}^1(P)$ is isomorphic (as a lcs) to ℓ^∞ with the subspace topology from $\mathbb{C}^{\mathbb{N}}$. Accordingly, $[L_s(X)]_P$ is not sequentially complete (by Proposition 14) as P is easily checked to be countably determined. Nevertheless, the conclusion of Theorem 1 is valid.

(B) There are situations where Theorem 1 applies, but not directly. For instance, let Y be any Mackey lcHs such that $L_s(Y)$ is sequentially complete (e.g. take for Y any sequentially complete, barrelled space) and let X denote Y equipped with its weak topology. Then typically (2.i)–(2.iii) do not hold for a general spectral measure $P : \Sigma \to L_s(X)$. Moreover, (H1)–(H3) either fail to hold or are difficult to verify. Nevertheless, the conclusion of Theorem 1 always holds, i.e. $\cap_{x\in X}\mathcal{L}^1(Px) = \mathcal{L}^1(P)$. Indeed, since Y has its Mackey topology it follows that $L(X) = L(Y)$ as linear spaces. Moreover, if P_Y denotes P considered as being $L(Y)$-valued, then the Orlicz-Pettis theorem together with $X' = Y'$ implies that $P_Y : \Sigma \to L_s(Y)$ is also a spectral measure. Since (H3) holds for P_Y it follows from Theorem 1 that $\cap_{y\in Y}\mathcal{L}^1(P_Y y) = \mathcal{L}^1(P_Y)$. But, it is clear from Lemma 2 that $\mathcal{L}^1(P_Y) = \mathcal{L}^1(P)$. Moreover, the definition of integrability with respect to a vector measure together with $X' = Y'$ implies that $\mathcal{L}^1(Pz) = \mathcal{L}^1(P_Y z)$, for every $z \in X = Y$. Hence, $\cap_{x\in X}\mathcal{L}^1(Px) = \mathcal{L}^1(P)$. ■

Proposition 14, with an argument along the lines of Example 17(A), provides one method for determining when (H3) fails. We also remark that if $\mathcal{L}^\infty(P) \nsubseteq \mathcal{L}^1(P)$, then $[L_s(X)]_P$ cannot be sequentially complete (cf. (3)). This simple test (which is not applicable to Example 17(A)) can sometimes be quite effective; see Example 3 and the following

Example 18. Let Ω, Σ and λ be as in Example 3 and let $X = \mathrm{sim}(\Sigma) \subseteq \mathcal{L}^\infty(\lambda)$ be equipped with the lcH-topology $\sigma(\mathrm{sim}(\Sigma), \mathcal{L}^1(\lambda))$. Define a spectral measure $P : \Sigma \to L_s(X)$ by $P(E)f = \chi_E f$ for $E \in \Sigma$ and $f \in X$. Then $\mathcal{L}^1(P) = \mathrm{sim}(\Sigma)$ and, for each $\varphi \in \mathcal{L}^1(P)$, the operator $P(\varphi)$ is multiplication in X by φ. Since $\mathcal{L}^\infty(P) \nsubseteq \mathcal{L}^1(P)$ we see that (H3) fails.

The sequential completion of X is $\widehat{X} = \mathcal{L}^\infty(\lambda)$ equipped with its weak-star topology. It follows that P is not equicontinuous since the extended spectral measure $\widehat{P} : \Sigma \to L_s(\widehat{X})$, given by $\widehat{P}(E)f = \chi_E f$ for $E \in \Sigma$ and $f \in \widehat{X}$, is not equicontinuous; see [15; Lemma 1.8] and [12; Proposition 4(i)]. Accordingly, X cannot be quasibarrelled. Since $P\mathbb{1}$ is the X-valued measure $E \mapsto \chi_E$ for $E \in \Sigma$, it is clear that $\mathcal{L}^1(P\mathbb{1}) = \text{sim}(\Sigma)$ and hence, that $\cap_{x\in X}\mathcal{L}^1(Px) = \text{sim}(\Sigma)$. So, the conclusion of Theorem 1 holds. ∎

Given a spectral measure $P : \Sigma \to L_s(X)$ the set function $P' : \Sigma \to L_s(X'_{\sigma(X',X)})$ defined by $P' : E \mapsto P(E)'$ (the dual operator), for $E \in \Sigma$, is also a spectral measure. Here $X'_{\sigma(X',X)}$ denote X' equipped with its weak-star topology $\sigma(X', X)$. It is clear that $\mathcal{L}^\infty(P) = \mathcal{L}^\infty(P')$ which poses the question of whether P' satisfies the conclusion of Theorem 1 whenever P does? Unfortunately, this is not the case in general. For, let X and P be as in Example 18, in which case the conclusion of Theorem 1 holds. Then X' is $\mathcal{L}^1(\lambda)$ equipped with the topology $\sigma(\mathcal{L}^1(\lambda), \text{sim}(\Sigma))$, which is precisely the lcHs of Example 3, and the dual spectral measure to P is also the spectral measure of Example 3. But, it was noted in Example 3 that the conclusion of Theorem 1 fails.

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