An equation $\dot{z} = z^2 + p(t)$ with no 2π-periodic solutions

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Abstract

The Mawhin conjecture - that there exists a 2π-periodic $p : \mathbf{R} \to \mathbf{C}$ such that $\dot{z} = z^2 + p(t)$ has no 2π-periodic solutions - is confirmed by the use of Fourier expansions.

In 1992 R.Srzednicki [4], [5] proved that for any 2π-periodic continuous $p : \mathbf{R} \to \mathbf{C}$ the equation $\dot{z} = \overline{z}^2 + p(t)$ has a 2π-periodic solution. J.Mawhin [3] conjectured that the similarly looking problem $\dot{z} = z^2 + p(t)$ could have no 2π-periodic solutions for some p. The first example of such p was constructed by J.Campos and R.Ortega [1]. This work was intended as an attempt to provide with another example by the use of a quite different method. During the preparation of this paper J.Campos [2] determined all the possible dynamics of this equation and found other examples.

Conjecture 1 *There exists $R_0 \in [1, 2]$ such that the equation*

$$\dot{z} = z^2 + Re^{it} \tag{1}$$

has no 2π-periodic solutions for $R = R_0$.

Let us define the sequence

$$a_1 = 1 , \quad a_n = \frac{1}{n} \sum_{k=1}^{n-1} a_k a_{n-k}. \tag{2}$$

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Conjecture 2 $\forall_{n\,>\,1}\;\; a_n^2 \;<\; a_{n-1}a_{n+1}$.

The main result of this paper is the following

Theorem 1 *Conjecture 2 implies Conjecture 1.*

The proof of this theorem will follow after two lemmas. It is easily seen that equation (1) is formally solved by

$$z_R(t) \;=\; \sum_{k=1}^{\infty}(-1)^k i e^{ikt} a_k R^k. \tag{3}$$

Lemma 1 *Let R_0 denote the radius of convergence of (3).*
Then (i) $R_0 \;\in\; [1,2]$,
(ii) $\forall_{R\in(-R_0,R_0)}\;\; z_R$ is a 2π-periodic solution of (1),
(iii) $\lim_{R\to R_0} i^{-1}z_R(\pi) \;=\; +\infty$.

Lemma 2 *If Conjecture 2 is true then*
(i) there is a seqence $R_n \;\in\; (0,R_0)$ convergent to R_0 such that the sequence $z_{R_n}(0)$ is convergent,
(ii) there exists $\lim_{R\to R_0-}\;\; \sum_{k=1}^{\infty}(-1)^k \frac{1}{k} a_k R^k$.

The lemmas will be proved later, now we use them to prove Theorem 1.

Proof of Theorem 1. To obtain a contradiction, suppose that there exists $s : \mathbf{R} \to \mathbf{C}$ which is a 2π-periodic solution of equation (1) for $R = R_0$. A standard argument shows that for R sufficiently close to R_0 there exists the solution $s_R : [0,2\pi] \to \mathbf{C}$ of (1) with initial condition $s_R(0) = s(0)$. Moreover, $\lim_{R\to R_0} s_R(t) = s(t)$, uniformly in $[0,2\pi]$. Let R_n be the sequence from Lemma 2 and $\omega = \lim_{n\to\infty} z_{R_n}(0)$. If $s(0) = \omega$ then $\lim_{n\to\infty} z_{R_n}(t) = s(t)$, uniformly in $[0,2\pi]$, contrary to Lemma 1 (iii). Thus $s(0) \neq \omega$ and $s_{R_n}(0) \neq z_{R_n}(0)$ for $n > n_0$. Functions s_{R_n}, z_{R_n} are two different solutions of Riccati equation (1). The standard computation shows that the function $u_n = \frac{1}{s_{R_n} - z_{R_n}}$ is a solution of the linear equation $\dot{u} = -2z_{R_n}u - 1$, so

$$u_n(2\pi) \;=\; \left[u_n(0) - \int_0^{2\pi} e^{2\int_0^t z_{R_n}(\tau)d\tau}dt\right] e^{-2\int_0^{2\pi} z_{R_n}(\tau)d\tau}.$$

From (3) we obtain

$$\int_0^{2\pi} z_{R_n}(\tau)d\tau = 0,\;\; \int_0^{t} z_{R_n}(\tau)d\tau = c_{0,n} + \sum_{k=1}^{\infty} c_{k,n}e^{ikt},$$

where $c_{0,n} = -\sum_{k=1}^{\infty} c_{k,n},\;\; c_{k,n} = (-1)^k\frac{1}{k}a_k R_n^k$. Thus

$$u_n(0) - u_n(2\pi) = 2\pi e^{2c_{0,n}}.$$

According to Lemma 2 (ii) $\lim_{n\to\infty} 2\pi e^{2c_{0,n}} \;>\; 0$, but $\lim_{n\to\infty}(u_n(0) - u_n(2\pi)) = \frac{1}{s(0)-\omega} - \frac{1}{s(2\pi)-\omega} = 0$, a contradiction.

Proof of Lemma 1. Taking $b_n(R) = a_n R^n$ we can rewrite (2) as

$$b_1(R) = R,\ b_n(R) = \frac{1}{n}\sum_{k=1}^{n-1} b_k(R)\ b_{n-k}(R).$$

Consider numbers $R > 0$, $n > 1$ and suppose that

$$\exists_{C(R)}\ \forall_{k<n}\ \ b_k(R) \leq \frac{C(R)}{k}. \tag{4}$$

Hence $b_n(R) \leq \frac{1}{n}\sum_{k=1}^{n-1} \frac{C(R)}{k} \frac{C(R)}{n-k} = \frac{(C(R))^2}{n^2}\sum_{k=1}^{n-1}\left(\frac{1}{k} + \frac{1}{n-k}\right)$,

$$b_n(R) \leq 2 \cdot C(R)^2 \cdot \frac{ln(n-1)+1}{n^2}, \tag{5}$$

$$b_n(R) \leq \frac{C(R)}{n}\ ,\ \ if\ only \tag{6}$$

$$\frac{ln(n-1)+1}{n} \leq \frac{1}{2C(R)}. \tag{7}$$

Consider $R = 1$ and take $C(1) = 1$. In this case we have (7) for every $n \geq 5$, (4) for $n = 5$ and (5), (6) for every $n > 1$, by induction. Consequently the series $\sum_{n=1}^{\infty} b_n(1)$ is convergent and $R_0 \geq 1$. The easy induction shows that $b_n(2) \geq 2$ for every n, so $R_0 \leq 2$, which gives (i). Since (ii) is evident, it remains to prove (iii) -that $\lim_{R\to R_0}\sum_{k=1}^{\infty} a_k R^k = +\infty$. It suffices to show that $\sum_{k=1}^{\infty} b_k(R_0) = +\infty$, because $a_k > 0$. Conversely, suppose that $\sum_{k=1}^{\infty} b_k(R_0) < +\infty$. It follows that $(\sum_{k=1}^{\infty} b_k(R_0))^2 = \sum_{n=2}^{\infty}\sum_{k=1}^{n-1} b_k(R_0) b_{n-k}(R_0) = \sum_{n=2}^{\infty} n \cdot b_n(R_0) < +\infty$. Hence $\exists_C \forall_n b_n(R_0) < \frac{C}{n}$. Choose n_0 such that $\frac{ln(n-1)+1}{n} < \frac{1}{2C}$ for $n \geq n_0$. Take $R_1 > R_0$ satisfying $b_k(R_1) < \frac{C}{k}$ for every $k < n_0$. Let $C(R_1) = C$. Then (7) holds for every $n \geq n_0$, (4) - for $n = n_0$ and (5),(6) hold for every $n > 1$. This shows that $\sum_{n=1}^{\infty} b_n(R_1)$ is convergent, which contradicts the fact that R_0 is the radius of convergence.

Proof of Lemma 2. By Conjecture 2, the sequence $\frac{a_{n+1}}{a_n}$ is increasing. Therefore $lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{1}{R_0}, a_{n+1}R_0^{n+1} < a_n R_0^n$. If $lim_{n\to\infty} a_n R_0^n = 0$ then according to the Abel theorem, we have

$$\lim_{R\to R_0-} z_R(0) = i \cdot \lim_{R\to R_0-}\sum_{k=1}^{\infty}(-1)^k a_k R^k = i \cdot \sum_{k=1}^{\infty}(-1)^k a_k R_0^k.$$

The same argument shows (ii). Now assume that $\lim_{n\to\infty} a_n R_0^n > 0$. Let

$$x = -a_1 R_0 + \sum_{n=1}^{\infty}\left(a_{2n}R_0^{2n} - a_{2n+1}R_0^{2n+1}\right),$$

$$y = \sum_{n=1}^{\infty}\left(-a_{2n-1}R_0^{2n-1} + a_{2n}R_0^{2n}\right),$$

$$x_R = -a_1 R + \sum_{n=1}^{\infty}\left(a_{2n} - a_{2n+1}R_0\right)R^{2n},$$

$$y_R = \sum_{n=1}^{\infty} \left(-a_{2n-1} + a_{2n} R_0\right) R^{2n-1}.$$

By Abel theorem, $\lim_{R \to R_0 -} x_R = x$ and $\lim_{R \to R_0 -} y_R = y$. Moreover, $x_R \leq \sum_{k=1}^{\infty} (-1)^k a_k R^k \leq y_R$ for $R \in (0, R_0)$, which gives (i).

Remark 1 $R_0 = 1.445796...$

Remark 2 *Using a new variable* $s = \frac{i}{z + i\sqrt{R}}$ *one can prove that (1) has a* 2π*-periodic solution for some* $R > R_0$.

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