Rank three geometries associated with $PSL(3,4)$

Harald Gottschalk*

1 Introduction

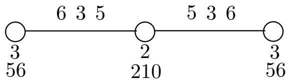
In [Bu86] F. Buekenhout started a research program in order to find a unifying combinatorial approach to all finite simple groups, inspired by J. Tits' famous theory of buildings (see e.g. [Ti74]), that is to classify diagram geometries fulfilling various conditions for all of these groups. This attempt has recently led to several collections of geometries (see [BCD95a], [BCD95b], [BDL94] and [BDL95]) using Cayley (see [De94]). The present paper should be regarded as a part of this program. Although $PSL(3,4)$ is a group of Lie-type and therefore wellknown in this context, it is of great interest since there are many connections between this group and some sporadic groups, e.g. the large Mathieu groups because of the famous construction of the 5-(24,8,1) design for M_{24} from the projective plane of order four (see [Lü69]), and $PSL(3,4)$ is a large subgroup of M_{22}. The geometries in this paper are also of another interest because they are examples of *building-like* geometries in the sense of [BuPa95], which are e.g. extensions of (generalized) polygons by linear spaces and thick diagram geometries.

We construct three geometries of rank three admitting $PSL(3,4)$ as a flag-transitive automorphism group in an easy combinatorial way using objects like hyperovals, Baer subplanes and unitals in $PG(2,4)$. Also, we describe these geometries by giving their diagrams and we determine their full automorphism and correlation groups. Our main result is the following.

*The author is supported by a grant of the Communauté française de Belgique.
Received by the editors April 1995.
Communicated by J. Doyen.
1991 *Mathematics Subject Classification :* Primary 51E24; Secondary 05B25.
Key words and phrases : Diagram geometries, hyperovals.

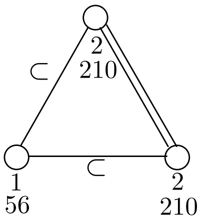
1.1 Theorem. *$PSL(3,4)$ acts flag-transitively on the following geometries of rank three.*

1. *A geometry Γ over*



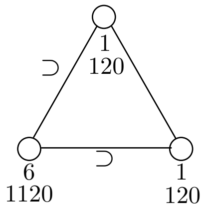
with $Aut\Gamma \simeq PSL(3,4)$ and $Cor\Gamma \simeq P\Sigma L(3,4)$.

2. *A geometry Δ over*



with $Aut\Delta \simeq P\Sigma L(3,4)$ and $Cor\Delta \simeq P\Sigma L(3,4) : 2$.

3. *A geometry Σ over*



with $Aut\Sigma \simeq PSL(3,4)$ and $Cor\Sigma \simeq P\Sigma L(3,4)$.

All geometries are firm, residually connected and satisfy the intersection property $(IP)_2$.

The paper is organized as follows. In the next section we recall the basic definitions of a geometry, its diagram and its automorphism group. In the third section we provide all knowlegde on hyperovals, Baer subplanes and unitals in the projective plane of order four, which is necessary for the constructions of the geometries for $PSL(3,4)$. After that, a construction of the projective space $PG(3,2)$ out of the plane $PG(2,4)$ is given which is used to determine the diagrams. The last section contains the constructions and descriptions.

2 Notation

For all definitions and notations in this section we refer to [Bu86]. Let I be any set with $|I| = n$, called the set of *types*. An *incidence structure* over I is a triple $\Gamma = (X, t, *)$ where $t : X \to I$ is a surjective mapping, the *type function*, and $*$ is a symmetric and reflexive relation on X, called the *incidence relation*. For any subset $Y \subset X$ the *type* of Y is the set $t(Y)$ and for every $i \in I$ the set $t^{-1}(i)$ is called the set of *i-varieties* of Γ. A *flag* of Γ is a subset F of X such that the elements of F are pairwise incident, a flag C with $t(C) = I$ is called a *chamber*. Γ is said to be a *geometry* if and only if

1. for any x and y in X with $x * y$, $t(x) = t(y)$ implies $x = y$, and
2. every flag is contained in a chamber.

The number n is called the *rank* of a geometry Γ. For every flag F the *residue* of F is defined as the geometry $\Gamma_F = (X_F, t_F, *_F)$ over the type set $I - t(F)$ where X_F is the set of all elements of X incident with each element of F, t_F is the restricted type function and $*_F$ the induced incidence. For any subset $J \subset I$, the *J-truncation* Γ^J is the geometry over J whose elements are all elements x of Γ with $t(x) \in J$ together with the induced incidence.

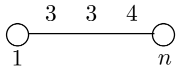
Remark. If $|I| = 2$ we also use the following definition for an incidence structure. We say that $\Gamma = (V, B, J)$ is a *design* (*incidence structure*) if and only if

1. $V \cap B$ is empty and
2. the *incidence relation* J is a subset of $V \times B$.

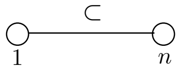
The elements of V (resp. B) are called *vertices* (resp. *blocks*). It is easy to see that both definitions are equivalent. Since a linear space is determined by its points, lines and their incidence, every projective space can be described as a geometry of rank two.

A permutation α of X is called an *automorphism* of Γ if for all $x, y \in X$, we find $x\alpha * y\alpha$ if and only if $x * y$, and α is a permutation of the set of all i-varieties for every $i \in I$. Let i and j be two types with $|t^{-1}(i)| = |t^{-1}(j)|$. Then a permutation β of X is said to be a *correlation* if $t^{-1}(i)\beta = t^{-1}(j)$, $t^{-1}(j)\beta = t^{-1}(i)$ and $t^{-1}(k)\beta = t^{-1}(k)$ for every $k \in I - \{i, j\}$ and β preserves the incidence relation. Clearly, if β and γ are correlations interchanging the same sets of i- and j-varieties, their product is an automorphism. A group of automorphisms of Γ is called *flag-transitive* if it acts transitively on the set of flags of Γ of every type $J \subset I$. We denote the group of all automorphism by $Aut\Gamma$ and Γ is said to be *flag-transitive* if $Aut\Gamma$ is flag-transitive. Clearly, all residues of a flag-transitive geometry of a given type $J \subset I$ are isomorphic and therefore we denote them by Γ_J.

The *diagram* of a flag-transitive geometry Γ is the complete graph on the type set I provided with some information on the incidence graphs of residues of rank two. More precisely, the edge between two vertices i and j is labelled d_i, g, d_j where g is the gonality of the incidence graph of a residue of type $\{i, j\}$ (i.e., the minimal length of a circle in the incidence graph of $\Gamma_{I-\{i,j\}}$ divided by two), d_i is its diameter starting with an element of type i and d_j is the j-diameter. Also under every vertex i we denote the cardinality of $\Gamma_{I-\{i\}}$ minus one, the so called *i-order* s_i of Γ and the number $|t^{-1}(i)|$. The edges labelled n, n, n are called *generalized n-gons.* Usually we denote them for $n \leq 4$ by drawing $n - 2$ edges between these vertices. If $n \geq 5$, we replace n, n, n by a single n. For



we write



to denote the complete graph on $n+2$ vertices.

A flag-transitive geometry of rank n is said to be *firm* if $s_i \geq 1$, *thin* if $s_i = 1$ and *thick* if $s_i \geq 2$ for all $i \in I$. It is called *residually connected* if its incidence graphs are connected for every residue Γ_J with $|J| \leq |I| - 2$. This is also called condition RC.

To check whether Γ is connected, we have the following lemma [Ti74].

2.1 Lemma. *Let Γ be a geometry over I and G be a flag transitive automorphism group of Γ. For every $i \in I$, G_i denotes the stabilizer of an element of type i. Then Γ is connected if and only if $< G_i : i \in I >= G$.*

Let Γ be a flag-transitive geometry. Then we say that Γ fulfills the *intersection property* $(IP)_2$ if in every residue $\Gamma_{I-\{i,j\}}$ the gonality is either at least 3 or this residue is a generalized digon.

3 Hyperovals, Baer subplanes and unitals in $PG(2,4)$

Let $\mathcal{P} = PG(2,4)$ be the projective plane of order four. We denote the elements of $\mathcal{P}$ (both points and lines) using capital letters, and the elements of contructed geometries with small letters in order to avoid confusion. A *hyperoval* $\mathcal{O}$ in $\mathcal{P}$ is a set of six points of the plane no three of which are collinear. Since $\mathcal{P}$ is a linear space where each point is on five lines, every line L meets the hyperoval in either two or no points. According to [Beu86], we call these lines *3-lines* (resp. *5-lines*) for $\mathcal{O}$. Let us denote by $\mathcal{T}$ the set of all 3-lines and by $\mathcal{F}$ that of all 5-lines. Then the structure $(\mathcal{O}, \mathcal{T}, \in)$ is the complete graph on six vertices and therefore we find $|\mathcal{T}| = 15$ and $|\mathcal{F}| = 6$. Following [Lü69] we can define an equivalence relation $\sim$ on the set $\mathbf{O}$ of all 168 hyperovals in $\mathcal{P}$ by

$$\mathcal{O}_1 \sim \mathcal{O}_2 :\Leftrightarrow |\mathcal{O}_1 \cap \mathcal{O}_2| \text{ is even}^1.$$

The equivalence classes of $\sim$ are three sets of 56 hyperovals, they are exactly the three orbits of $PSL(3,4)$ on $\mathbf{O}$. Two elements $\mathcal{O}_1$ and $\mathcal{O}_2$ of $\mathbf{O}$ intersect in no, one, two or three points, thus through each quadrangle (a set of four points no three of which are collinear) in $\mathcal{P}$ there is only one hyperoval. Also, through each triangle there are three hyperovals (one of each class) and through a point pair we find twelve (four of each class). We make the following observation concerning elements of a class $\mathcal{C}$ of hyperovals and pairs of 5-lines.

[1] A purely combinatorial proof for this can be found in [vLi84].

3.1 Lemma. *Let $\mathcal{C}$ be a class of hyperovals in $\mathcal{P}$. Then the following holds.*

1. *If (L, M) is a pair of 5-lines for $\mathcal{O}_1 \in \mathcal{C}$, then there exists a hyperoval $\mathcal{O}_2$ in $\mathcal{C}$ intersecting $\mathcal{O}_1$ non-trivially such that (L, M) is a pair of 5-lines for $\mathcal{O}_1$ and $\mathcal{O}_2$.*

2. *If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two elements of $\mathcal{C}$ with non-trivial intersection, then there is one and only one pair of 5-lines for $\mathcal{O}_1$ and $\mathcal{O}_2$.*

Proof. Let $\mathcal{O}_1$ be an element of $\mathcal{C}$ and (L, M) be a pair of 5-lines. We denote $\mathcal{O}_1$ as a set of points of $\mathcal{P}$, say $\mathcal{O}_1 = \{A, B, C, D, E, F\}$. Then we find three 3-lines through $L \cap M$, say AB, CD and EF. If A', B' and C', D' are the remaining points of AB (resp. CD), then $\mathcal{O}_2 = \{A', B', C', D', E, F\}$ is a hyperoval, because, w.l.o.g., $AE \cap CD = C' = BF \cap CD$, $AF \cap CD = D' = BE \cap CD$ and $CE \cap AB = A' = DF \cap AB$, $CF \cap AB = B' = DE \cap AB$. Note that if E' and F' are the remaining points on EF, the set $\{A', B', C', D', E', F'\}$ is not a hyperoval, because the upper argument yields three collinear points in this set.

Let now $\mathcal{O}_1$ and $\mathcal{O}_2$ be two hyperovals in $\mathcal{P}$ intersecting non-trivially. Therefore their intersection contains two points, say P and Q. Then we can apply the construction given above for every point R on PQ, $R \notin \{P, Q\}$, and the hyperoval $\mathcal{O}_1$. Thus we get get three hyperovals of $\mathcal{C}$ through P and Q different from $\mathcal{O}_1$. Since there are only four elements of $\mathcal{C}$ through all point-pairs, one of these must be $\mathcal{O}_2$ and the existence of a common pair of 5-lines is shown.

It remains to show the uniqueness of this pair. Let us assume that there are two such pairs. Then the pairwise intersections of all involved lines form a set of four (resp. six) points outside $\mathcal{O}_1 \cup \mathcal{O}_2$ and we find that $\mathcal{P}$ has at least $10+4+2+3\cdot 3 = 25$ (resp. $10 + 6 + 4 \cdot 2 = 24$) points, a contradiction. ∎

Remark. The construction of the common pair of 5-lines shows also that we cannot find such a pair for three elements of $\mathcal{C}$.

The question of a possible converse of 3.1 arises. To look for that, we recall the construction of the Steiner-system for M_{22} (see [Lü69]). Let V be the set of all 21 points of $\mathcal{P}$ together with a new point called *infinity*. The set of blocks is the following. We add the point infinity to every line of $\mathcal{P}$ and take a class $\mathcal{C}$ of hyperovals. These form the block set B of a design $\mathcal{M}_{22} = (V, B, I)$, where I is the induced incidence, admitting an automorphism group isomorphic to M_{22}. Now we can prove a converse of 3.1.

3.2 Lemma. *Let L and M be two lines and $\mathcal{C}$ be a class of hyperovals of $\mathcal{P}$. Then we find exactly four elements of $\mathcal{C}$ disjoint from L and M whose pairwise intersections contain two points.*

Proof. The construction of $\mathcal{M}_{22}$ yields the equivalence of a pair of lines and a pair of intersecting hyperovals in $\mathcal{C}$. Therefore the application of 3.1.2 shows the existence of a pair of hyperovals disjoint from L and M. Again, the construction in the proof of 3.1.1 can be used to find the two remaining hyperovals. Clearly, no other element of $\mathcal{C}$ can be disjoint from the pair (L, M) since the lines through $L \cap M$ cover $\mathcal{P}$. ∎

Remark. Of course we find two other elements of the block set of $\mathcal{M}_{22}$ for a pair of intersecting hyperovals besides the common pair of 5-lines disjoint from that pair. These two are also a pair of intersecting hyperovals.

A *Baer subplane* of $\mathcal{P}$ is a set $\mathcal{B}$ of seven points and seven lines whose induced incidence structure is isomorphic to $PG(2,2)$. Since every quadrangle determines a unique Baer subplane, every hyperoval is related to such a subplane. The detailed construction of such a pair is given e.g. in [Beu86] or [vLi84]. Also, if $\mathcal{C}$ is an equivalence class of hyperovals, $PSL(3,4)$ is transitive just on the related quadrangles (see [HuPi85]) and therefore the set $\mathbf{B}$ of all 360 Baer subplanes splits into three orbits. These orbits can also be obtained by the definition of an equivalence relation $\sim$ on $\mathbf{B}$. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are two Baer subplanes, we say $\mathcal{B}_1 \sim \mathcal{B}_2$ if and only if they intersect in an odd number of points. This is a wellknown fact, see e.g. [vLi84] or [HuPi85]. The group $P\Gamma L(3,4)$ induces S_3 on the set of the three equivalence classes of Baer subplanes, because $PGL(3,4)$ is transitive on the quadrangles in $\mathcal{P}$ (see [HuPi85]) and therefore on the set of all Baer subplanes. Hence the Baer involutions (involutions in $P\Gamma L(3,4) - PGL(3,4)$ whose fixed points are a Baer subplane) are not central in $PGL(3,4)$ or $PSL(3,4)$ and the group $P\Sigma L(3,4)$ interchanges two of these classes. Clearly, the same holds for the three classes of hyperovals. We establish the following connection between a class $\mathcal{C}$ of hyperovals and the classes of Baer subplanes. A Baer subplane is in the class related to $\mathcal{C}$ if and only if the intersection with all elements of $\mathcal{C}$ has an even number of points. Also, we state here the fact that for a given Baer subplane there are eight disjoint subplanes in every class (see e.g. [vLi84]). Since $PG(2,2)$ is selfdual, it is natural to ask whether a Baer subplane is also selfdual under a correlation of $PG(2,4)$. In fact, the following lemma holds.

3.3 Lemma *Let $\mathcal{B}$ be a Baer subplane of $\mathcal{P}$. Then $\mathcal{B}$ is selfdual under a correlation of $\mathcal{P}$.*

Proof. Let α be a correlation of $\mathcal{P}$. Then the image of $\mathcal{B}$ under α is also a Baer subplane, say $\mathcal{B}'$. If β (resp. β') is the Baer involution of $\mathcal{B}$ (resp. $\mathcal{B}'$), it is sufficient to show that $\beta = \beta'$ to prove the lemma. Since $\mathcal{B}\alpha = \mathcal{B}'$, we find $\beta^\alpha = \beta'$. Thus $\beta = \beta'$ if and only if $\beta\alpha = \alpha\beta$.

The group $CorPG(2,4)/PGL(3,4)$ is a group of order four which contains two normal subgroups generated by involutions, namely by a Baer involution and a correlation. Therefore $CorPG(2,4)/PGL(3,4) \simeq 2^2$, hence Baer involutions commute with correlations. ∎

A *unital* in $\mathcal{P}$ is a set $\mathcal{U}$ of nine points no four of which are collinear, such that every line in the plane meets $\mathcal{U}$. A unital can be constructed in the following way (see [Beu86] or [vLi84]). Let $\{A, B, C\}$ be a triangle in $\mathcal{P}$. Then the lines AB, AC and BC determine nine points outside the triangle, in fact these nine points form a unital $\mathcal{U}$. As we see, these lines do not intersect in $\mathcal{U}$, they are parallel in the linear space $\mathcal{U}$, hence a unital is also an affine plane of order three. In the same way, each remaining parallel class in $\mathcal{U}$ determines a triangle outside, thus the twelve points outside the unital can be regarded as the disjoint union of four triangles, say

$\{A, B, C\}$, $\{D, E, F\}$, $\{P, Q, R\}$ and $\{U, V, W\}$, related to the four parallel classes. Clearly, each choice of two of these triangles determines a hyperoval. Therefore, if $\mathcal{O}_1$ and $\mathcal{O}_2$ are two disjoint hyperovals, the remaining nine points of $\mathcal{P}$ are a unital. This shows also that we cannot find a common pair of 5-lines for *disjoint* hyperovals.

Let now $\mathbf{U}$ be the set of all unitals. Then, according to the ATLAS, $|\mathbf{U}| = 280$ and $PSL(3,4)$ is acting transitively on $\mathbf{U}$ (see [Co85]). At the end of this section we prove a lemma which shows a connection between the three types of objects in $\mathcal{P}$ described here.

3.4 Lemma. *Let $\mathcal{Y} = \{A, B, C\}$ be a triangle in $\mathcal{P}$. Then there are exactly three Baer subplanes (one in each class of Baer subplanes) through A and the three points of BC not in $\mathcal{Y}$ meeting $\mathcal{Y}$ only in A.*

Proof. Since $\mathcal{Y}$ is a triangle, the nine points on the three lines through the edges in its complement form a unital $\mathcal{U}$. Let us denote the remaining points on BC by P, Q, R. Then the lines AB, AC, BC, AP, AQ and AR cover $9 + 3 + 3 \cdot 3 = 21$ points, hence $\mathcal{P}$. Thus the nine points outside $\mathcal{U}$ and $\mathcal{Y}$ split into three triangles related to the remaining parallel classes. Since these triangles form two hyperovals in each class, the lines AP, AQ and AR contain only one point of each. Now, every of these triangles together with $\{A, P, Q, R\}$ are a Baer subplane because P, Q and R are points of $\mathcal{U}$ and therefore each is on one line of every parallel class, hence each is collinear with two points of the three triangles. This yields the existence of the Baer subplanes. Since they intersect pairwise in four points, they are in different classes. The uniqueness follows from the fact that the triangles form a quadrangle with A. ■

We call these Baer subplanes *1-Baer subplanes* (1-BS) for a triangle $\mathcal{Y}$. If we look for the total number of triangles such that a given Baer subplane is a 1-BS for these, we start with a line of this subplane. Then we find four of these triangles through every remaining point of the subplane and the two points on that line outside. Since there are seven lines in the subplane, we get $4 \cdot 7 = 28$ triangles.

3.5 Corollary. *Let $\mathcal{Y}$ be a triangle in $\mathcal{P}$ and $\mathcal{B}_1$ and $\mathcal{B}_2$ be two 1-BS for $\mathcal{Y}$ intersecting the triangle in different points. Then $\mathcal{B}_1$ and $\mathcal{B}_2$ are either disjoint or they intersect in a triangle.*

Proof. The construction of the 1-BS holds clearly for all points of $\mathcal{Y}$. Let now $\mathcal{B}_1$ and $\mathcal{B}_2$ be two 1-BS through different points of $\mathcal{Y}$. Then they are constructed either from different triangles outside $\mathcal{Y}$ and their intersection is empty, or from the same triangle and this triangle is the intersection. Note that $\mathcal{B}_1$ and $\mathcal{B}_2$ are in different classes only when their intersection is empty. ■

4 Constructing $PG(3,2)$ out of $PG(2,4)$

Let $\mathcal{P} = PG(2,4)$ be the projective plane of order four and let $\mathcal{O}$ be a hyperoval in $\mathcal{P}$. We construct the symplectic generalized quadrangle over $GF(2)$ with the help of $\mathcal{O}$ and extend this contruction to get the whole space $PG(3,2)$.

If X is any (finite) set, we denote the set of all unordered pairs of X by $\binom{X}{2}$. Let us now concider the incidence structure $\mathcal{S} = (\mathcal{T}, \binom{\mathcal{F}}{2}, I)$ for a hyperoval $\mathcal{O}$, where the incidence relation I is defined as follows: if x is an element of $\mathcal{T}$ (a 3-line L for $\mathcal{O}$) and y is a pair (M, N) of 5-lines, then xIy if and only if $L \cap M \cap N$ consists of a single point of $\mathcal{P}$. We prove now that $\mathcal{S}$ is a finite generalized quadrangle (GQ), hence an incidence structure where

- for each two different vertices v and w there is at most one block b incident with v and w,
- if v is any vertex, then there are $t+1$ blocks through v,
- if b is any block, then b is incident with $s+1$ points,
- if (v, b) is a non-incident vertex-block-pair, then there is exactly one vertex w on b collinear with v.

The pair (s, t) is called the *order* of the GQ.

4.1 Lemma. *The incidence structure $\mathcal{S}$ is a GQ of order* $(2, 2)$.

Proof. Let v and w be two vertices of $\mathcal{S}$, respectively two 3-lines L and M for $\mathcal{O}$. If $L \cap M$ is a point of $\mathcal{P}$ on $\mathcal{O}$, then there is no block of $\mathcal{S}$ through v and w. Suppose now that $L \cap M \notin \mathcal{O}$. Then there exists only one line N in $\mathcal{P}$ such that $(\mathcal{O} \cap L) \mathbin{\dot\cup} (\mathcal{O} \cap M) \mathbin{\dot\cup} (\mathcal{O} \cap N) = \mathcal{O}$. Therefore there is a unique pair of 5-lines through $L \cap M$, hence a unique block in $\mathcal{S}$ through v and w. Again, let v be any vertex of $\mathcal{S}$, thus a 3-line L in $\mathcal{P}$. As above, we find through each of the three points of L outside $\mathcal{O}$ two other 3-lines and only one pair of 5-lines, hence each vertex of $\mathcal{S}$ is incident with 3 blocks. If conversly b is a block of $\mathcal{S}$, say the pair (L, M) of 5-lines, then we find three 3-lines through $L \cap M$. Let now (v, b) be a non-incident vertex-block-pair of $\mathcal{S}$. Again, we identify v with a 3-line L and b with a pair (M, N). Therefore we find $L \cap M \neq L \cap N$ and there is a unique point P on L not on $\mathcal{O}$, M or N. Let $Q = M \cap N$. Then $PQ \cap \mathcal{O}$ is not empty, hence PQ corresponds to a vertex w of $\mathcal{S}$ on b collinear with v. ∎

From the uniqueness of the GQ of order $(2, 2)$, it follows that $\mathcal{S}$ is the symplectic GQ over $GF(2)$ (see [PaTh84]).

Remark. Note that this is a realization of the construction of this GQ using the group S_6.

Let $\mathcal{D}$ be the set of all triples of 3-lines for $\mathcal{O}$ such that an element of $\mathcal{D}$ can be identified with a triangle on $\mathcal{O}$ and therefore $|\mathcal{D}| = \binom{6}{3} = 20$. Now, we consider the incidence structure $\Pi = (\mathcal{T}, \binom{\mathcal{F}}{2} \cup \mathcal{D}, J)$ where the incidence J is the same as in $\mathcal{S}$ for a 3-line and a pair of 5-lines. A 3-line is said to be incident with a triple of lines in $\mathcal{D}$ if it appears in this triple. Therefore we have 15 vertices and 35 blocks in Π.

4.2 Lemma. Π *is the projective space* $PG(3,2)$.

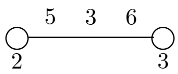
Proof. By construction Π contains the GQ of order $(2,2)$. Therefore we only have to show that Π is a linear space, because $PG(3,2)$ is the only linear space with 15 points containing this GQ (see [PaTh84]). Let v and w be two vertices of Π, say the two 3-lines L and M for $\mathcal{O}$. Now we have to distinguish two cases.

1. L and M intersect in a point not on $\mathcal{O}$. Therefore there is a unique pair of 5-lines through $L \cap M$ (see 4.1).

2. L and M intersect on $\mathcal{O}$. Here the two 3-lines determine a unique triangle on $\mathcal{O}$, hence a unique element of $\mathcal{D}$.

Thus Π is a linear space and the lemma is proved. ■

4.3 Lemma. $\Pi' = (\mathcal{T}, \mathcal{D}, J')$ *with the induced incidence* J' *is a geometry over the following diagram.*



Proof. Since A_6 acts doubly transitively on six points and therefore on the hyperoval, it is an automorphism group of this geometry. The fact that Π is a linear space yields a gonality of at least three for Π. Therefore we can find this geometry in [BDL95]. ■

Remark. In [Beu86] A. BEUTELSPACHER also constructs $PG(3,2)$ out of $PG(2,4)$ using a hyperoval. Let $\mathcal{O}$ be a hyperoval in $\mathcal{P}$. Then the incidence structure $(\mathcal{P} - \mathcal{O}, \mathcal{T} \cup \mathcal{D}, \in)$ is isomorphic to $PG(3,2)$ where $\mathcal{T}$ is the set of all 3-lines for $\mathcal{O}$ and $\mathcal{D}$ is the set of all triangles outside $\mathcal{O}$.

5 The construction of geometries of rank 3 for $PSL(3,4)$

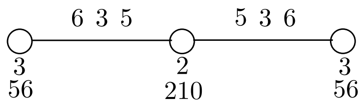
Let I be the set $\{0,1,2\}$. We construct three firm RC-geometries over I fulfilling $(IP)_2$ using hyperovals and Baer subplanes in $\mathcal{P}$.

For the first geometry Γ we take two of the three classes of hyperovals as elements of type 0 and 2. The $\binom{21}{2} = 210$ point-pairs in $\mathcal{P}$ are the elements of type 1. The incidence is defined as follows. Between elements of type 0 (resp. 2) and those of type 1 we take the induced incidence, an element of type 0 is incident with an element of type 2 if they intersect in a triangle in $\mathcal{P}$. Clearly, we also have defined every element of Γ to be incident with itself (this will be assumed in every construction but ought to be mentioned once). Let now x be an element of type 0. Then it is incident with 15 point-pairs (elements of type 1) and each of these is incident with four elements of type 2. If x is of type 1, then it is incident with four of type 0 and four of type 2. Thus every flag is contained in a chamber and Γ is a geometry with us also the total number of $56 \cdot 15 \cdot 4 = 210 \cdot 4 \cdot 4 = 3360$ chambers. Since $PSL(3,4)$ is doubly transitive on the points of $\mathcal{P}$, it is obvious that all of its elements also are automorphisms of Γ. Conversely, every automorphism of Γ induces an automorphism of $\mathcal{P}$, hence we find $Aut\Gamma$ is $PSL(3,4)$ or $P\Sigma L(3,4)$ since $PGL(3,4)$ is transitive on all quadrangles in $\mathcal{P}$

and therefore transitive on all hyperovals. Since $P\Sigma L(3,4)$ interchanges two classes of hyperovals, we get $Aut\Gamma \simeq PSL(3,4)$ and $Cor\Gamma \simeq P\Sigma L(3,4)$. According to the Atlas (see[Co85]), the stabilizer of a hyperoval in $\mathcal{P}$ is isomorphic to A_6. Since this group is 2-transitive on its stabilized hyperoval and on the 15 points outside, it is transitive on all chambers through this hyperoval. Therefore $PSL(3,4)$ acts flag-transitively on Γ.

Let x be an element of Γ of type 0 (resp. 2). Then its residue Γ_0 (Γ_2) contains the $\binom{6}{2} = 15$ elements of type 1. Since through each triangle on it there is a unique element of type 2 (resp. 0), the residue has 20 elements of this type and Γ_0 (Γ_2) is the geometry described in 4.3.

For an element of type 1 we find four elements of type 0 an 2 in Γ_1. Here, every element of type 0 intersects every element of type 2 in at least two points in $\mathcal{P}$ thus their intersection is a triangle. This implies that Γ_1 is a generalized digon. Therefore $(IP)_2$ holds and we Γ is also residually connected since A_6 is maximal in $PSL(3,4)$ (see [Co85]) and obviously it does not contain the stabilizers of elements of type 1 or 2 (e.g. the stabilizer of a point-pair is a subgroup of a line stabilizer), thus 2.1 holds. To summarize: Γ is a geometry over the following diagram.



Let us now consider one equivalence class $\mathcal{C}$ of 56 hyperovals in $\mathcal{P}$. We define the elements of $\mathcal{C}$ to be the 0-varieties of an incidence structure Δ over I. The set of 1-varieties (resp. 2-varieties) are all possible 210 pairs of points in $\mathcal{P}$ (resp. pairs of lines). The incidence between a hyperoval and a point-pair is the induced one. A hyperoval and a line-pair are said to be incident if and only if they have no common point in $\mathcal{P}$ (the line-pair is a pair of 5-lines for the hyperoval). For a pair of points $x = (P, Q)$ and a line-pair $y = (L, M)$ we define $x * y$ if and only if $|PQ \cap L \cap M| = 1$ in $\mathcal{P}$ and (L, M) is a pair of 5-lines for two hyperovals of $\mathcal{C}$ through (P, Q).

If x is an element of type 0 of Δ, it is incident with 15 point-pairs. From 3.1.2 it follows that each of these pairs is incident with $\binom{4}{2} = 6$ line-pairs. But only three of them are also incident with x by 3.1.1. An element y of type 1 (a point-pair) is then incident with four hyperovals and again we find three line-pairs incident with y and one hyperoval. By 3.2 the same holds for each line-pair. Therefore Δ is a geometry with $56 \cdot 15 \cdot 3 = 210 \cdot 4 \cdot 3 = 2520$ chambers.

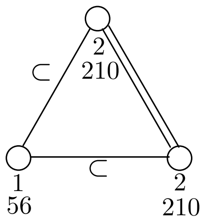
As above, each automorphism of Δ induces an automorphism of $\mathcal{P}$ and therefore again $Aut\Delta \simeq PSL(3,4)$ or $Aut\Delta \simeq P\Sigma L(3,4)$. But here the Baer involution interchanging the two classes of hyperovals, different from $\mathcal{C}$, is also an automorphism of Δ, thus $Aut\Delta \simeq P\Sigma L(3,4)$.

Let α be a correlation of $\mathcal{P}$. Then α maps point-pairs on line-pairs and vice versa. Since the dual of a hyperoval is the set $\mathcal{F}$, a hyperoval in the dual plane, and α is of order 2 , we find $Cor\Delta \simeq P\Sigma L(3,4) : 2$.

Again the transitivity of A_6 on a hyperoval and its complement in $\mathcal{P}$ yields the transitivity on all chambers through the hyperoval and $PSL(3,4)$ is a flag-transitive automorphism group of Δ.

From our construction of Δ we see immediately, by application of 3.1.1 and 4.1 that for each hyperoval the residue is the GQ of order $(2,2)$. Also, 3.1.2 yields

Δ_1 to be the complete graph on four vertices and by 3.2 the same holds for Δ_2. Therefore $(IP)_2$ holds for Δ and RC holds by the same arguments as above, thus Δ is a geometry over the following diagram.



For the last geometry we fix again a class $\mathcal{C}$ of hyperovals. Then the set $\mathbf{U}$ of all unitals determines $280 \cdot 4 = 1120$ triangles on the elements of $\mathcal{C}$ which are all unordered triangles in $\mathcal{P}$. We define the triangles to be the elements of type 0 of an incidence structure Σ. As elements of type 1 and 2 we choose the two classes of Baer subplanes in $\mathcal{P}$ not related to $\mathcal{C}$. The incidence is defined as follows. A triangle is incident with a Baer subplane if and only if this subplane is a 1-BS for it. If we take an element of type 1 and one of type 2, say two Baer subplanes $\mathcal{B}_1$ and $\mathcal{B}_2$, they are incident when $\mathcal{B}_1 \cap \mathcal{B}_2$ is empty.

If we take a triangle, it follows from 3.5 that it is incident with three elements of type 1 and 2 where each of type 1 is incident with two of type 2 and vice versa, thus these six elements form also a triangle. If x is of type 1 in Σ (a Baer subplane $\mathcal{B}$), there are 28 elements of ype 0 incident with x, all of them incident with two subplanes disjoint with $\mathcal{B}$. Since the same arguments hold for an element of type 2, Σ is a geometry with $1120 \cdot 3 \cdot 2 = 6720 = 120 \cdot 28 \cdot 2$ chambers.

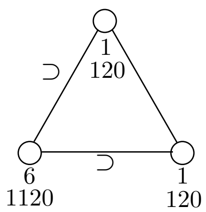
Let $\{x, y\}$ be a flag of type $\{1, 2\}$, hence two disjoint Baer subplanes. Then the complement of $\{x, y\}$ in $\mathcal{P}$ is another Baer subplane (see [Ueb95]). Since a triangle is is selfdual under a correlation, it follows from 3.3 that the incidence in $\mathcal{P}$ can be recontructed from that in Σ only up to duality, but a correlation induces the indentity on Σ. Therefore $Aut\Sigma \simeq PSL(3,4)$ and $Cor\Sigma \simeq P\Sigma L(3,4)$.

The stabilizer of a Baer subplane in $\mathcal{P}$ in $PSL(3,4)$ is a group isomorphic to $PSL(3,2)$ (see [Co85]) and it is transitive on all 14 points of $\mathcal{P}$ outside this subplane. Therefore we find again that it is transitive on all chambers through the Baer subplane and $PSL(3,4)$ acts flag-transitively on Σ.

Also, $PSL(3,2)$ is maximal in $PSL(3,4)$ and, clearly, a stabilizer of a Baer subplane cannot contain the stabilizer of one in another class. So again 2.1 holds and Σ is residually connected.

As above, the application of 3.5 yields Σ_0 to be a triangle. In Σ_1 we find 28 triangles and eight Baer subplanes where every triangle is incident with exactly two subplanes (by 3.5). Thus, this residue is the complete graph on eight points.

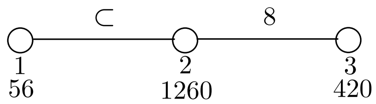
As a summary, Σ is an RC-geometry with $(IP)_2$ over the following diagram.



This proves theorem 1.1. ∎

We conclude this paper with some remarks.

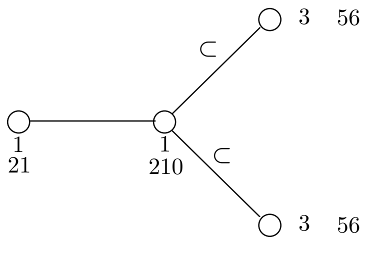
1. The geometry Δ descibed here is essentially due to WEISS. In [We91] he shows that there is only one geometry Π over



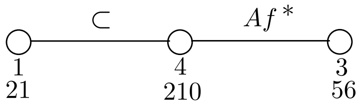
admitting $PSL(3,4).2^2$ as an automorphism group. His construction is using purely group-theoretical methods. From Π, our geometry Δ can be obtained via an *unfolding* (see [BuPa95], where we find misprinted 1- and 2-orders of Δ). The process of *folding* down Δ to obtain Π is the following. We define a geometry Π' over $\{0,+,-\}$, where the set of 0-varieties is that of Δ. The elements of type $+$ are all flags F of Δ with $t^{-1}(F) = \{1,2\}$ and those of type $-$ are all elements of type 1 and 2 of Δ. Then Π' is a geometry over the same diagram as Π. The uniqueness of Π yields $\Pi' \simeq \Pi$ and, of course, $Aut\Pi \simeq Cor\Delta \simeq P\Sigma L(3,4) : 2$. Hence our construction dertermines the structure of $Aut\Pi$ more precisely.

2. All three examples of geometries associated with $PSL(3,4)$ fulfill also another condition. They are *residually weakly primitive*, that is they admit a flag-transitive automorphism group G such that the stabilizer G_i is maximal for at least one $i \in I$, and the same holds for all residues and their stabilizers. The three geometries dicussed in this paper appeared during the author's work on the classification of all geometries for $PSL(3,4)$ with this condition, which will be given elsewhere.

3. According to [Pa95], our geometry Γ is this paper can be obtained as a truncation of the following rank four geometry for $PSL(3,4)$, which is also residually weakly primitive.



The construction of this geometry is the following [Pa95]. Let $I = \{0,1,2,3\}$ and the set of 0-varieties of a geometry be the set of points of $\mathcal{P} = PG(2,4)$. The elements of type 1 are the point-pairs in $\mathcal{P}$, those of type 2 and 3 two families of hyperovals. The incidence is the natural one, resp. two hyperovals are said to be incident if and only if they intersect in a triangle. This yields a geometry over the given diagram, described in [CHP90]. According to [Bu95], it was previously known to BUEKENHOUT but not published. Of course, Γ is the $\{1,2,3\}$-truncation of it. The $\{0,1,2\}$-truncation (resp. $\{0,1,3\}$-truncation) is a geometry over



which is described in [Sp84]. For recent work on geometries over diagrams like this see [Huy95]. In the $\{0,2,3\}$-truncation, $(IP)_2$ does not hold.

4. Since the large Mathieu groups act on extensions of $PG(2,4)$, it is natural to look for extensions of the discussed examples of geometries admitting M_{22}, M_{23} or M_{24} as flag-transitive automorphism groups. This question is studied by the author.

Acknowledgement. I would like to thank Francis Buekenhout for many valuable discussions and helpful comments on the original manuscript.

References

[Beu86] Beutelspacher, A. $21-6=15$*: A connection between two distinguished geometries.* Amer. Math. Monthly **93** (1986), 29-41.

[Bu86] Buekenhout, F. *The geometry of the finite simple groups*; in *Buildings and the geometry of diagrams* (ed. L.A. Rosati). Lect. Notes in Math. **1181**, 1-78, Springer-Verlag, Berlin-Heidelberg-New York (1986).

[Bu95] Buekenhout, F. *Personal communication* (1995).

[BCD95a] Buekenhout, F., Cara, Ph., Dehon, M. *A census of strange flag-transitive rank 2 geometries.* In preparation.

[BCD95b] Buekenhout, F., Cara, Ph., Dehon, M. *Geometries of small almost simple groups based on maximal subgroups.* Preprint (1995).

[BDL94] Buekenhout, F., Dehon, M., Leemans, D. *All geometries of the Mathieu group M_{11} based on maximal subgoups.* Preprint (1994).

[BDL95] Buekenhout, F., Dehon, M., Leemans, D. *An atlas of residually weakly primitive geometries for small groups.* In preparation.

[BuPa95] Buekenhout, F., Pasini, A. *Finite digram geometries extending buildings*; in *Handbook of Incidence Geometry* (ed. F. Buekenhout). Elsevier, Amsterdam (1995).

[CHP90] Cameron, P., Hughes, D.R., Pasini, A. *Extended generalized quadrangles.* Geom. Dedicata **35** (1990), 193-228.

[Co85] Conway J., et al. *An Atlas of finite groups.* Clarendon Press, Oxford (1985).

[De94] Dehon, M. *Classifying geometries with* CAYLEY. J. Symbolic Computation **17** (1994), 259-276.

[HuPi85] Hughes, D.R., Piper, F.C. *Design theory*. Cambridge University Press, Cambridge (1985).

[Huy95] Huybrechts, C. *A generalization of results due to Sprague and Lefèvre-Van Nylpelseer for finite* $[L.Af^*]$*-geometries*. To appear in Geom. Dedicata (1995).

[Lü69] Lüneburg, H. *Transitive Erweiterungen endlicher Permutationsgruppen*. Lect. Notes in Math. **84**, Springer-Verlag, Berlin-Heidelberg-New York (1969).

[Pa95] Pasini, A. *Personal communication* (1995).

[PaTh84] Payne, S.E., Thas, J.A. *Finite generalized quadrangles*. Research Notes in Math. **110**, Pitman, London (1984).

[Sp84] Sprague, A.P. *Extended dual affine planes*. Geom. Dedicata **16** (1984), 107-121.

[Ti74] Tits, J. *Buildings of spherical type and finite* BN*-pairs*. Lect. Notes in Math. **386**, Springer-Verlag, Berlin-Heidelberg-New York (1974).

[Ueb95] Ueberberg, J. *Projective Planes and Dihedral Groups*. Submitted for the proceedings of the conference *Combinatorics '94*, Rome, Montesilvano (1995).

[vLi84] van Lint, J.H. *On ovals in* $PG(2,4)$ *and the McLaughlin graph*. EUT-Report, 84-WSK-03 (1984), 234-255.

[We91] Weiss, R. *A geometric characterization of the groups* M_{12}*, He and Ru*. J. Math. Soc. Japan **43** (1991), 795-814.

Service de Géométrie, C.P. 216
Université Libre de Bruxelles
B-1050 Bruxelles
Belgium

Mathematisches Institut
Arndtstrasse 2
D-35392 Giessen
Germany
e-mail: harald.gottschalk@math.uni-giessen.de