

The triangular extensions of a generalized quadrangle of order $(3,3)$

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Abstract

We show that the triangular extension of a generalized quadrangle of order $(3,3)$ is unique. The proof depends upon certain computer calculations.

1 Introduction and the result

Extensions of finite generalized quadrangles (EGQ, for short), or, more generally, of polar spaces, play an important role as incidence geometries admitting interesting automorphism groups, such as sporadic simple, or some classes of (extensions of) classical groups. Buekenhout and Hubaut [3] initiated the study of extensions of polar spaces from a geometric point of view by proving some characterization theorems, in particular they classified locally polar spaces such that the lines of the residual polar space are of size 3. They also classified locally polar spaces admitting a classical group acting on point residues, later on these results were generalized in a more general framework of flag-transitive diagram geometries, see the survey [22] by Pasini and Yoshiara for an extensive bibliography. However, very few characterizations are known which do not assume group actions. For polar lines of size 3, see already mentioned [3] and [2] by Buekenhout. Blokhuis and Brouwer [1] and P.Fisher [11] classified EGQ(3,1), The author [16, 18, 19] characterized extensions

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of polar spaces related to some 3-transposition groups, including Fischer's sporadic simple groups. In [17] he proved the uniqueness of $\text{EGQ}(3,9)$ and classified its further extensions.

Here we shall be concerned with triangular $\text{EGQ}(3,3)$. For basic definitions and a general account on EGQ see Cameron, Hughes and Pasini [4]. A triangular $\text{EGQ}(s, t)$ may and will be viewed as a graph Γ such that the subgraph $\Gamma(u)$ induced on the neighbourhood of any vertex u is isomorphic to the collinearity graph of a generalized quadrangle of order (s, t) , or $\text{GQ}(s, t)$, for short. Concerning $\text{GQ}(s, t)$, a standard reference is [23]. There are two nonisomorphic $\text{GQ}(3,3)$, dual to each other. One, usually denoted by $W(3)$, is the point-line system of the totally isotropic, with respect to a nondegenerate symplectic form, points and lines of the 3-dimensional projective space over $\text{GF}(3)$ ($\text{PG}(3,3)$, for short). The other one, usually denoted by $Q_4(3)$, is defined similarly with a nondegenerate symmetric bilinear form instead of the symplectic one and $\text{PG}(4,3)$ instead of $\text{PG}(3,3)$.

Let \mathcal{U}_n be the graph defined on the nonisotropic points on a n -dimensional $\text{GF}(4)$ -space $T = T(\mathcal{U}_n)$ carrying a nondegenerate hermitian form, two points being adjacent if they are perpendicular. Note that \mathcal{U}_4 is isomorphic to the collinearity graph of $W(3)$, and \mathcal{U}_{n+1} is locally \mathcal{U}_n .

We say that a graph Γ is locally \mathcal{D} (or Δ), where \mathcal{D} is a family of graphs (resp. Δ a graph), if for any vertex u of Γ the subgraph $\Gamma(u)$ is isomorphic to a member of \mathcal{D} (resp. to Δ). In our case \mathcal{D} consists of the collinearity graphs of $W(3)$ and $Q_4(3)$.

Theorem 1.1 *Let Γ be a triangular $\text{EGQ}(3,3)$ (in other words, Γ is locally \mathcal{D}). Then Γ is isomorphic to \mathcal{U}_5 .*

Under the additional assumption that a classical group is induced on $\Gamma(u)$ for any vertex u , this statement was proved in [3]. Later on, this was improved in [12, 25] in the slightly more general framework of the classification of flag-transitive $c.C_2$ -geometries, still involving a strong assumption on group action.

2 Proof

Let Γ be a triangular $\text{EGQ}(3,3)$. Our approach is based on the observation made in [3] that given a point u of Γ and a point v at distance 2 from u , their common neighbourhood $\Gamma(u, v)$ (which will be often called a μ -graph of $\Gamma(u)$), is a *hyperoval* (or a *local subspace*, in the terminology of [3]) in $\Delta = \Gamma(u)$, that is a subset Φ of points of Δ such that each line of Δ meets either 0 or 2 points of Φ . Hyperovals of GQ were studied by several authors, see e.g. [13, 21]. However the results achieved are concerned mainly with various extreme cases, and nothing like a classification of hyperovals in GQ , which is required in our approach, exists.

So we classify the hyperovals of $Q_4(3)$ and $W(3)$, using a computer. Then we rule out most of the hyperovals of $Q_4(3)$, using some simple criteria. As an immediate corollary we have that $\Gamma(x) \cong \Gamma(y)$ for any distance two pair of points x, y of Γ such that $|\Gamma(x, y)|$ is of certain size. It gives us an opportunity to eliminate most of the hyperovals of $W(3)$.

The remaining ones are exactly the 45 hyperovals which appear in \mathcal{U}_5 as common neighbourhoods mentioned above. Then we assume that $\Gamma(u) = \Delta \cong W(3)$. We deduce that Γ is a strongly regular graph having the same parameters as \mathcal{U}_5 . Then we establish that $\Gamma(x) \cong \Delta$ for *any* $x \in \Gamma$. Moreover, we see that Γ has *quadruples*, that is, for each nonadjacent pair of points x, y there are exactly two other points z, z' such that $\Gamma(x, y) = \Gamma(x, y, z, z')$. This defines on Γ the structure of a partial linear space with line size 4. One can then check that the latter partial linear space is such that the lines and the affine planes on any point form a finite GQ. These objects were classified in [8]. The application of [8, 5] completes the proof that $\Gamma \cong \mathcal{U}_5$ (alternatively, we demonstrate how to use the classification of generalized Fischer spaces [6, 10] to get the same result).

The remaining case, where $\Gamma(x) \cong Q_4(3)$ for any x , is dealt with similarly. It turns out that this assumption leads to a contradiction.

2.1 Preliminaries

The determination of all the hyperovals in a GQ(3,3) Δ was based on an almost straightforward backtrack exhaustive search. It is natural to regard a hyperoval Ω as the subgraph of Δ induced by Ω . Clearly if Ω is disconnected then each connected component of Ω is a hyperoval, as well. So we look for the connected hyperovals only, and then, if possible, glue components together. We note, however, that all the hyperovals in Δ turn out to be connected. The main way to reduce the number of objects found by the search was the use of the group $G = \text{Aut}(\Delta)$. Indeed, for the set of orbits of G on the hyperovals of Δ it suffices to find a representative R_k for each orbit O_k . Moreover, the following idea proved to be highly successful.

Let S be a graph which is a subgraph of Ω for any hyperoval Ω of Δ (for instance, $S \cong K_2$). Let $\mathcal{S} = \{S_j\}$ be a set of representatives of the orbits of S on the subgraphs of Δ isomorphic to S . Then a set $\mathcal{R} = \{R_k\}$ of representatives of G -orbits on the hyperovals may be chosen in such a way that each $R_k \in \mathcal{R}$ contains some $S_j \in \mathcal{S}$.

So the problem is to find such S that the set \mathcal{S} is not huge and, on the other hand, the number of different hyperovals containing a given $S_k \in \mathcal{S}$ is not huge, as well. Since any Ω is a triangle-free graph of valence 4, we choose as S the *4-claw*, that is the subgraph induced on the union of $\{x\}$ and $\Omega(x)$. We find a set \mathcal{S} . Then for each $S_k \in \mathcal{S}$ we perform the exhaustive search of the hyperovals containing S_k . The resulting set \mathcal{R} need not be a minimal one, that is, it may contain several representatives for one G -orbit. Thus, finally, we construct a minimal set \mathcal{R}' .

The computer calculations were carried out using the GAP system for algebraic computations [14] along with the package GRAPE for computations in graphs [24]. The latter uses the package NAUTY for computations of automorphisms and isomorphisms of graphs [15].

Proposition 2.1 *The hyperovals Φ of $\Delta \cong \mathcal{U}_4 \cong W(3)$ are as follows.*

1. 432 of size 20. The 20 points outside Φ are collinear to 8 points of Φ .
2. 540 of size 16.
3. 720 of size 12. There are 2 points outside Φ collinear to 6 points of Φ .

4. 45 of size 8. The 32 points outside Φ are collinear to 2 points of Φ . Let $v \in \Phi$ and Φ' be a hyperoval containing v . Then $\Phi(v) = \Phi'(v)$ implies $\Phi' = \Phi$. Let Φ' be a hyperoval of type 2. Then $\Phi \cap \Phi' \not\cong K_2$. There is a one-to-one correspondence between the hyperovals of size 8 and the isotropic points of $T = T(\Delta)$ given by $\Phi = \Delta \cap p^\perp$, where $p \in T$ is an isotropic point.

The group $\text{Aut}(\Delta)$ acts transitively on the hyperovals of each type. □

Note that the first part of Proposition 2.1, contradicts the first (technical) part of the statement of [3, Proposition 8]. Note that the second part of that statement remains valid, as we shall see later.

Proposition 2.2 *The hyperovals Φ of $\Delta \cong Q_4(3)$ are as follows.*

1. 1080 of size 14. There are 8 points outside Φ collinear to 6 points of Φ . There are only 4 hyperovals intersecting Φ in $3K_2$.
2. 360 of size 18. There are 12 points outside Φ collinear to 6 points of Φ . All the hyperovals intersecting Φ in $3K_2$ are of type 1.
3. 324 of size 20. The points outside Φ are collinear to 8 points of Φ . All the hyperovals intersecting Φ in $4K_2$ are of type 2.
4. 135 of size 16. There are 16 points outside Φ collinear to 4 points of Φ , and the remaining 8 points are collinear to 8 points of Φ . The hyperovals intersecting Φ in $4K_2$ are of type 1 or 2.
5. 216 of size 10. There are 20 points outside Φ collinear to 2 points of Φ , and the remaining 10 points are collinear to 4 points of Φ . There are 60 (resp. 20) hyperovals of type 5 (resp. of type 1) intersecting Φ in K_2 .
6. 270 of size 12. There are 24 points outside Φ collinear to 4 points of Φ , and the remaining 4 points are not collinear to the points of Φ . There are exactly 24 hyperovals of type 6 intersecting Φ in $2K_2$.

The group $\text{Aut}(\Delta)$ acts transitively on the hyperovals of each type. □

Lemma 2.3 *In the notation of Proposition 2.2, only hyperovals of types 5 or 6 may appear as μ -graphs of EGQ .*

Proof. It follows from Proposition 2.2, 1, that any hyperoval Φ of type 1 cannot appear as μ -graph. Indeed, there must be at least 8 hyperovals intersecting Φ in $3K_2$, but there are only 4 such ones.

Next, if Φ is of type 2 there must be other μ -graphs intersecting Φ in $3K_2$, but all of them, by Proposition 2.2, must be of type 1, a contradiction.

Similarly, we reject hyperovals of types 3 and 4. □

Now the following statement is immediate.

Corollary 2.4 *Let x, y be two points of Γ at distance 2 such that $|\Gamma(x, y)| = 20$. Then $\Gamma(x) \cong \Gamma(y) \cong W(3)$. □*

2.2 A point of $W(3)$ -type exists

Here we assume that $\Gamma(u) = \Delta \cong W(3)$ for some $u \in \Gamma$.

Lemma 2.5 *In the notation of Proposition 2.1, only hyperovals of types 2 or 4 may appear as μ -graphs.*

Proof. First, we show that the hyperovals of type 1 cannot appear as μ -graphs. Let $\Phi = \Gamma(u, v)$ be a type 1 hyperoval. By Corollary 2.4, we have $\Gamma(v) \cong W(3)$. The following facts obtained by means of a computer will be used.

1) There are two orbits O_1, O_2 of lengths 30 and 10, respectively, in the action of the stabilizer H of Φ in $G = \text{Aut}(\Delta)$ on the set of edges of Φ .

2) There are 25 hyperovals intersecting Φ in the disjoint union of 4 copies of K_2 . All of them are of type 1. The group H has two orbits Ω_1, Ω_2 of lengths 20 and 5, respectively, on this set of hyperovals. If $\Psi \in \Omega_1$ then it contains exactly one edge from O_2 . If $\Psi \in \Omega_2$ then all the edges from $\Psi \cap \Phi$ belong to O_2 . Given $e \in O_1$, there exist exactly two hyperovals $\Psi \in \Omega_1$ such that $e \subset \Psi$.

By Proposition 2.1 there are 20 vertices in $\Xi = \Gamma(v) \setminus \Gamma(u)$ such that for each $x \in \Xi$ we have that $\Gamma(x) \cap \Phi$ is the disjoint union of 4 copies of K_2 . Hence $\Gamma(u, x)$ belongs to $\Omega = \Omega_1 \cup \Omega_2$. So we have constructed a mapping ϕ from Ξ to Ω . It is an injection, since the disjoint union of 4 copies of K_2 determines four of the lines of Δ , and Δ is a partial linear space. We have to choose 20 of the 25 elements of Ω . Since the line size of $GQ(3,3)$ is 4, for each edge e of Φ there exist exactly two $\Psi \in \phi(\Xi)$ such that $e \subset \Psi$. Now it follows from 2) that $\Omega_1 \subseteq \phi(\Xi)$. Hence $\phi(\Xi) = \Omega_1$.

Now the graph $\Gamma_2(u)$ is isomorphic to a connected component of the graph whose vertex set is the set Φ^G (i.e. the hyperovals of type 1), and the edge set is $\{\Phi^G, \Psi^G\}$, where $\Psi \in \Omega_1$. It is easy to check, either by computer or by exploiting its G -invariance, that the latter graph is connected. This is a contradiction, since the latter graph has 432 vertices, whereas $\Gamma_2(u)$ has 54. Thus Φ cannot be of type 1.

Let Φ be of type 3. There exists a two-element subset Ξ of $\Gamma(v) \setminus \Gamma(u)$ such that $\Gamma(x) \cap \Phi$ is the disjoint union of 3 copies of K_2 for any $x \in \Xi$. On the other hand, computer calculations show that each hyperoval Ψ such that $\Phi \cap \Psi$ is the disjoint union of 3 copies of K_2 is of type 1. This is the contradiction. \square

The following is well known.

Lemma 2.6 *Let Ξ be a $GQ(3,3)$. Assume that Ξ has a local hyperoval of type 4, that is, isomorphic to $GQ(1,3)$. Alternatively, assume that Ξ possess a triple x, y, z , of pairwise noncollinear points such that $\Gamma(x, y) = \Gamma(x, z)$. Then $\Xi \cong W(3)$. \square*

Lemma 2.7 *There exists $v \in \Gamma_2(u)$ such that $\Gamma(u, v)$ is of type 4.*

Proof. Suppose that this is not the case. Hence, by Lemma 2.5, for any $v \in \Gamma_2(u)$ the hyperoval $\Gamma(u, v)$ is of type 2. We have

$$|\Gamma_2(u)| = 40 \cdot 27/16,$$

which is not an integer. This is a contradiction. \square

By Lemma 2.7, there exists $v \in \Gamma_2(u)$ such that $\Gamma(u, v)$ is of type 4. By Lemma 2.1, 4, for each $w \in \Gamma(v) \setminus \Gamma(u)$ the subgraph $\Gamma(u, w)$ is a hyperoval of type 4. Note that $|\Gamma(v) \setminus \Gamma(u)| = 32$ coincides with the number of hyperovals intersecting Ω in a K_2 , and all such hyperovals are of type 4. The graph Σ defined on the type 4 hyperovals by the rule that two vertices are adjacent if the intersection of the corresponding hyperovals equals K_2 is isomorphic to the complement of the collinearity graph of $\text{GQ}(4,2)$, in particular it is connected. Hence for any x belonging to the connected component Ξ of $\Gamma_2(u)$ containing v , one has that $\Gamma(u, x)$ is of type 4.

Clearly Ξ is a cover of Σ . Hence *each* type 4 hyperoval of $\Gamma(u)$ is a μ -graph of Γ . Since $\mu(\Sigma) = 24$, which is greater than the size of any of hyperovals, Ξ is a connected proper cover of Σ . It implies that for any $x \in \Xi$ there exists $y \in \Xi$ such that $\Gamma(u, x) = \Gamma(u, y)$. Let $w \in \Gamma(u, x) = \Omega$. Since Ω is a graph of valence 4, we have for $\Theta = \Gamma(w)$ that $\Theta(u, x) = \Theta(u, y)$. Note that x is not adjacent to y . Hence by Lemma 2.6, $\Theta \cong W(3)$. Therefore there exists a fourth point $t \in \Theta$ not collinear to u, x , or y , such that $\Theta(u, x) = \Theta(u, t)$. By Lemma 2.1 4), $\Gamma(u, t) = \Gamma(u, x)$. Thus for *any* type 4 hyperoval Ω of Δ there exist *three distinct* points x_1, x_2, x_3 of $\Gamma_2(u)$ such that $\Gamma(u, x_i) = \Omega$ for $i = 1, 2, 3$. Now, counting in two ways the edges between $\Gamma(u)$ and $\Gamma_2(u)$, one has that for any $x \in \Gamma_2(u)$ the hyperoval $\Gamma(u, x)$ is of type 4. In particular, $|\Gamma_2(u)| = |\Gamma| - 40 - 1 = 135$.

Note that for any $z \in \Delta$ we have $\Gamma(z) \cong W(3)$. Since there exists $p \in \Gamma_2(z)$ such that $\Gamma(z, p)$ is of type 4, the repetition of the above argument gives us that for any $q \in \Gamma_2(z)$ the hyperoval $\Gamma(z, q)$ is of type 4. Therefore Γ is a strongly regular graph (SRG, for short) with the same parameters as \mathcal{U}_5 .

To summarize, we state the following.

Proposition 2.8 *Let Γ be a triangular EGQ(3,3) such that $\Gamma(u) \cong W(3)$ for some $u \in \Gamma$. Then $\Gamma(v) \cong W(3)$ for any $v \in \Gamma$. Moreover, Γ is an SRG with the same parameters as \mathcal{U}_5 , and for any distance two pair of vertices x, y there exist two more vertices $w = w_{xy}, w' = w'_{xy}$ such that $\Gamma(x, y) = \Gamma(x, w) = \Gamma(x, w')$. \square*

Using the above mentioned quadruples of points of Γ , we may define the structure of a partial linear space \mathcal{L} of line size 4 on Γ . The idea to consider \mathcal{L} is due to Hans Cuyper [7].

To complete the current case it suffices to show that \mathcal{L} is a (finite) locally GQ with affine planes. This means the following. Consider the set of minimal-by-inclusion subspaces generated by the pairs of intersecting lines of \mathcal{L} (such subspaces are usually called *planes*). We require all the planes which are linear spaces to be affine planes. Now the incidence system of lines and affine planes through a given point should be isomorphic to a (finite) GQ.

Proposition 2.9 *Let Γ be a triangular EGQ(3,3) such that $\Gamma(u) \cong W(3)$ for some u . Then $\mathcal{L} = \mathcal{L}(\Gamma)$ is a locally GQ(4,2) with affine planes, and therefore $\Gamma \cong \mathcal{U}_5$.*

Proof. Choose $w \in \Gamma$. Let $l = wu, m = wv$ be two distinct lines of \mathcal{L} on w such that u and v are nonadjacent, and let $p(l), p(m)$ be the corresponding isotropic points (cf. Proposition 2.1) of $T(\Gamma(w))$.

Let $x \in l, y \in m$ such that x and y are not equal to w . Since x and y lie at distance 2 in Γ , there is a line through them. Where are the other points, say z , on this line? Clearly $z \neq w$. We claim that $z \notin \Gamma(w)$, as well. Suppose it is false. Then by Proposition 2.1 z is adjacent to some vertex t of $\Gamma(w, x)$. Since $\Gamma(t)$ is a subspace of \mathcal{L} and xz is a line of this subspace, we have $y \in \Gamma(t)$, a contradiction to $\Gamma(w, x, y) = \emptyset$. So $z \in \Gamma_2(w)$, moreover $\Gamma(w, x, z) = \Gamma(w, y, z) = \emptyset$. Hence $p(z)$ belongs to the totally isotropic line of $T(w)$ generated by $p(x)$ and $p(y)$. We have shown that all the points in the subspace of \mathcal{L} generated by l and m and distinct from w belong to the set $\Pi = \{t \in \Gamma_2(w) \mid p(t) \perp p(x), p(t) \perp p(y)\}$. Clearly, Π generates a linear space on 16 points with line size 4, that is the affine plane of order 4. On the other hand, l and m generate a subspace of $\langle \Pi \rangle$. But any two lines in this plane generate it. So $\langle l, m \rangle \cong AG(2, 4)$.

It is easy to check that a pair of lines through w corresponding to the pair of nonorthogonal isotropic points of $T(\Gamma(w))$ does not generate a linear subspace of \mathcal{L} . Thus the only planes of \mathcal{L} are the affine ones. Moreover, the planes on w are in one-to-one correspondence with the totally isotropic lines of $T(\Gamma(w))$. So \mathcal{L} is a locally GQ(4,2) with affine planes.

The rest of the proof is straightforward and consists of implementation of the corresponding classification result [8], which says that the locally (finite) GQ with affine planes are in one-to-one correspondence with the standard quotients of the corresponding affine polar spaces, see [5, 9]. It is easy to deduce from [5, 8] that our object is the desired \mathcal{U}_5 . □

Remark. Alternatively, it suffices to show that \mathcal{L} is a generalized Fischer space, and then apply their classification. A generalized Fischer space is a connected partial linear space such that each of its planes is either affine or dual affine. Also, we know that there are only finitely many lines (points) on a given point (resp. line) and the cocollinearity graph of \mathcal{L} (that is, Γ) is connected. Such spaces are called finite and irreducible. Let us prove that \mathcal{L} is a generalized Fischer space. It was shown above that if $l = wu, m = wv$ are two distinct lines on w such that u and v are not adjacent in Γ , then $\langle l, m \rangle \cong AG(2, 4)$. Thus it suffices to prove that if u and v are adjacent, then $\langle l, m \rangle \cong AG^*(2, 3)$. Indeed, in this case we may pick $t \in \Gamma(w, u, v)$ and consider $\Gamma(t)$ as a subspace of \mathcal{L} . Since $\langle l, m \rangle \cap \Gamma(t)$ is isomorphic to $AG^*(2, 3)$, so is $\langle l, m \rangle$.

Now it easily follows from the classification of finite, irreducible Fischer spaces given in [6, 10] that $\Gamma \cong \mathcal{U}_5$.

2.3 The remaining case

Here we assume that $\Gamma(u) \cong Q_4(3)$ for any point u of Γ . It follows from Lemma 2.6 that Γ has distinct μ -graphs, that is, if $\Gamma(x, y) = \Gamma(x, z)$, where y and z both at distance two from x , then $y = z$.

We claim that hyperovals of type 5 must appear as μ -graphs of EGQ. Assume to the contrary that all the μ -graphs are arising from hyperovals of type 6. By a standard counting argument, there must be 90 such μ -graphs. It follows from

Proposition 2.2 that there are exactly 24 μ -graphs intersecting the given one in $2K_2$, and all with such intersection must be taken from the set X of type 6 hyperovals. So we may define an $O_5(3)$ -invariant graph on X of valence 24, two vertices being adjacent if the corresponding hyperovals intersect in $2K_2$. A union of connected components of this graph must be of size 90. But the maximal possible size of the blocks of imprimitivity of $O_5(3)$ on X is 10. This implies a contradiction, since the valence of X is greater than 10. We are done.

Finally, we prove that, to the contrary, hyperovals of type 5 cannot appear as μ -graphs of EGQ. We need one more statement rectifying the embedding of a type 5 hyperoval Φ in Δ .

Lemma 2.10 *Let $\Phi \subset \Delta$, X be the set of vertices of Δ outside Φ such that each vertex $x \in X$ satisfies $\Delta(x) \cap \Phi \cong K_2$. Φ is isomorphic to the two-fold antipodal cover of K_5 with the antipodal equivalence relation $\phi = \phi_\Phi$. X is a connected graph of valence 6, $x, y \in X$ being adjacent if $\Delta(y) \cap \Phi \cap \phi(\Delta(x) \cap \Phi)$ is of size one. \square*

Now assume that given $u \in \Gamma$, $v \in \Gamma_2(u)$, we have $\Gamma(u, v)$ of type 5. Let $W \subset \Gamma(v) \setminus \Gamma(u)$ such that $\Gamma(u, v, w) \cong K_2$ for any $w \in W$. The subgraph W is isomorphic to X defined in Lemma 2.10. Then $\Gamma(u, w)$ is of type 5, by Proposition 2.2 and Lemma 2.1. The set Y of type 5 hyperovals of $\Gamma(u)$ intersecting $\Gamma(u, v)$ in K_2 is of size 60 (Proposition 2.2), and the stabilizer of $\Gamma(u, v)$ in $O_5(3)$ acts transitively on Y . So in our attempt to select 20 of them we could start from any element of Y . Let Φ_1 be such a hyperoval. We try to form a graph isomorphic to the graph X defined in Lemma 2.10. There are exactly 6 elements Φ' of Y such that $\Phi' \cap \Gamma(u, v) \cap \phi_{\Gamma(u, v)}(\Phi_1 \cap \Gamma(u, v)) \cong K_1$ and $\Phi_1 \cap \Phi' \cong K_2$ or $2K_2$ (the latter is a necessary condition for the vertices in W associated with Φ_1 and Φ' to be adjacent, see Proposition 2.2, for the former see Lemma 2.10). Since X is of valence 6, we are forced to pick up all the 6 possible elements of Y . Proceeding in this manner (i.e. considering Φ' instead of Φ_1 , etc.), we however do not end up with 20 elements of Y , but with all 60 of them. Therefore it is impossible to assign to each vertex in W a hyperoval from Y , a contradiction.

The consideration of the case $\Delta \cong Q_4(3)$ is complete. Hence the proof of Theorem 1.1 is complete.

Note added in proof.

A step towards giving a computer-free proof of the result of this paper was made in A.A.Makhnev. Finite locally GQ(3,3)-graphs (in Russian). Siberian Math. J. 35(1994) 1314-1324.

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