Smash Products for G-sets, Clifford Theory and Duality Theorems

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Introduction

The categorical approach to the theory of group graded rings has been successful in recent years and in particular the use of smash products has provided several new ideas, cf. [4] and later [3], [10], [11], ... Another tool, the use of which is prompted upon us by the problems connected to the consideration of induced modules for a G-graded ring R with respect to a non-normal subgroup H of G, is the G-set theory, cf. [11]. The latter considerations stem from a general Clifford theory for group graded rings initiated by E. Dade, cf. [6] and [7] and extended recently in [9] and [12] to almost complete generality as far as graded modules are being concerned. However the G-set graded modules remained to be studied and in view of the Clifford theory with respect to a non-normal subgroup and the interest of the generalized Hecke algebras appearing in this theory, cf. [12], it is worthwhile to develop in some detail a theory of smash products for G-sets and its use in Clifford theory and duality theorems as in [4], [2].

In Section 2 of this paper we let R be a G-graded ring and A a G-set; the main results in this section, Theorem 2.1. and Theorem 2.7. yield a category isomorphism between $R\#A$-mod, that is the category of left modules over the smash product of R and A, and (G, A, R)-gr, that is the category of A-graded left modules over the G-graded ring R.

In Section 3 we extend E. Dade's Clifford theory to the case of G-set gradations; the main results in this section are Theorem 3.5. and Theorem 3.10. (the G-set

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Clifford's theorem). In analogy to E. Dade's functor $R\overline{\otimes}_{R^{(H)}}$- we construct here a generalized epi-mono functor, with respect to an H-set B and a G-set A,

$$R\overline{\otimes}_{R^{(H)}}- : (H, B, R^{(H)})\text{-gr} \to (G, A, R)\text{-gr}$$

We study when this functor (or the functor T^H in the sense of [11]) is an equivalence of categories.

Section 4 is concerned with duality theorems and here the main result is Theorem 4.2. As a consequence we do obtain that $(R\#A) * T \cong M_A(R)^f$, where T is a subgroup of $\mathrm{Aut}_G(A)$ making A into a transitive right T-set, $(R\#A)*T$ is a suitably defined crossed product of the smash product $R\#A$ and the group $T, M_A(R)^f$ is the matrix ring of finite A by A-matrices over the ring R. This yields an extension of some results of [2], [10].

Notation and terminology stems from [13].

1 Preliminaries

Throughout this paper G will be a multiplicative group with identity element 1 and $R = \oplus_{\sigma \in G} R_\sigma$ is a G-graded ring the identity element of which is also written 1 (but this will cause no ambiguity). A non-empty set A is a (left) G-set if there exists a left action of G on A given by a map $G \times A \to A, (\sigma, a) \mapsto \sigma a$, such that $1 \cdot a = a$ for all $a \in A$ and $(\sigma\tau)a = \sigma(\tau a)$ for all $\sigma, \tau \in G$ and $a \in A$. Modules will be left modules unless otherwise specified.

If A is a G-set then an R-module M is an A-graded R-module, or a graded R-module of type A, if $M = \oplus_{a \in A} M_a$ as additive groups and $R_\sigma M_a \subset M_{\sigma a}$ for all $\sigma \in G, a \in A$.
The category (G, A, R)-gr consists of the graded R-modules of type A and for such objects M and N the morphisms from M to N are given by $\mathrm{Hom}_{(G,A,R)-\mathrm{gr}}(M, N) = \{f : M \to N, f \text{ is } R-\text{linear and } (M_a)f \subset N_a \text{ for all } a \in A\}$. In [11] this category is studied in some detail, we recall from [11] that the category (G, A, R)-gr is a Grothendieck category whenever A is finite.

If A is a G-set then we may define the (generalized) smash product $R\#A$ as the free left R-module with basis $\{p_a, a \in A\}$ and multiplication defined by bilinear extension of : $(r_\sigma p_a)(s_\tau p_b) = (r_\sigma s_\tau)p_b$ if $\tau b = a$ and otherwise $(r_\sigma p_a)(s_\tau p_b) = 0$. In [11] the smash product was considered for finite A, our notion agrees with [10]. The ring $R\#A$ is always associative. In case A is finite, $R\#A$ has an identity $1 = \sum_{a \in A} p_a$ and $R \to R\#A, x \mapsto \sum_{a \in A} x p_a$, is an injective ring morphism. In general $R\#A$ does not have an identity element but it is a ring with local units, that is : every finite subset of $R\#A$ is contained in a subring of the form $e(R\#A)e$ where e is an idempotent of $R\#A$. Now $R\#A$-mod is the category of unitary (left) $R\#A$-modules in the sense that for $M \in R\#A$-mod we have $M = (R\#A)M$. Note also that $\{p_a, a \in A\}$ is a set of orthogonal idempotents of $R\#A$. For basic facts concerning $R\#A$ we refer to [4], [10], [11], [12].

2 Smash Products for G-sets

With notation as in Section 1 we have :

2.1. Theorem. Let $R = \oplus_{\sigma\in G} R_\sigma$ be a G-graded ring, A a G-set. Then : 1. $R\#A$-mod is a Grothendieck category with projective generator ${}_{R\#A}R\#A$.

2. The category (G, A, R)-gr is isomorphic to the category $R\#A$-mod.

Proof : 1. It is easy to check that $R\#A$-mod is a category with coproduct and that it satisfies $AB5$ [8], since $R\#A$ is a ring with local units. Let $f : M \to N$ be a nonzero morphism in $R\#A$-mod. Then there exists a $y \in \Lambda$ and $m \in M$ such that $(p_y m)f \neq 0$. Define a map $g :_{R\#A} R\#A \to M, rp_z \to rp_y m$ if $z = y$ and 0 otherwise. Then g is a morphism in $R\#A$-mod such that $gf \neq 0$. Hence ${}_{R\#A}R\#A$ is a generator of $R\#A$-mod and 1. holds.

2. If $N \in R\#A$-mod then $N = \oplus_{x\in A} p_x N$ (as additive groups). Define $N' = N$ as groups, $N'_x = p_x N$ for all $x \in A$, and $rn := (rp_x)n$ for all $r \in R, n \in N'_x$. Then $N' \in (G, A, R)$-gr. If $\phi : N \to P$ is a morphism in $R\#A$-mod, define $\phi' : N' \to P', n \mapsto (n)\phi$ for each $n \in N$, then ϕ' is a morphism in (G, A, R)-gr. Hence $(\ )' :\ R\#A\text{-mod} \to (G, A, R)\text{-gr},\ N \to N'$, is a covariant functor. Otherwise, let $M \in (G, A, R)$-gr then putting $M'' = M$ as an R-module we get $M'' \in R\#A$-mod by defining, for $r \in R, x \in A, m \in M, (rp_x)m = rm_x$.

If $\psi : M \to Q$ is a morphism in (G, A, R)-gr, define $\psi'' : M'' \to Q'', m \mapsto (m)\psi$ for each $m \in M$, then ψ'' is a morphism in $R\#A$-mod. Thus $(\ )'' :\ (G, A, R)$-gr $\to R\#A$-mod, $M \mapsto M''$, is a covariant functor.

Using the same argument as used in [3] or [4], we can prove that the functor $(\ )' : R\#A$-mod $\to (G, A, R)$-gr is an isomorphism with inverse ( )'. □

2.2. Remarks. 1. If $A = G$ and G acts on A by left translation, then $R\#G$ is exactly the generalization of the smash product defined by M. Beattie [3] and hence Theorem 2.6. of [3] holds.

2. If A is finite, then Theorem 2.13 of [11] follows from Theorem 2.1.2.

3. By Theorem 2.1.1., $({}_{R\#A}R\#A)'$ is a projective generator of (G, A, R)-gr. But $R\#A = \oplus_{y\in A} p_y(R\#A) = \oplus_{x\in A}(\oplus_{y\in A}(\oplus_{\sigma x=y} R_\sigma))p_x$ and $\oplus_{y\in A}(\oplus_{\sigma x=y} R_\sigma)p_x \cong \oplus_{y\in A}(\oplus_{\sigma x=y} R_\sigma) = R(x)$ (the x-shift of R) in (G, A, R)-gr. Thus $\oplus_{x\in A} R(x)$ is a projective generator of (G, A, R)-gr and hence Theorem 2.8. of [11] holds.

For a fixed $x \in A, G_x = \{\sigma \in G | \sigma x = x\}$ is the x-stabilizer in G. Then $R^{(G_x)} = \oplus_{\sigma\in G_x} R_\sigma$ is a subring of R. For any $y \in A$, let $V_y = \oplus_{\sigma y=x} R_\sigma, W_y = \oplus_{\sigma x=y} R_\sigma$, then the following assertions hold.

2.3. $V_x = R^{(G_x)} = W_x$, and V_y (resp. W_y) is a left (resp. right) unitary $R^{(G_x)}$-module.

2.4. $V(= R) = \oplus_{y\in A} V_y$ (as $R^{(G_x)}$-modules) and $W(= R) = \oplus_{y\in A} W_y$ belong to (G, A, R)-gr and become (left) unitary $R\#A$-modules by defining, for $r \in R, y \in A, w \in W : (rp_y)w = rw_y$.

2.5. $V_{\sigma y}R_\sigma \subseteq V_y$ for all $\sigma \in G, y \in A$, and V becomes right unitary $R\#A$-module by defining, for $v \in V, r \in R, y \in A$:

$$v(r\rho_y) := (vr)_y(\text{ in } V)$$

2.6. V(resp.W) is an $R^{(G_x)} - R\#A-$(resp. an $R\#A - R^{(G_x)}$-bimodule).
Of course we must be careful now to distinguish the rings $V = R$ and $W = R$ from the right $R\#A$-module V and left $R\#A$-module W respectively.

For $v \in V, w \in W$, define (,): $V \otimes_{R\#A} W \to R^{(G_x)}, (v,w) = (v,w)_{G_x}$ (in R), and define [,] $W \otimes_{R^{(G_x)}} V \to R\#A, (w,v) = \sum_{z\in\Lambda} wv_zp_z$. We only show that [,] is right $R\#A$-linear and the compatibility conditions are satisfied. For the remaining computations we refer to [8]. For $v, v' \in V, w, w' \in W, r \in R, y \in A$: $[w, v(rp_y)] = [w,(vr)_y] = w(vr)_yp_y$, and $[w,v](rp_y) = \sum_z(wv_zp_z)(rp_y) = \sum_z\sum_{\rho y=z} wv_zr_\rho p_y = w(\sum_z\sum_{\rho y=z} v_zr_\rho)p_y = w(vr)_yp_y = [w, v(rp_y)]$, so that [,] is right $R\#A$-linear.
Also : $[w,v]w' = \sum_z(wv_zp_z)w' = \sum_z wv_zw'_z = w(v,w')$, and

$$\begin{aligned} v'[w,v] &= v'\sum_z wv_z\rho_z = \sum_z (v'wv_z)_z(\text{ in } V) \\ &= (v'w)_{G_x}\sum_z v_z = (v',w)v \end{aligned}$$

Thus the compatibility conditions are satisfied. Therefore we have :

2.7. Theorem. The sextuple $\{R^{(G_x)}, V, W, R\#A, (\ , \ ), [\ , \ ]\}$ is a Morita context. The Morita context is strict if and only if for each $y \in A, \sum_{\rho y=x} R_{\rho^{-1}}R_\rho = R_1$.

Proof : We only need to prove the last statement. By [1, Theorem 2.2.], the Morita context is strict, i.e. the functors $W \otimes_{R^{(G_x)}} - : R^{(G_x)}$-mod $\to R\#A$-mod and $V \otimes_{R\#A} - : R\#A$-mod $\to R^{(G_x)}$-mod are inverse equivalences of categories, if and only if [,] and (,) are surjective if and only if [,] is surjective since R has unit and hence (,) is surjective. If [,] is surjective and then for any $y \in A, p_y = \sum_{i=1}^n [w_i, v_i] = \sum_{i=1}^n \sum_{z\in A} w_iv_{iz}p_z, 1 = \sum_{i=1}^n w_iv_{iy} = \sum_{i=1}^n\sum_{\rho y=x} w_{i\rho^{-1}}v_{i\rho}$, where $w_{i\rho^{-1}} \in R_{\rho^{-1}}, v_{i\rho} \in R_\rho$, so $\sum_{\rho y=x} R_{\rho^{-1}}R_\rho = R_1$.

Conversely, if $\sum_{\rho y=x} R_{\rho^{-1}}R_\rho = R_1$ for any $y \in A$, then $1 = \sum_{i=1}^n\sum_{\rho y=x} w_{i\rho^{-1}}v_{i\rho}$ for some $w_{i\rho^{-1}} \in R_{\rho^{-1}}, v_{i\rho} \in R_\rho$, and $rp_y = \sum_{i=1}^n\sum_{\rho y=x}[rw_{i\rho^{-1}}, v_{i\rho}]$ for all $r \in R$, so [,] is surjective. □

2.8. Corollary. Let R be a G-graded ring, A a finite transitive G-set and $x \in A$. If $\sum_{\rho y=x} R_{\rho^{-1}}R_\rho = R_1$ for all $y \in A$ then $R\#A$ and $R^{(G_x)}$ are Morita equivalent. (Since in this case $R\#A$ is ring with unit) □

2.9. Remark. 1. Corollary 3.11 of [11] immediately follows from Theorem 2.7.
2. If H a subgroup of $G, A = G/H$ the set of left H-cosets in G with the usual G-action on it defined by translation and $x = \{H\}$, then Corollary 2.19 of [11] holds, i.e. if R is a strongly G-graded ring and H is a subgroup of finite index in G, then $R\#G/H$ and $R^{(H)}$ are Morita equivalent.
3. Let $A = G/H, x = \{H\}$, then Corollary 3.12 of [11] follows from Theorem 2.7.

3 Direct Clifford Theory

Let A be a G-set, H a subgroup of G and let B be a subset of A such that $\sigma B \subset B$ for all $\sigma \in H$. We may define a functor $T^B : (G,A,R)\text{-gr} \to (H,B,R^{(H)})\text{-gr}$ as follows : for $M \in (G,A,R)$-gr, $T^B(M) = M^{(B)} = \oplus_{b\in B} M_b$, and if $f : M \to N$ is a morphism in (G,A,R)-gr then $T^B(f) = f \mid M^{(B)}$. In view of Theorem 3.7 of [11] the functor T^B has a left adjoint S^B and a right adjoint S_B. In case $\sigma B \subset B$ for $\sigma \in G$ implies $\sigma \in H$ then $T^B \circ S_B = T^B \circ S^B$ is the identity of $(H,B,R(H))$-gr.

We now extend the construction of the epi-mono functor $R\overline{\otimes}_{R_1}$- given by E. Dade to the generality of our situation, obtaining $R\overline{\otimes}_{R^{(H)}} - : (H,B,R^{(H)})\text{-gr} \to (G,A,R)$-gr. For any $M \in (G,A,R)$-gr, the B-null socle $S(B)(M)$ is a graded submodule of M with x-components :

$$
\begin{aligned}
(3.1) \qquad S(B)(M)_x &= \{m \in M_x | R_{h\rho^{-1}} m = 0 \text{ for all } h \in H\} \text{ if } \rho^{-1}x \in B,\\
&= M_x \text{ if } Gx \cap B = \phi
\end{aligned}
$$

Then the following properties hold :

(3.2) $S(B)(M)$ is the largest (under inclusion) graded submodule N of M such that $N^{(B)} = 0$

(3.3) $S(B)(M/S(B)(M)) = 0$.

(3.4) If $f : M \to N$ is a morphism in (G,A,R)-gr, then $S(B)(M)f \subseteq S(B)(N)$. For any $N \in (H,B,R^{(H)})$-gr, by [11], $S^B(N) = R \otimes_{R^{(H)}} N = \oplus_{x\in A}(R \otimes_{R^{(H)}} N)_x \in (G,A,R)$-gr, where $(R \otimes_{R^{(H)}} N)_x = \sum_{\sigma y = x} R_\sigma \otimes N_y$.

Now we define $R\overline{\otimes}_{R^{(H)}} N = R \otimes_{R^{(H)}} N/S(B)(R \otimes_{R^{(H)}} N)$. If $f : N \to P$ is a morphism in $(H,B,R^{(H)})$-gr we define $R\overline{\otimes}_{R^{(H)}} f : R\overline{\otimes}_{R^{(H)}} N \to R\overline{\otimes}_{R^{(H)}} P, r\overline{\otimes} n \mapsto r\overline{\otimes}(n)f$ for all $r\overline{\otimes} n = r \otimes n + S(B)(R \otimes_{R^{(H)}} N) \in R\overline{\otimes}_{R^{(H)}} N$.

3.5. Theorem. With notation as above, $R\overline{\otimes}_{R^{(H)}} - : (H,B,R^{(H)})\text{-gr} \to (G,A,R)$-gr is an additive functor which preserves direct sums. Furthermore,

1. if $\sigma B \subseteq B, \sigma \in G$, implies $\sigma \in H$, then $T^B \circ R\overline{\otimes}_{R^{(H)}}$-= the identity of $(H,B,R^{(H)})$-gr and T^B induces an equivalence from $(H,B,R^{(H)})$-gr to (G,A,R)-gr $(B) = \{M \in (G,A,R)\text{-gr} \mid S(B)(M) = 0 \text{ and } M = RM^{(B)}\}$ with inverse $R\overline{\otimes}_{R^{(H)}} -$.

2. if $\sum_{h\in H} R_{\rho h^{-1}} R_{h\rho^{-1}} = R_1$ for all $\rho \in G$, then S^B, S_B and $R\overline{\otimes}_{R^{(H)}} -$ are isomorphic.

3. if $A = \bigcup_{\sigma\in G} \sigma B$ and the above conditions 1 and 2 hold, then the functor T^B is an equivalence from $(H,B,R^{(H)})$-gr to (G,A,R)-gr with inverse S^B, S_B or $R\overline{\otimes}_{R^{(H)}} -$.

Proof : 1. For any $N \in (H,B,R^{(H)})$-gr, we have $(R\overline{\otimes}_{R^{(H)}} N)^{(B)} = \sum_{x\in B}(R\overline{\otimes}_{R^{(H)}} N)_x = R^{(H)} \otimes_{R^{(H)}} N \to^{\sim} N$ in $(H,B,R^{(H)})$-gr, and $R\overline{\otimes}_{R^{(H)}} N \in (G,A,R)\text{-gr}(B)$. On the other hand, for any $M \in (G,A,R)\text{-gr}(B)$, define $\varphi : R \otimes_{R^{(H)}} M^{(B)} \to M, r \otimes m \mapsto rm$ for all $r \in R, m \in M^{(B)}$, then φ is a natural epimorphism in (G,A,R)-gr. By $S(B)(M) = 0$, $\text{Ker}(\varphi) = S(B)(R \otimes_{R^{(H)}} M^{(B)})$ and hence $R\overline{\otimes}_{R^{(H)}} M^{(B)} \simeq M$ in (G,A,R)-gr. Thus 1 holds.

2. If $\sum_{h\in H} R_{\rho h^{-1}}R_{h\rho^{-1}} = R_1$ for all $\rho \in G$, then for any $M \in (G,A,R)$-gr, $S(B)(M)_x = 0$ if $Gx \cap B \neq \phi$ and M_x otherwise. Therefore $S^B(N) = R\overline{\otimes}_{R^{(H)}}N$ for all $N \in (H,B,R^{(H)})$-gr. Similar to the proof of Proposition 3.10 of [11], we have that S^B and S_B are isomorphic, so that 2 holds.

3. Since $A = \bigcup_{\sigma\in G}\sigma B, S(B)(M) = 0$ for all $M \in (G,A,R)$-gr, and $M_{\rho x} = (\sum_{h\in H} R_{\rho h^{-1}}R_{h\rho^{-1}})M_{\rho x} \subseteq \sum_{h\in H} R_{\rho h^{-1}}M^{(B)} \subseteq RM^{(B)}$ for all $\rho \in G, x \in B$. Hence $M = RM^{(B)}$, and (G,A,R)-gr $= (G,A,R)$-gr(B) and 3 holds.

3.6. Remark. 1. Let A be a G-set, B a subset of A. Then $A_1 = \bigcup_{\sigma\in G}\sigma B$ and $A_2 = A - A_1$ are G-subsets of A. By Proposition 2.6 of [11], the category (G,A,R)-gr is equivalent to the product (G,A_1,R)-gr $\times(G,A_2,R)$-gr. One will see that $R\overline{\otimes}_{R^{(H)}}-$, S^B and S_B are only the functor from $(H,B,R^{(H)})$-gr to (G,A_1,R)-gr.

2. If the condition "R is a strongly G-graded ring" is replaced by $\sum_{h\in H} R_{\rho h^{-1}}R_{h\rho^{-1}} = R_1$ for all $\rho \in G$, then Proposition 3.10 of [11] also holds.

3. If $A = G, B = H \subset G$, then the above conditions 1 and 3 hold. In this case, the constructions of the H-null socle and the epi-mono functor $R\overline{\otimes}_{R^{(H)}}-$ are the same as the ones appearing in [6], Sect. 5.

3.7. Corollary. Let A be a G-set, H a subgroup of G. If B is a subset of A such that $\sigma B \subseteq B$, for $\sigma \in G$ if and only if $\sigma \in H$, then

1. If N is a simple object in $(H,B,R^{(H)})$-gr, then so is $R\overline{\otimes}_{R^{(H)}}N$ in (G,A,R)-gr.

2. Let M be a simple object in (G,A,R)-gr, then $M^{(B)}$ is either 0 or a simple object in $(H,B,R^{(H)})$-gr. In the latter case $R\overline{\otimes}_{R^{(H)}}M^{(B)} \to^{\sim} M, r\overline{\otimes}m \mapsto rm$ for all $r \in R, m \in M^{(B)}$, and hence $M = RM^{(B)}$.

3. Let M be an injective object in (G,A,R)-gr. If $S(B)(M) = 0$ then $M^{(B)}$ is an injective object in $(H,B,R^{(H)})$-gr, too.

Proof : 1. Follows from $S(B)(R\overline{\otimes}_{R^{(H)}}N) = 0$.

2. If $M^{(B)} \neq 0$ then $M = RM^{(B)} \in (G,A,R)$-gr(B) and the results hold.

3. Since $RM^{(B)}$ is an injective object in (G,A,R)-gr(B) we have the result.

3.8. Remark. 1. If $A = G$ and $B = H \subset G$, then Lemmas 6.1 and 6.3 and Theorem 6.4 of [6] follow from Theorem 3.5 and Corollary 3.7.

2. If $A = G$ and $B = H = 1$, then Propositions 2.2, 2.3 and 6.1 of [7] follow Corollary 3.7.

3. Let $B = \{x\}$ and $H = G_x$. If M is a simple object in (G,A,R)-gr, then M_x is either 0 or a simple $R^{(G_x)}$-module.

4. If $G_x = H$ for all $x \in A$, then any graded simple R-module (of type A) M is a semisimple $R^{(H)}$-module and H is a normal subgroup of G.

3.9. Corollary. Let R be a G-graded ring, H a subgroup of G. Then the functor $T^H : R\text{-gr} \to R^{(H)}\text{-gr}$, $M = \oplus_{\sigma\in G} M_\sigma \mapsto M^{(H)} = \oplus_{\sigma\in H} M_\sigma$, is an equivalence if and only if $\sum_{h\in H} R_{\rho h^{-1}} R_{h\rho^{-1}} = R_1$ for all $\rho \in G$. One may compare this result to Theorem 2.8 in [5].

An object M of (G, A, R)-gr is said to be (G, A)-torsionfree if M is $|G_x|$-torsionfree for any $x \in A$ such that G_x is finite. For any $M \in R$-mod we let $CD[M]$ be the class $\{X \in R\text{-mod}$ such that there exists an R-exact sequence $M^{(\mathcal{J})} \to M^{(\mathcal{T})} \to X \to 0$, for some sets $\mathcal{J}, \mathcal{T}\}$. Recall that to a Gabriel topology $\mathcal{F}$ there corresponds a quotient category $(R, \mathcal{F})$-mod, being the full subcategory of R-mod consisting of the $\mathcal{F}$-closed modules.

3.10. Theorem. Let R be a G-graded ring and A a G-set such that for all $x \in A$ the stabilizer G_x is finite and let M be an A-graded semisimple R-module in (G, A, R)-gr which is (G, A)-torsionfree. Put $E = \mathrm{End}_R(M)$, then $\mathcal{F} = \{L$, left ideal of E such that $ML = M\}$ is a (left) Gabriel topology of E. Moreover, the functors

$$\mathrm{Hom}_R(M, -) : CD[M] \to (E, \mathcal{F}) - \mathrm{mod},$$

$$M \otimes_E - : (E, \mathcal{F}) - \mathrm{mod} \ \to CD[M]$$

are inverse equivalences of categories. In particular, if M is finitely generated as an R-module, then $\mathcal{F} = \{E\}$ and $\mathrm{Hom}_R(M, -)$ induces an equivalence from $CD[M]$ to E-mod.

Proof. In view of Theorem 1.3. of [9] we only have to establish that M is Σ-quasiprojective as an R-module.

On the other hand, Proposition 2.6. and Corollary 2.7. of [11] entail that (G, A, R)-gr is equivalent to the product of the categories $(G/G_x, R)$-gr where x varies over a set of representatives for the G-orbits in A. So we may assume $A = \bigcup G/G_x$ is a disjoint union and $M \in (G, A, R)$-gr corresponds to $T(M) = \pi_x(G/G_x, R)$-gr such that $T(M) = \oplus_x M(x)$ as R-modules, where for all x we have $M(x) \in (G/G_x, R)$-gr. It is easily checked that, when M is simple, resp. semisimple, in (G, A, R)-gr then $M(x)$ is simple, resp. semisimple, in $(G/G_x, R)$-gr for all x.

Since M is (G, A)-torsionfree it follows that $M(x)$ is $|G_x|$-torsionfree. Theorem 2.1.2. of [12] allows to deduce that for these $x, M(x)$ is a direct summand of $N(x)$ in the category $(G/G_x, R)$-gr, where $N(x)$ is a semisimple G-graded R-module. Thus there exists a semisimple G-graded R-module N such that M is a direct summand of N in R-mod. It follows that M is Σ-quasiprojective since N is Σ-quasiprojective as an R-module.

4 Duality theorem

Let $R = \oplus_{\sigma \in G} R_\sigma$ be a G-graded ring, A a G-set, T a subgroup of $\mathrm{Aut}_G(A)$, i.e. for $t \in T, t : A \to A, x \mapsto xt$ is bijective and $\sigma(xt) = (\sigma x)t$ for all $\sigma \in G, x \in A$. Then T acts as a group of ring automorphisms on $R\#A$ on the left and right by ${}^t(ap_x) = ap_{xt^{-1}}$ and $(ap_x)^t = ap_{xt}$.
If A is finite and T-transitive, then $(R\#A)^T = R(= R(\sum_{x\in A} p_x))$.

Now we may form the skew group ring $(R\#A) * T$, generated by $a_\sigma p_x * t, a_\sigma \in R_\sigma, x \in A, t \in T$, with

$$(a_\sigma p_x * t)(b_\tau p_y * s) = (a_\sigma p_x)(b_\tau p_{y^{t^{-1}}}) * ts$$

4.1. Lemma. $(R\#A) * T$ is a ring with local units.

Proof. For any finite set of $w_i = a_{\sigma_i} p_{x_i} * T_i \in (R\#A) * t, i = 1, ..n$, we let $F = \{x \in A | x = \sigma_i x_i$ or $x_i t_i$ for some $i\}$ and $f = \sum_{x\in F} p_x$. Then $f * 1$ is an idempotent in $(R\#A) * T$ and $(f * 1)w_i(f * 1) = w_i, i = 1, ..., n$.

Put $e_{x,xt} = p_x * t$. Then $e_{x,xt}e_{y,ys} = e_{x,ys} = e_{x,xts}$ if $xt = y$ and 0 otherwise, hence $\{e_{x,xt} | x \in A, t \in T\}$ is a system of matrix units. On the other hand, $e_{x,x}(a_\sigma p_z * t) = a_\sigma p_z * t$ if $\sigma z = x$ and 0 otherwise, and $(a_\sigma p_z * t)e_{y,y} = a_\sigma p_z * t$ if $zt = y$ and 0 otherwise, so that $e_{x,x}((R\#A) * T)e_{y,y} = \sum_{\substack{\sigma z = x \\ zt = y}} R_\sigma p_z * t \neq 0$ if and only if $y \in GxT$ or $x \in GyT$. Denote $D_x = GxT = \{\sigma xt | \sigma \in G, t \in T\}$ for $x \in A$, then $\{e_{y,yt} | y \in D_x, t \in T\}$ is a complete system of matrix units if and only if $D_x = xT$.
Therefore we have the following

4.2. Theorem. With notation as above, assuming that a T-orbit $xT(x \in A)$ is always a G-subset of A, i.e. $GxT \subseteq xT$. Then $(R\#A) * T \cong \oplus_{x\in I} M_{xT}(R)^f$ (as rings)
where I a subset of A such that $A = \bigcup_{x\in I} xT$ (disjoint union), $M_{xT}(R)^f$ the ring of matrices over R with rows and columns indexed by xT and with finitely many nonzero entries.

Proof. Denote $X_x = \sum_{y,z\in xT} e_{y,y}((R\#A) * T)e_{z,z}$. Then X_x is a subring of $(R\#A) * T$ and

$$(R\#A) * T = \oplus_{x\in I} X_x \text{ (as rings)}$$

Furthermore, $X_x = \sum_{y,z} e_{y,y} X_x e_{z,z}$. Since $\{e_{y,yt} | y \in xT, t \in T\}$ is a complete system of matrix units, by Lemma 2.1 of [10], $X_x \cong M_{xT}(B)^f$, where $B = e_{x,x}X_x e_{x,x} = \sum_{\substack{\sigma y = x \\ yt = x}} R_\sigma p_y * t$.
Obviously, $\{\sigma \in G | \sigma y = x = yt$ for some $y \in A, t \in T\} = G$, and if we define $\alpha : B \to R, a_\sigma p_y * t \mapsto a_\sigma$, then α is a ring isomorphism since $(a_\sigma p_y * t)(b_\tau p_z * s) = a_\sigma b_\tau p_{zt} * ts$
for $a_\sigma p_y * t, b_\tau p_z * s \in B$. $\square$

4.3. Corollary. Let R be a G-graded ring, A a G-set, T a subgroup of $\mathrm{Aut}_G(A)$ such that A is a transitive right T-set. Then $(R\#A) * T \cong M_A(R)^f$.

In particular, if A is finite then $(R\#A) * T$ and R are Morita equivalent. $\square$

4.4. Examples. 1. Let R be a G-graded ring, H a subgroup contained in the centre of G. Then H is a left G-set with the usual conjugation G-action and it may be regarded as a subgroup of $\mathrm{Aut}_G(H)$ by translation, i.e. $h \in \mathrm{Aut}_G(H), h : x \mapsto xh$ for $x \in H$, and $(R\#A) * H \cong M_A(R)^f$.
2. If G is a subgroup of a group K. Then K is a left G-set with the G-action given by translation and it may be regarded as a subgroup of $\mathrm{Aut}_G(K)$ by translation (same as above 1.) and hence $(R\#K) * K \cong M_K(R)^f$.

In particular, if $K = G$ then we have Theorem 2.2. of [2] and Theorem 2.3. of [10].

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