Topology and closed characteristics of K-contact manifolds.

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Abstract

We prove that the characteristic flow of a K-contact form has at least n+1 closed leaves on a closed 2n+1-dimensional manifold. We also show that the first Betti number of a closed sasakian manifold with finitely many closed characteristics is zero.

1 Preliminaries

A contact form on a 2n+1-dimensional manifold M is a 1-form α such that the identity

$$\alpha \wedge (d\alpha)^n \neq 0$$

hold everywhere on M. Given such a 1-form α, there is always a unique vector field ξ satisfying $\alpha(\xi) = 1$ and $i_\xi d\alpha = 0$. The vector field ξ is called the *characteristic vector field* of the contact manifold (M, α) and the corresponding 1-dimensional foliation is called a contact flow.

The 2n-dimensional distribution $D(x) = \{v \in T_xM/\alpha(x)(v) = 0\}$ is called the *contact distribution*. It carries a 1-1 tensor field J such that $J^2 = -I_{2n}$, where I_{2n} is the identity 2n by 2n matrix. The tensor field J extends to all of TM by requiring $J\xi = 0$.

Also, the contact manifold (M, α) carries a nonunique riemannian metric g adapted to α and J in the sense that the following identities are satisfied

$$d\alpha(X, Y) = g(X, JY)$$

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and

$$\alpha(X) = g(\xi, X)$$

for any vector field X and Y on M. Such a metric g is called a *contact metric*. When the vector field ξ is Killing relative to the contact metric g, the triple (M, α, g) is called a K-contact manifold. If in addition, the identity

$$(\nabla_X J)Y = g(X,Y)\xi - \alpha(Y)X \tag{1}$$

is satisfied, then the contact metric structure is called *sasakian*. We refer to [3] for details on contact metric structures.

A flow on a manifold M is said to be almost regular ([16]) if each point on M belongs to a flow box pierced by the flow at most a finite number of times. As an easy consequence of a theorem of Wadsley ([17]), any almost regular contact flow on a compact manifold is riemannian. Therefore, by Proposition 1 in [12], any almost regular compact contact manifold is K-contact.

The existence of at least 2 closed characteristics has been proven for circle invariant contact forms in [1]. Using a result of Weinstein ([18]) about critical manifolds of a circle invariant function, we will provide sharper lower bounds for the number of closed characteristics for the class of contact flows mentioned above.

The k^{th} Betti numbers of a compact sasakian $2n+1$-dimensional manifold are zero or even for odd k smaller or equal to n ([4]). In particular, the first Betti number of such a manifold is zero or even ([15]) and Theorem 2 of this paper presents a sufficient dynamical condition for a closed sasakian manifold to be simply connected. We point out that the interplay between first Betti numbers and leaf closures of K-contact flows has been previously reported in [9].

2 Critical manifolds of circle invariant functions

Let M be a compact manifold with an effective circle action on it. In general, the quotient Σ of M by the circle action is not a manifold in the ordinary sense, it is a V-manifold ([18]) in the sense of Satake ([13]).

Definition 1 *The Lusternik-Schnirelman category of a space M, cat(M), is the minimum number of contractible open sets that can cover M.*

Definition 2 *The cuplength of a space M, denoted cuplength(M), is the maximum number m of positive degree cohomology classes $[\omega_1]$, $[\omega_2]$,...,$[\omega_m]$ such that $\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_m \neq 0$ on M.*

By the Lusternik-Schnirelman theory ([14]), one has

$$cat(M) \geq cuplength(M) + 1. \tag{2}$$

We refer to the work of Weinstein in [18] for the proof of the the following proposition:

Proposition 1 (Weinstein) *Let M be a compact manifold with an effective circle action and let Σ be the corresponding quotient V-manifold. Then any circle invariant function has at least $cat(\Sigma)$ critical circles.*

3 Circle invariant contact forms

Let M be a 2n+1 dimensional manifold with a circle invariant contact form α. If Z denotes the infinitesimal generator of the circle action, then it is shown in [1] that critical circles of the circle invariant function $\alpha(Z)$ are closed characteristics of α. For completeness, we briefly present the proof of this fact.

Let us denote by $\mathbf{S}$ the function $-\alpha(Z) = -i_Z\alpha$. Then, taking into account that the vector field Z leaves α invariant, we observes that

$$d\mathbf{S} = -di_Z\alpha = -L_Z\alpha + i_Zd\alpha = i_Zd\alpha.$$

Hence, a point $p \in M$ is critical if and only if Z and ξ are proportional at p. This implies in turn that the Z and ξ orbits through p are the same since Z and ξ commute.

When the restriction of α to each orbit of the circle action is nonsingular, a sharper lower bound to the number of those closed characteristics can be obtained. This is the case, as we shall see later, when α is a K-contact form.

Theorem 1 *Let M be a 2n+1 dimensional compact manifold with a circle invariant contact form α. If Z is the infinitesimal generator of the circle action and $\alpha(Z) \neq 0$ everywhere on M, then α has at least n+1 closed characteristics.*

Proof. As already mentionned above, critical circles of the circle invariant function $\mathbf{S} = \alpha(Z)$ are closed characteristics of α ([1]). Let $\beta = \frac{1}{\alpha(Z)}\alpha$. Then β is also a contact form. Since

$$d\beta = (-\frac{1}{(\alpha(Z))^2})di_Z\alpha \wedge \alpha + \frac{1}{\alpha(Z)}d\alpha \tag{3}$$

and $L_Z\alpha = 0$, we see that

$$i_Zd\beta = -\frac{Z\alpha(Z)}{(\alpha(Z))^2}\alpha + \frac{1}{\alpha(Z)}(di_Z\alpha + i_Zd\alpha) = 0, \tag{4}$$

that is, Z is the characteristic vector field of β. As a consequence, the quotient V-manifold Σ of M by the circle action generated by Z carries a symplectic form and since $(d\beta)^n \neq 0$, one sees that $cuplength(\Sigma) \geq n$. Now, by Proposition 1 and inequality (2), the function $\mathbf{S} = \alpha(Z)$ has at least $n+1$ critical circles, which are also closed characteristics of α.∎

Suppose now that M is a 2n+1-dimensional K-contact manifold with K-contact form α, characteristic vector field ξ and K-contact metric g. Since M is compact, its isometry group $\mathcal{I}(M)$ is a compact Lie group by a classical theorem of Myers and Steenrod ([8]). Let φ_t denote the real 1-parameter group of isometries generated by ξ. The closure of φ_t in $\mathcal{I}(M)$ is a torus group, hence one can find arbitrary close to ξ, a periodic Killing vector field Z which commutes with ξ and such that $\alpha(Z) \neq 0$ everywhere on M. A straightforward calculation shows that Z leaves α invariant. This shows also that K-contact forms satisfies the hypotheses of Theorem 1 and we have the following corollary:

Corollary 1 *Let M be a compact K-contact $2n+1$-dimensional manifold. Then the characteristic flow has at least $n+1$ closed leaves.*

Examples of K-contact flows with exactly $n+1$ closed characteristics are known on the $2n+1$-dimensional sphere ([10]). We ask ourselves the question: If a closed K-contact $2n+1$-dimensional manifold M has a finite number of closed characteristics, is the first Betti number of M necessarily zero? The affirmative answer to this question in dimension 3 follows trivially from Corollary 2 in [9] which says that a 3-dimensional closed K-contact manifold with nonzero first Betti number has all its characteristics closed. Using the material in the next section, we will prove the following theorem:

Theorem 2 *Let M be a closed sasakian manifold with a finite number of closed characteristics. Then the first Betti number of M is zero.*

4 Morse theory on sasakian manifolds

In this section, we will apply elementary Morse theory to the function that generates closed leaves of sasakian flows. We begin the section with some terminology, refering to [5] for more details on Morse theory.

A connected submanifold $N \subset M$ is called a nondegenerate critical manifold of a function f on M if the following conditions are satisfied:

(i). Each point $p \in N$ is a critical point of f.

(ii). The hessian of f, $Hess_f$, is nondegenerate in directions normal to N.

The normal bundle νN of a nondegenerate critical manifold N is decomposed into a positive and negative part

$$\nu N = \nu^+ N \oplus \nu^- N$$

where $\nu_p^+ N$ and $\nu_p^- N$ are the positive and negative eigenspaces of the hessian of f, $Hess_f$. The fiber dimension of $\nu^- N$, denoted by λ_N, is referred to as the index of N relative to f. A function f all of whose critical manifolds are nondegenerate is said to be a *clean function* ([6]). Let θ^- denote the orientation bundle of $\nu^- N$ and

$$P_t(M;\mathbf{R}) = \sum t^k dim H^k(M;\mathbf{R})$$

the Poincaré series of M. If one defines the Morse series ([5]) of f relative to the coefficient field $\mathbf{R}$ by

$$\mathcal{M}_t(f) = \sum t^{\lambda_N} P_t(N;\theta^- \otimes \mathbf{R}) \tag{5}$$

where the sum runs over all critical manifolds of f, then the Morse inequalities hold:

$$\mathcal{M}_t(f) \geq P_t(M;\mathbf{R}). \tag{6}$$

The inequalities (6) implies that $\mathcal{M}_t(f)$ majorizes $P_t(M;\mathbf{R})$ coefficient by coefficient.

Let M be a closed sasakian manifold with characteristic vector field ξ. We choose a vector field Z as in Theorem 1 so that closed characteristics are critical manifolds of the function

$$\mathbf{S} = \alpha(Z).$$

As in [11], let F_ξ denote the set of periodic points of the chararcteristic vector field ξ. F_ξ is a union of closed characteristics and each connected component of F_ξ is a totally geodesic odd dimensional closed submanifold of M ([11]). In fact, one can easily show that each connected component of F_ξ is a regular sasakian submanifold of M.

We now proceed to compute the hessian $Hess_{\mathbf{S}}$ of $\mathbf{S} = \alpha(Z)$ in directions perpendicular to a connected component N of F_ξ. Let $p \in N$ and v, w be two tangent vectors perpendicular to N at p. We extend v and w into local vector fields V and W by parallel translation along geodesics emanating from p. In particular, the identities $(\nabla V)(p) = 0 = (\nabla W)(p)$ are valid and will be used repeatedely as well as the identity $\nabla_X \xi = -JX$ valid on any K-contact manifold ([3]) and the fact that Z is a Killing vector field commuting with ξ.

$$\begin{aligned}
Hess_{\mathbf{S}}(p)(v,w) &= (V(Wg(\xi, Z)))(p) && (7)\\
&= (V(-g(JW,Z) + g(\xi, \nabla_W Z)))(p) && (8)\\
&= (-g(\nabla_V JW, Z) - g(JW, \nabla_V Z) - Vg(\nabla_\xi Z, W))(p) && (9)\\
&= (-g(\nabla_V JW, Z) - g(JW, \nabla_V Z) + Vg(JZ, W))(p) && (10)\\
&= (-g(\nabla_V JW, Z) - g(JW, \nabla_V Z) + g(\nabla_V JZ, W) && (11)\\
&\quad +g(JZ, \nabla_V W))(p) && (12)\\
&= (-g((\nabla_V J)W, Z) - g(J\nabla_V W, Z) - g(JW, \nabla_V Z) && (13)\\
&\quad +g((\nabla_V J)Z, W) + g(J\nabla_V Z, W))(p) && (14)\\
&= (-2g(V,W)\alpha(Z) + \alpha(W)g(V,Z) && (15)\\
&\quad -2g(JW, \nabla_V Z) + g(V,Z)\alpha(W))(p) \text{ by identity (1)} && (16)\\
&= -2g(v,w)(\alpha(Z))(p) - 2g(Jw, \nabla_v Z) \text{ evaluating at } p && (17)
\end{aligned}$$

Let $\alpha(Z)(p) = k$ and put $Z = k\xi + \delta$ where δ is a Killing vector field vanishing all along N. Then points at which δ is zero lie on periodic orbits of ξ and one has the identity:

$$\nabla_v Z = -kJv + \nabla_v \delta. \qquad (18)$$

Lemma 1 *(i). The tangent vector $\nabla_v \delta$ is nonzero and perpendicular to N.*

(ii). The hessian $Hess_{\mathbf{S}}$ of $\alpha(Z)$ along N is given by

$$Hess_{\mathbf{S}}(p)(v,w) = -2g(Jw, \nabla_v \delta)$$

and is nondegenerate in directions perpendicular to N.

Proof. Our proof of the first part of assertion (i) was inspired from the work of Kobayashi ([7]). If $\nabla_v \delta$ were zero, then, since $\delta(p) = 0$, the vector field δ would be zero along the geodesic γ tangent to v at p, hence every orbit that intersects γ

would be closed, contradicting the fact that v is perpendicular to N. Let η be an arbitrary vector field tangent to N in a neighborhood of p. Then

$$\begin{aligned} g(\nabla_v \delta, \eta) &= g(\nabla_v Z, \eta) - kg(\nabla_v \xi, \eta) \\ &= -g(v, \nabla_\eta Z) + kg(Jv, \eta) = 0. \end{aligned}$$

The first and second terms in the last identity are zero because N is totally geodesic and Jv is perpendicular to N. This proves that $\nabla_v \delta$ is perpendicular to N and completes the proof of assertion (i). Now combining identity (17) with identiry (18), we see that

$$Hess_{\mathbf{S}}(p)(v, w) = -2g(Jw, \nabla_v \delta)$$

and it is nondegenerate in directions perpendicular to N since $\nabla_v \delta$ is nonzero for every v in those directions. This completes the proof of Lemma 1.

Proposition 2 *The function* $\mathbf{S} = \alpha(Z)$ *is clean and each of its critical manifolds has even index.*

Proof. By Lemma 1, each critical submanifold of $\alpha(Z)$ is nondegenerate. It remains to prove the assertion about even index. To that end, let v be any direction perpendicular to N at p. Then, using identity (17),

$$\begin{aligned} Hess_{\mathbf{S}}(p)(Jv, Jv) &= -2g(Jv, Jv)\alpha(Z) - 2g(J^2 v, \nabla_{Jv} Z) \\ &= -2g(v, v)\alpha(Z) + 2g(v, \nabla_{Jv} Z) \\ &= -2g(v, v)\alpha(Z) - 2g(Jv, \nabla_v Z) = Hess_{\mathbf{S}}(p)(v, v). \end{aligned}$$

This clearly establishes the fact that if $Hess_{\mathbf{S}}$ is negative definite in the direction v, it is also negative definite in the direction Jv, hence the indices are all even.∎

5 Proof of Theorem 2

Suppose now that M is a sasakian closed manifold with a finite number of closed characteristics. Since, by Proposition 2, $\alpha(Z)$ is a clean function all of whose critical manifolds have even indices, $\alpha(Z)$ has a unique local maximum corresponding to a single component of the critical set ([6], page 501). Since M is compact, the unique local maximum is actually a global one and $\alpha(Z)$ has also a unique global minimum corresponding to a single component of the critical set. In other words, $\alpha(Z)$ has a unique critical circle with index 0. Let us denote that circle by N_0. The Morse series (5) can be written

$$\mathcal{M}_t(\alpha(Z)) = P_t(N_0; \theta^- \otimes \mathbf{R}) + \sum t^{\lambda_N} P_t(N; \theta^- \otimes \mathbf{R})$$

where λ_N are even integers each of which is at least equal to 2. But since $\nu^- N_0 = N_0$, one has

$$P_t(N_0; \theta^- \otimes \mathbf{R}) = P_t(N_0; \mathbf{R}) = 1 + t.$$

Hence, the Morse inequalities (6) become

$$1 + t + o(t^2) \geq 1 + t\ dim H^1(M; \mathbf{R}) + t^2 dim H^2(M; \mathbf{R}) + ...$$

and we derive the inequality

$$1 \geq dim H^1(M; \mathbf{R}).$$

But since Tachibana ([15]) has shown that the first Betti number of a compact sasakian manifold is zero or even, we conclude that under the conditions of Theorem 2, the dimension of $H^1(M; \mathbf{R})$ is zero.∎

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