Adjoints, Multi–adjoints, Pluri–adjoints.

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Abstract

Starting from a family of categories $\mathcal{A}_p$, where p ranges over a set P of indices, and functors $G_p : \ \mathcal{A}_p \longrightarrow \mathcal{X}$, $p \in P$, where $\mathcal{X}$ is a category and each G_p has a left adjoint F_p, we construct a category $\mathcal{A}$ and a functor $G : \ \mathcal{A} \longrightarrow \mathcal{X}$ that has a multi-adjoint in the sense of Y. Diers, and then a category $\overline{\mathcal{A}}$ and a functor $\overline{G} : \overline{\mathcal{A}} \longrightarrow \mathcal{X}$ that has a pluri-adjoint in the sense of the authors. These constructions show, at least in this instance, how the transition from adjoints to multi-adjoints to pluri-adjoints is performed. The problem of the existence and the characterization of a general class of pluri-adjoints arising in this manner invites further study.

1. Let P be a set, and let $(\mathcal{A}_p)_{p\in P}$ be a family of small categories. Let $\mathcal{X}$ be a small category. Assume that, for each $p \in P$, there is given a functor $G_p : \ \mathcal{A}_p \longrightarrow \mathcal{X}$ that has a left adjoint $F_p : \ \mathcal{X} \longrightarrow \mathcal{A}_p$, $F_p \dashv G_p$. Let $\mathcal{A}$ be the disjoint union of the categories $\mathcal{A}_p$, $p \in P$, that is, the coproduct in **Cat** (the category of small categories and functors between them) of the categories $\mathcal{A}_p$, $p \in P$. Recall that an *object* of $\mathcal{A}$ is then a pair (A, p) with A an object of $\mathcal{A}_p$ and $p \in P$. A *morphism* in $\mathcal{A}$ between (A, p) and (B, q) is a morphism $A \longrightarrow B$ in $\mathcal{A}_p$ if $p = q$ (otherwise there are no morphisms $(A, p) \longrightarrow (B, q)$). Thus,

$$\mathcal{A}((A,p),(B,q)) = \begin{cases} \mathcal{A}_p(A,B) & \text{if } p = q, \\ \emptyset & \text{if } p \neq q. \end{cases}$$

Received by the editors July 1994
Communicated by A. Warrinier
AMS Mathematics Subject Classification : 18A40, 18A35
Keywords : Multi-adjoints, pluri-adjoints.

Composition is clearly reduced to composition of morphisms in each $\mathcal{A}_p$. The *identity* of (A, p) in $\mathcal{A}$ is id_A in $\mathcal{A}_p$. We shall abbreviate (A, p) to A when the context permits.

We shall define a functor

$$G: \ \mathcal{A} \longrightarrow \mathcal{X}$$

as the functor $[G_p]_{p\in P}$ arising from the functors G_p , $p \in P$, by the universal property of the coproduct. Explicitly, for $(A, p) \in \mathrm{Ob}\,\mathcal{A}$, we shall define $G(A, p) = G_p(A)$, and if $f: \ (A, p) \longrightarrow (B, p)$ is a morphism in $\mathcal{A}$ (recall that the second components of the two objects must be equal for a morphism to exist), then, by definition, $G(f) = G_p(f)$. The G thus defined is indeed a functor. For example, if $(A, p) \xrightarrow{f} (B, p) \xrightarrow{g} (C, p)$ are morphisms in $\mathcal{A}$, then $G(g\circ f) = G_p(g\circ f) = G_p(g)\circ G_p(f) = G(g)\circ G(f)$.

Proposition 1.1. *The functor G defined above has a canonical left multi-adjoint, in the sense of Diers* [**1**,**2**].

Proof. Let $X \in \mathrm{Ob}\mathcal{X}$. To construct a *universal family* of morphisms [cf. loc. cit.] from X to G, consider the family $(F_p(X))_{p\in P}$ and the corresponding family $(F_p(X), p)_{p\in P}$ of objects of $\mathcal{A}$. For each p, let

$$\eta^p(X): \ X \longrightarrow G_p(F_p(X)) = G(F_p(X), p)$$

be the unit of the adjunction $F_p \dashv G_p$. To show that the family $(\eta^p(X))_{p\in P}$ is universal (in the sense of the definition of the multi-adjoint), let $(A, p) \in \mathrm{Ob}\mathcal{A}$ and let

$$f: X \longrightarrow G(A, p) = G_p(A)$$

be a morphism in $\mathcal{X}$. Since $F_p \dashv G_p$, there is a unique morphism $g: \ F_p(X) \longrightarrow A$ in $\mathcal{A}_p$ such that the composite

$$X \xrightarrow{\eta^p(X)} G_p(F_p(X)) \xrightarrow{G_p(g)} G_p(A)$$

is equal to f. In terms of $\mathcal{A}$ and G, this means that the composite

$$X \xrightarrow{\eta^p(X)} G(F_p(X), p) \xrightarrow{G(g)} G(A, p)$$

is equal to f. But, of course, since (A, p) is given, p is unique and then, as shown, g is unique. ∎

2. In order to go on with our construction, we need to make additional assumptions on the categories $\mathcal{A}_p$ and $\mathcal{X}$. First, we shall assume that $\mathcal{X}$ is a finitely complete category (that is, has all finite limits) and that it has a *zero object* (that is, the terminal object, which is the product of the empty family, is also initial), denoted 0. As a consequence of this assumption, $\mathcal{X}$ has a family of *zero morphisms*. That is, for each pair of objects X, Y of $\mathcal{X}$, there is the composite morphism $0_{X,Y}: \ X \longrightarrow 0 \longrightarrow Y$, where the first, resp. second, arrow arises from the quality of 0 as a terminal, resp. initial, object. We note that the composite of each such zero morphism with any morphism is also a zero morphism. In addition, we shall assume that each $\mathcal{A}_p$, $p \in P$, has a *zero object* 0_p and, of course, a class of *zero morphisms*. (In case any of these categories did not have a zero object, we can certainly add one.)

Now we are ready to construct a new category, denoted $\overline{\mathcal{A}}$. The *objects* of $\overline{\mathcal{A}}$ are finite sequences $A = (A_{p_1}, A_{p_2}, \ldots, A_{p_n})$ of objects A_{p_i} of $\mathcal{A}_{p_i}$, where the order of components does not really matter, but where it is assumed that components that lie on different spots come from different $\mathcal{A}_p$'s. Two such sequences, the above A and $B = (B_{q_1}, B_{q_2}, \ldots, B_{q_m})$, will be considered *equal* and will be identified iff each one is obtained from the other one by adding or deleting zero objects from categories $\mathcal{A}_p$. If all components of such a sequence are zero objects, then that sequence will consequently be equal to the *empty sequence*, which will be a zero object of $\overline{\mathcal{A}}$ (this will become clear when we define the morphisms). If $A = (A_{p_1}, A_{p_2}, \ldots, A_{p_n})$ and $B = (B_{p_1}, B_{p_2}, \ldots, B_{p_n})$ are two objects (by the above identification of sequences, we can always assume that in A and B we have the same set of indices), the *morphisms* $A \longrightarrow B$ in $\overline{\mathcal{A}}$ will be all sequences $f = (f_{p_1}, f_{p_2}, \ldots, f_{p_n})$, where $f_{p_i} : A_{p_i} \longrightarrow B_{p_i}$ in $\mathcal{A}_{p_i}$, $i = 1, 2, \ldots, n$. If in the above A one has $n = 0$, then there is exactly one morphism from each object to A and exactly one from A to each object. *Composition* of morphisms will be performed componentwise, and the *identity* of an object will be the sequence of identities of the components in their respective categories. We note that $\mathcal{A}$ is naturally a subcategory of $\overline{\mathcal{A}}$, namely, the one whose objects and morphisms have exactly one component.

Now, we shall define the functor

$$\overline{G} : \overline{\mathcal{A}} \longrightarrow \mathcal{X}.$$

First, let us remark that each G_p sends the zero object 0_p of $\mathcal{A}_p$ to the zero object of $\mathcal{X}$. This follows from the fact that each G_p , $p \in P$, has a left adjoint and the zero object of each one of this categories is terminal. Then the effect of G on each zero object is the same as the effect of the corresponding G_p on it (cf. Section **1.**), that is, the zero object of $\mathcal{X}$. Then, if $A = (A_{p_1}, A_{p_2}, \ldots, A_{p_n})$ is any object of $\overline{\mathcal{A}}$, we define

$$\overline{G}(A) = \prod_{i=1}^{n} G(A_{p_i}) \tag{1}$$

Up to isomorphism, the right hand side of (1) does not depend on the way the object A is represented. Since, if we add/delete zero components to/from the sequence A, this will add/delete zero factors to/from the product. But the zero object is terminal, and thus, for each $X \in \mathrm{Ob}\mathcal{X}$, we have $X \times 0 \simeq X$. In particular, $\overline{G}(0) = 0$. For morphisms, the definition is similar. If $f = (f_{p_1}, f_{p_2}, \ldots, f_{p_n})$ is a morphism of $\overline{\mathcal{A}}$, then, by definition,

$$\overline{G}(f) = \prod_{i=1}^{n} G(f_{p_i})$$

and $\overline{G}$ carries zero morphisms to the corresponding zero morphisms. It is easily checked that $\overline{G}$ thus defined is indeed a functor. Moreover, for objects and morphisms from $\mathcal{A}$ (sequences with just one component), $\overline{G}$ coincides with G.

Proposition 2.1. $\overline{G}$ *has a pluri-adjoint* (cf. [**4**, **5**]).

Proof. Let $X \in \mathrm{Ob}\mathcal{X}$, $X \neq 0$. By assumption, each functor G_p has a left adjoint F_p and, hence, for each p, there is the unit $\eta^p(X) : X \longrightarrow G_p(F_p(X) = \overline{G}(F_p(X))$. We shall take as objects of the index category $\mathcal{T}(X)$ at X all finite subsets of P. Let $\tau = \{p_1, p_2, \ldots, p_n\}$ be such an object. Then the functor $\Theta(X) : \mathcal{T}(X) \longrightarrow \overline{\mathcal{A}}$ will assign to τ the object $F^\tau(X) = (F_{p_1}(X), \ldots, F_{p_n}(X))$. If $n = 0$, then $\tau = \emptyset$ and $F^\tau(X)$ is the zero object of $\overline{\mathcal{A}}$. For each $\tau \in \mathrm{Ob}\mathcal{T}(X)$, we define the morphism

$$h_\tau :\ X \longrightarrow \overline{G}F^\tau(X) = \overline{G}(F_{p_1}(X), \ldots, F_{p_n}(X)) = \prod_{i=1}^{n} G(F_{p_i}(X)$$

by $h_\tau = \langle \eta^{p_1}(X), \eta^{p_2}(X), \ldots, \eta^{p_n}(X)\rangle$, that is, the morphism obtained from $\eta^{p_1}(X)$, $\eta^{p_2}(X)$, …, $\eta^{p_n}(X)$ by the universal property of the product. If X is the zero object, then $\mathcal{T}(X)$ has by definition a single object, say $*$, and then $F^*(0) = 0$ with $h_* : 0 \longrightarrow \overline{G}F^*(0) = 0$ the unique possible morphism. And, of course, the morphisms $\sigma \longrightarrow \tau$ in $\mathcal{T}(X)$ are those morphisms $u : F^\sigma(X) \longrightarrow F^\tau(X)$ of $\overline{\mathcal{A}}$ for which $\overline{G}(u) \circ h_\sigma = h_\tau$.

We will verify that the above data satisfy the conditions for a pluri-adjoint (cf. [**5**]).

(1). *Global covering property.* Let $g : X \longrightarrow \overline{G}A$ be given in $\mathcal{X}$. If $A = (A_{p_1}, A_{p_2}, \ldots, A_{p_n})$, take $\tau = \{p_1, p_2, \ldots, p_n\}$. Since G has a multi-adjoint with $(\eta^p(X))_{p\in P}$ as universal family (cf. **1**.), there is a unique morphism $\varphi_i : F_{p_i} \longrightarrow A_{p_i}$ in $\mathcal{A}$ such that the composite

$$X \xrightarrow{g} \overline{G}(A) = \prod_{k=1}^{n} G(A_{p_k}) \xrightarrow{\mathrm{proj}_i} G(A_{p_i}) \tag{2}$$

is equal to $G(\varphi_i) \circ \eta^{p_i}(X)$, where proj_i is the i^{th} projection of the product. Then

$$\prod_{i=1}^{n} G\varphi_i : \prod_{i=1}^{n} G(F_{p_i}(X)) = \overline{G}F^\tau(X) \longrightarrow \prod_{i=1}^{n} GA_{p_i} = \overline{G}(A)$$

has the property that $\prod_{i=1}^{n} G\varphi_i \circ h_\tau = g$. Indeed, this can be checked by "projecting onto the factors." If "proj" is the standard symbol for projection, then

$$\mathrm{proj}_i \circ \prod_{k=1}^{n} G\varphi_k \circ h_\tau = G\varphi_i \circ \mathrm{proj}_i \circ h_\tau = G\varphi_i \circ \eta^{p_i}(X) = \mathrm{proj}_i \circ g.$$

(2). *Inductive injectivity.* Let $\tau \in \mathrm{Ob}\mathcal{T}(X)$ where $X \in \mathrm{Ob}\mathcal{X}$, let $A \in \mathrm{Ob}\mathcal{A}$, and let $f_1, f_2 :\ F^\tau(X) \longrightarrow A$ be such that

$$\overline{G}f_1 \circ h_\tau = \overline{G}f_2 \circ h_\tau. \tag{3}$$

By adding zero components to $F^\tau(X)$ and A, we can assume that $A = (A_{p_1}, A_{p_2}, \ldots, A_{p_n})$, $F^\tau(X) = (V_{p_1}, V_{p_2}, \ldots, V_{p_n})$ (same set of indices), where some of the components V_{p_i} are $F_{p_i}(X)$ and some are zero objects. Similarly, $f_1 = (u_1, u_2, \ldots, u_n)$,

$f_2 = (v_1, v_2, \ldots, v_n)$. Denote $\mathrm{proj}_i \circ h_\tau = \zeta_i$, where $\mathrm{proj}_i : \overline{G}F^\tau(X) \longrightarrow G(V_{p_i})$ is the canonical projection. Then ζ_i is either $\eta^{p_i}(X)$ (if $V_{p_i} = F_{p_i}(X)$) or the corresponding zero morphism (if V_{p_i} is a zero object). Projecting (3) onto the ith factor, we get $G(u_i) \circ \zeta_i = G(v_i) \circ \zeta_i$ for $i = 1, 2, \ldots, n$. If $\zeta_i = \eta^{p_i}(X)$, we get $u_i = v_i$ by the uniqueness condition for the multi-adjoint; if $\zeta_i = 0$, then V_{p_i} is a zero object and hence $u_i = v_i$. Thus, $f_1 = f_2$. Hence we can take $\sigma = \tau$ and $r = \mathrm{id}_{F^\tau(X)}$ to get $f_1 \circ r = f_2 \circ r$.

(3). *Directedness.* Let $X \in \mathrm{Ob}\mathcal{X}$, and let $\tau, \rho \in \mathrm{Ob}\mathcal{T}(X)$. We can assume that $F^\tau(X) = (V_{p_1}, V_{p_2}, \ldots, V_{p_n})$, $F^\rho(X) = (W_{p_1}, W_{p_2}, \ldots, W_{p_n})$ (same set of indices), where some of the V_{p_i}'s and/or W_{p_i}'s might be zero. (It is, however, convenient, and possible, to assume that, for each i, at least one of V_{p_i} and W_{p_i} is nonzero since otherwise we can delete both of them.) Let $\xi = (p_1, p_2, \ldots, p_n)$ and hence $F^\xi(X) = (F_{p_1}(X), F_{p_2}(X), \ldots, F_{p_n}(X))$. We shall define the morphisms

$$\begin{aligned} u &= (u_1, u_2, \ldots, u_n) : F^\xi(X) \longrightarrow F^\tau(X), \\ v &= (v_1, v_2, \ldots, v_n) : F^\xi(X) \longrightarrow F^\rho(X) \end{aligned}$$

in the following way: If, for a given i, both V_{p_i} and W_{p_i} are nonzero, we have $V_{p_i} = W_{p_i} = F_{p_i}(X)$, and we define both u_i and v_i as the identity of $F_{p_i}(X)$; if $V_{p_i} \neq 0$ and $W_{p_i} = 0$, we define u_i as above and v_i as the unique morphism to 0; in case $V_{p_i} = 0$ and $W_{p_i} \neq 0$, we interchange the roles of u_i and v_i.

We claim that

$$\overline{G}(u) \circ h_\xi = h_\tau \text{ and } \overline{G}(v) \circ h_\xi = h_\rho. \tag{4}$$

Indeed, for the first equality, projecting the left-hand side onto the ith factor, we get:

$$\mathrm{proj}_i \circ \overline{G}(u) \circ h_\xi = G(u_i) \circ \mathrm{proj}_i \circ h_\xi = G(u_i) \circ \eta^{p_i}(X).$$

The latter is either $\eta^{p_i}(X)$ if $V_{p_i} \neq 0$ or the zero morphism if $V_{p_i} = 0$. On the other hand, $\mathrm{proj}_i \circ h_\tau$ is the same thing. This proves the first equality (4). The proof for the second equality is similar.

Because of (4), u and v are morphisms of the category $\mathcal{T}(X)$, namely, $u : \xi \longrightarrow \tau$, $v : \xi \longrightarrow \rho$. ∎

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