Hemirings, Congruences and the Hewitt Realcompactification

S.K. Acharyya K.C. Chattopadhyay G.G. Ray*

Abstract

The present paper indicate a method of obtaining the Hewitt realcompactification υX of a Tychonoff space X, by considering a distinguished family of maximal regular congruences, viz., those which are real, on the hemiring $C_+(X)$ of all the non-negative real valued continuous functions on X.

1. Introduction

The structure space $W(R)$ of a hemiring R, as the set of all maximal regular congruences on R equipped with the hull-kernel topology, has been introduced in 1990 by Sen and Bandyopadhyay [5], who have shown that $W(R)$ is a T_1 topological space and it is T_2 only under certain restrictions. In a previous paper [1] the present authors proved that in case R contains the identity, $W(R)$ is compact and for any Tychonoff space X, the structure space of the hemiring $C_+(X)$ of all the non-negative real valued continuous functions on X is precisely the Stone-Čech compactifications βX of X. In this paper we have focused our attention on a particular type of congruences, viz., the real maximal regular congruences on $C_+(X)$. Given any maximal regular congruence ρ on $C_+(X)$, we have shown that a partial ordering '$\leq$' on the quotient hemiring $C_+(X)/\rho$ can be so defined that $C_+(X)/\rho$ becomes a totally ordered hemiring, which further contains an order isomorphic copy of the hemiring $\mathbb{R}_+$ via a canonical map. ρ is called real if $C_+(X)/\rho$ is isomorphic to $\mathbb{R}_+$, otherwise it is called hyper-real. Next we have shown that a real congruence ρ on $C_+(X)$ is charac-

*Research of the third author is supported by U.G.C., New Delhi
Received by the editors September 1993
Communicated by Y. Félix
AMS Mathematics Subject Classification : Primary 54D60, 54C45, Secondary : 54C30
Keywords : Hemirings, Congruence, Realcompactification, C-embedding.

terized by the property that the set $\{\rho(n) : n \in I\!N\}$ is cofinal in $C_+(X)/\rho$, where $I\!N$ is the set of all natural numbers and for each n in $I\!N$, $\rho(n)$ denotes the residue class in the hemiring $C_+(X)/\rho$ which contains the function $\underline{n}$, taking value n constantly on X. This result has further led us to show an intrinsic feature of real congruences on $C_+(X)$ in terms of their associated z-filters on X. Using all this result we have finally succeeded in proving that the set of all real maximal regular congruences on $C_+(X)$ with the hull-kernel topology in vX, the Hewitt realcompactification of X.

2. Partially ordered hemirings

Definition 2.1 *Following [4] we define a non-empty set R with two distinct compositions '$+$' and '$.$' a hemiring, if it satisfies all the axioms of a ring except possibly the one that ensures the existence of additive inverses of the members of R; and which satisfies the additional axiom:*

$$a.0 = 0.a = 0 \quad \forall\ a \in R.$$

Definition 2.2 *Following [5] we define a congruence on a hemiring R to be an equivalence relation ρ on R which satisfies the following conditions:*

$$\forall x, y, z \in R, (x, y) \in \rho \Rightarrow (x + z, y + z) \in \rho,$$

$$(x.z, y.z) \in \rho \text{ and } (z.x, z.y) \in \rho.$$

The congruence ρ is called cancellative if,

$$\forall\ x, y, z \in R, (x + z, y + z) \in \rho \Rightarrow (x, y) \in \rho.$$

A cancellative congruence ρ on a hemiring R is called regular if there exist elements e_1, e_2 in R such that

$$\forall a \in R, (a + e_1.a, e_2.a) \in \rho \ \text{ and } \ (a + a.e_1, a.e_2) \in \rho.$$

Evidently each cancellative congruence on a hemiring with unity 1 is regular.

For details of these concepts we refer to [4] and [5]. For further results and notations regarding residue classes of a hemiring modulo maximal regular congruences we refer to [1] because they will frequently be used in this article.

Definition 2.3 *A hemiring $(H, +, .)$ equipped with a partial order '$\leq$' is called a partially ordered hemiring if the following conditions are satisfied: $\forall a, b, c, d \in H$*

1. $a \leq b \Leftrightarrow a + c \leq b + c$
2. $a \leq c$ *and* $b \leq d \Rightarrow a.d + c.b \leq a.b + c.d.$

Definition 2.4 *A congreunce ρ on a partially hemiring H is called convex if for all a, b, c, d in H,*

$$(a, b) \in \rho \text{ and } a \leq c \leq d \leq b \Rightarrow (c, d) \in \rho.$$

The following tells precisely when the residue class hemiring of a partially ordered hemiring modulo a regular congruence on it can be partially ordered in some natural way.

Theorem 2.5 *Let H be a partially ordered hemiring, ρ be a regular congruence on H. In order that H/ρ be a partially ordered hemiring, according to the definition: $\rho(a) \leq \rho(b)$ if and only if there exist x, y in H such that $(x, y) \in \rho$ and $a+x \leq b+y$, it is necessary and sufficient that ρ is convex.*

Proof. First assume that ρ is convex. To prove the antisymmetry assume that $\rho(a) \leq \rho(b)$ and $\rho(b) \leq \rho(a)$ where a, b belong to H. Then there exist (x_i, y_i) in $\rho, i = 1, 2$ such that $a + x_1 \leq b + y_1$ and $b + x_2 \leq a + y_2$. This implies that $a + x_1 + x_2 \leq b + y_1 + x_2 \leq a + y_1 + y_2$. Since $(a + x_1 + x_2, a + y_1 + y_2)$ belongs to ρ, in view of the convexity of ρ, we have $(a + x_1 + x_2, b + y_1 + x_2)$ belongs to ρ. Since ρ is cancellative, this implies that $(a + x_1, b + y_1)$ belongs to ρ which gives $(a + x_1 + y_1, b + x_1 + y_1)$ belongs to ρ and this yields (a, b) belongs to ρ, i.e., $\rho(a) = \rho(b)$. The reflexivity and transitivity of '$\leq$' on H/ρ is trivial and hence their proofs are omitted.

It can easily be verified that for any a, b, c in $H, \rho(a) \leq \rho(b)$ if and only if $\rho(a) + \rho(c) \leq \rho(b) + \rho(c)$. So to complete the proof we need to check only that for a, b, c, d in $H, \rho(a) \leq \rho(c)$ and $\rho(b) \leq \rho(d)$ implies that $\rho(a).\rho(d) + \rho(c).\rho(b) \leq \rho(a).\rho(b)+\rho(c).\rho(d)$. Let us take a, b, c, d in H such that $\rho(a) \leq \rho(c)$ and $\rho(b) \leq \rho(d)$. So there exist $(x_1, y_1), (x_2, y_2)$ in ρ such that $a + x_1 \leq c + y_1$ and $b + x_2 \leq d + y_2$. Then, since H is partially ordered hemiring, we have

$$(a + x_1).(d + y_2) + (c + y_1).(b + x_2) \leq (a + x_1).(b + x_2) + (c + y_1).(d + y_2)$$

i.e.,

$$\begin{aligned}&(a.d + c.b) + (a.y_2 b + x_1.d + x_1.y_2 + y_1.x_2 + c.x_2 + y_1.b)\\ &\leq (a.b + c.d) + (a.x_2 + y_1.d + y_1.y_2 + x_1.x_2 + c.y_2 + x_1.b)\end{aligned}$$

Since (x_1, y_1) and (x_2, y_2) belong to ρ we have that all of $(a.y_2, a.x_2), (x_1.d, y_1.d)$, $(x_1.y_2, y_1.y_2), (y_1.x_2, y_1.y_2), (c.x_2, c.y_2)$ and $(y_1.b, x_1.b)$ are members of ρ. Thus,

$$(a.y_2 + x_1.d + x_1.y_2 + y_1.x_2 + c.x_2 + y_1.b, a.x_2 + y_1.d + y_1.y_2 + x_1.x_2 + c.y_2 + x_1.b) \in \rho.$$

Hence,

$$\rho(a.d + c.b) \leq \rho(a.b + c.d),$$

i.e.,

$$\rho(a).\rho(d) + \rho(c).\rho(b) \leq \rho(a).\rho(b) + \rho(c).\rho(d).$$

Thus H/ρ is a partially ordered hemiring.

Conversely, if H/ρ is a partially ordered hemiring according to the given definition, then it is easy to verify that ρ is convex. □

Remark 2.6 *For a, b in H we write $\rho(a) < \rho(b)$ if $\rho(a) \leq \rho(b)$ and $\rho(a) \neq \rho(b)$.*

3. Congruences on the lattice ordered hemiring $C_+(X)$

In what follows X will stand for a Tychonoff space. $C(X)$ denotes the ring of all real valued continuous functions on X. For a real number $r, \underline{r}$ denotes the constant function on X such that $\underline{r}(x) = r$ for all x in X. We take $\mathbb{R}_+$ to be the hemiring of all non-negative real numbers and $C_+(X) = \{f \in C(X) : f(x) \geq 0 \ \forall x \in X\}$. Then $C_+(X)$ is a lattice ordered hemiring with usual definition of '$+$', '$.$' and '$\leq$' and for any two f, g in $C_+(X)$, $f \vee g$ and $f \Lambda g$ are defined by,

$$(f \vee g)(x) = max\{f(x), g(x)\} and$$

$$(f \Lambda g)(x) = min\{f(x), g(x)\} \ \forall x \in X.$$

Obviously $f \vee g$ and $f \Lambda g$ belong to $C_+(X)$.

Convention. Each congruence on $C_+(X)$ considered in this paper will be assumed to be regular and further every scuh congruence ρ will stand for a proper one i.e., for which $\rho \neq C_+(X) \times C_+(X)$.

We recall some notions and results pertaining to the congruences on the hemiring $C_+(X)$. For a detailed discussion see [1].

Theorem 3.1 *If ρ is a congruence on $C_+(X)$ then $E(\rho) = \{E(f,g) : (f,g) \in \rho\}$ is a z-filter on X, where $E(f,g) = \{x \in X : f(x) = g(x)\}$ is the agreement set of f and g.*

Definition 3.2 *A congruence ρ on $C_+(X)$ is called*

1. *a z-congruence if for all f, g in $C_+(X), E(f,g)$ belongs to $E(\rho)$ implies that (f,g) belongs to ρ.*
2. *a prime congruence if for all f, g, h, k in $C_+(X), (f.h + g.k, f.k + g.h) \in \rho$ implies either $(f,g) \in \rho$ or $(h,k) \in \rho$.*
3. *a maximal congruence if there does not exist any congruence σ on $C_+(X)$ which properly contains ρ*

Theorem 3.3 *If $\mathcal{F}$ is a z-filter on X, then*

$$E^{-1}(\mathcal{F}) = \{(\{,\}) \in \mathcal{C}_+(\mathcal{X}) \times \mathcal{C}_+(\mathcal{X}) : \mathcal{E}(\{,\}) \in \mathcal{F}\}$$

is a z-congruence on $C_+(X)$.

Theorem 3.4 *The assignment $\rho \rightarrow E(\rho)$ establishes a one-to-one correspondence between the set of all z-congruences on $C_+(X)$ and that of all z-filters on X.*

Theorem 3.5 *If ρ is a maximal congruence on $C_+(X)$ then $E(\rho)$ is a z-ultrafilter on X and conversely if $\mathcal{F}$ is a z-ultrafilter on X then $E^{-1}(\mathcal{F})$ is a maximal congruence on $C_+(X)$.*

We now state two results which are not included in [1]. Their proofs follow immediately from the following fact:

$$E(f_1, g_1) \cup E(f_2, g_2) = E(f_1.f_2 + g_1.g_2, f_1.g_2 + f_2.g_1)$$

for all f_1, f_2, g_1, g_2 in $C_+(X)$.

Theorem 3.6 *If ρ is a prime z-congruence on $C_+(X)$, then $E(\rho)$ is a prime z-filter on X. Conversely, for any prime z-filter $\mathcal{F}$ on $X, E^{-1}(\mathcal{F})$ is a prime z-congruence on $C_+(X)$.*

Theorem 3.7 *Each maximal congruence on $C_+(X)$ is both a prime congruence and z-congruence.*

4. Order structure on the quotient hemiring of $C_+(X)$

Our contemplated main result of this paper demands some study on the order structure of the quotient hemiring of $C_+(X)$ modulo maximal congruences. The following is the first proposition towards such an end.

Theorem 4.1 *A z-congruence ρ on $C_+(X)$ is convex.*

Proof. Let (f, g) belong to ρ and h_1, h_2 in $C_+(X)$ be such that $f \leq h_1 \leq h_2 \leq g$. Since $E(f, g) \subset E(h_1, h_2)$ and $E(f, g)$ belongs to $E(\rho), E(h_1, h_2)$ belongs to $E(\rho)$. Clearly then (h_1, h_2) belong to ρ because ρ is a z-congruence. □

The following two results show that the order structure of the quotient hemiring $C_+(X)/\rho$ has some connection with agreement sets of the members of ρ. (Compare with similar results in the Sec. 5.4 of [3] for the quotient ring $C(X)/I$, where I is a z-ideal in $C(X)$.).

Theorem 4.2 *Let ρ be a z-congruence on $C_+(X)$ and f, g belong to $C_+(X)$. Then $\rho(f) \leq \rho(g)$ if and only if $f \leq g$ on some member of $E(\rho)$. On the other hand if $f < g$ at each point of some member of $E(\rho)$, then $\rho(f) < \rho(g)$.*

Proof. Let $\rho(f) \leq \rho(g)$. Then there exists (h_1, h_2) in ρ with $f + h_1 \leq g + h_2$. Therefore $f \leq g$ on the set $E(h_1, h_2)$ in $E(\rho)$. Conversely, let $f \leq g$ on Z where Z is a member of $E(\rho)$. Then there exists (h_1, h_2) in ρ such that $Z = E(h_1, h_2)$. Put $h = (f - g) \vee \underline{0}$. Then h belongs to $C_+(X)$ and $E(h, \underline{0})$ contains $E(h_1, h_2)$. Since ρ is a z-congruence, this implies that $(\underline{0}, h)$ belongs to ρ. We assert that $f + \underline{0} \leq g + h$. Hence $\rho(f) \leq \rho(g)$.

For the remaining part of this theorem assume that $f < g$ everywhere on some Z in $E(\rho)$. Then $E(f.g) \cap Z = \phi$ which implies that (f, g) does not belong to ρ. Therefore $\rho(f) \neq \rho(g)$. But by the first part of this theorem, we have $\rho(f) \leq \rho(g)$. Hence $\rho(f) < \rho(g)$. □

Theorem 4.3 *Let f, g belong to $C_+(X)$ and ρ be a maximal congruence on $C_+(X)$ with $\rho(f) < \rho(g)$. Then there exists a set Z in $E(\rho)$ at each point of which $f < g$.*

Proof. The result follows by using Theorem 4.2 and arguing similarly as in the Proof of 5.4 (b) of [3]. □

A question may be raised - what are the z-congruences ρ on $C_+(X)$ which makes the partially ordered hemiring $C_+(X)/\rho$ a totally ordered one? A sufficient condition is provided in the following.

Theorem 4.4 *If ρ is a prime z-congruence on $C_+(X)$, then $C_+(X)/\rho$ is a totally ordered hemiring. The same assertion is true in particular therefore for a maximal congruence.*

Proof. We need to verify only that for arbitrary f, g in $C_+(X), \rho(f)$ and $\rho(g)$ are comparable with respect to the relation '$\leq'$. Now $Z_1 = \{x \in X : f(x) \leq g(x)\}$ and $Z_2 = \{x \in X : g(x) \leq f(x)\}$ are zero sets in X such that $Z_1 \cup Z_2 = X$. By Theorem 3.4, $E(\rho)$ is a prime z-filter on X. Hence either Z_1 belongs to $E(\rho)$ or Z_2 belongs to $E(\rho)$. But $f \leq g$ on Z_1 and $g \leq f$ on Z_2. By Theorem 4.2 we have either $\rho(f) \leq \rho(g)$ or $\rho(g) \leq \rho(f)$. □

The following proposition is basic towards the initiation of real and hyper-real congruences on $C_+(X)$. The proof is a routine verification and hence omitted.

Theorem 4.5 *Let ρ be maximal congruence on $C_+(X)$. Then the mapping $\psi : r \rightarrow \rho(r)$ establishes an order preserving isomorphism of the totally ordered hemiring $\mathbb{R}_+$ into the totally ordered hemiring $C_+(X)/\rho$.*

This theorem leads to the following

Definition 4.6 *A maximal congruence ρ on $C_+(X)$ is called*

1. *real if $\psi(\mathbb{R}_+) = C_+(X)/\rho$,*

2. *hyper-real if it not real.*

Therefore Theorem 3.7 of [1] can be restated as follows:

Theorem 4.7 *For each point x in X, the fixed congruence $\rho_x = \{(f, g) \in C_+(X) \times C_+(X) : f(x) = g(x)\}$ on $C_+(X)$ is real.*

The following is criterion for a maximal congruence on $C_+(X)$ to be a real one.

Theorem 4.8 *A maximal congruence ρ on $C_+(X)$ is real if and only if the set $\{\rho(n) : n \in I\!N\}$ is cofinal in the totally ordered hemiring $C_+(X)/\rho$.*

To prove this we need the following lemma.

Lemma 4.9 *For any maximal congruence ρ on $C_+(X)$ each non-zero element in $C_+(X)/\rho$ has a multiplicative inverse.*

Proof. Let f belong to $C_+(X)$ be such that $\rho(f) \neq \rho(\underline{0})$. Since ρ is a z-congruence, this ensures that $E(f, \underline{0})$ does not belong to $E(\rho)$. Since $E(\rho)$ is z-ultrafilter on X one can find (h_1, h_2) in ρ with $E(f, \underline{0}) \cap E(h_1, h_2) = \phi$. Let $h = \mid h_1 - h_2 \mid$ and $g = 1/(f + h)$. Then $h, g \in C_+(X)$ and $E(f, g, \underline{1}) = E(h_1, h_2)$. Since (h_1, h_2) belongs to ρ and ρ is a z-congruence, $(f.g, \underline{1})$ belongs to ρ. Thus $\rho(f).\rho(g) = \rho(\underline{1})$.

Proof of the theorem. Since n is confinal in the totally ordered hemiring $I\!R_+$, the necessity part of the theorem becomes trivial.

Assume therefore that the set $\{\rho(\underline{n}) : n \in I\!N\}$ is cofinal in the totally ordered hemiring $C_+(X)/\rho$. We first show that the set $\{\rho(\underline{q}) : q \in Q_+\}$ is dense in the totally ordered hemiring $C_+(X)/\rho$, where Q_+ denotes the set of all non-negative rationals. Let f, g belongs to $C_+(X)$ be such that $\rho(f) < \rho(g)$. Then we assert that there is a positive integer n such that $\rho(f) + \rho(\underline{1/n}) < \rho(g)$. If possible, let for all $n \in I\!N$

$$\rho(f) + \rho(\underline{1/n}) \geq \rho(g) \cdots\cdots 4.8.1.$$

Set,

$$B = \{b \in C_+(X)/\rho : \rho(f) + b < \rho(g)\}.$$

Since $\rho(f) \leq \rho(g)$, by Theorem 4.3 one can find Z in $E(\rho)$ such that $f(x) < g(x)$ for each x in Z. Put $h = ((g - f) \vee \underline{0})/2$. Then $f(x) < f(x) + h(x) < g(x)$ for all x in Z. By the second part of Theorem 4.2 we have $\rho(f) < \rho(f) + \rho(h) < \rho(g)$. This shows that $\rho(h) \neq \rho(\underline{0})$ and $\rho(h) \in B$. Thus B contains non-zero elements of $C_+(X)/\rho$. Let b be an arbitrary non-zero element of B. Then by Lemma 4.9, b has a multiplicative inverse, b^{-1}, in $C_+(X)/\rho$. Inequality 4.8.1 gives us

$$\rho(f) + b < \rho(g) \leq \rho(f) + \rho(\underline{1/n}) \ \forall n \in I\!N.$$

This shows that $b < \rho(\underline{1/n})$ for all $n \in I\!N$, and hence $b^{-1} \geq \rho(\underline{n})$ for all $n \in I\!N$ This is contradiction to the assumption that $\{\rho(\underline{n}) : n \in I\!N\}$ is cofinal in $C_+(X)/\rho$. Thus there is a positive integer n for which $\rho(f) + \rho(\underline{1/n}) < \rho(g)$, so that

$$\rho(\underline{n}).\rho(f) + \rho(\underline{1}) < \rho(\underline{n}).\rho(g) \cdots\cdots 4.8.2$$

Let m be the smallest integer such that $\rho(\underline{n}).\rho(f) < \rho(\underline{m})$ and hence in view of 4.8.2 we have

$$\rho(\underline{n}).\rho(f) < \rho(\underline{m}) < \rho(\underline{n}).\rho(g).$$

Thus, $\rho(f) < \rho(\underline{m/n}) < \rho(g)$. Therefore $\{\rho(\underline{q}) : q \in Q_+\}$ is dense in $C_+(X)/\rho$.

Let us define a map $\Phi : C_+(X)/\rho \to I\!R_+$ by the following rule: let $f \in C_+(X)$. If there is a $q \in Q_+$ such that $\rho(f) = \rho(\underline{q})$ then we put $\Phi(\rho(f)) = q$. Otherwise set,

$$L_f = \{s \in Q_+; \rho(\underline{s}) < \rho(f)\} \cup \{q : q \textit{ is a negative rational}\}$$

$$U_f = \{s \in Q_+ : \rho(f) < \rho(\underline{s})\}.$$

Then (L_f, U_f) defines a Dedekind section of the set of rationals and accordingly determines a unique real number t, say, which is clearly non-negative. We put in this case $\Phi(\rho(f)) = t$.

In order to show that Φ is an isomorphism of $C_+(X)/\rho$ onto $I\!R_+$ we choose f, g in $C_+(X)$ arbitrarily. Then for any four non-negative rational numbers p, q, r, s, satisfying

$$\rho(p) \leq \rho(f) < \rho(r) \text{ and } \rho(q) \leq \rho(g) < \rho(s),$$

one, in view of Theorems 4.2 and 4.3 can easily verify that

$$p + q \leq \Phi(\rho(f)) + \Phi(\rho(g)) < r + s$$

and

$$p + q \leq \Phi(\rho(f) + \rho(g)) < r + s.$$

The last pair of inequalities together with the denseness of $\{\rho(\underline{q}) : q \in Q_+\}$ in $C_+(X)/\rho$ clearly ensures that

$$\Phi(\rho(f) + \rho(g)) = \Phi(\rho(f)) + \Phi(\rho(g)).$$

By an argument similar to one used above we can show that

$$\Phi(\rho(f).\rho(g)) = \Phi(\rho(f)).\Phi(\rho(g)).$$

Let f, g belong to $C_+(X)$ such that $\rho(f) < \rho(f)$. Since $\{\rho(\underline{q}) : q \in Q_+\}$ is dense in $C_+(X)/\rho$, in view of the definition of Φ it follows that $\Phi(\rho(f)) < \Phi(\rho(g))$. Thus Φ is an order preserving isomorphism of $C_+(X)/\rho$ onto $I\!R_+$ and hence ρ is a real maximal congruence on $C_+(X)$. □

From the above Theorem we can say that for any hyperreal maximal congruence ρ on $C_+(X)$ there exists an $f \in C_+(X)$ for which $\rho(f) \geq \rho(\underline{n})$ for all $n \in I\!N$. We call such a $\rho(f)$ an inifnitely large element of $C_+(X)/\rho$. The multiplicative inverse of an infinitely large element is called an infinitely small element of $C_+(X)/\rho$. One can check that the multiplicative inverse of an infinitely small element is infinitely large. Thus a hyper-real congruence on $C_+(X)$ is characterised by the presence of infinitely large (or infinitely small) elements in the residue class hemiring.

The following proposition correlates hyper-real congruences on $C_+(X)$ with unbounded functions on this hemiring.

Theorem 4.10 *Let ρ be a maximal congruence on $C_+(X)$ and $f \in C_+(X)$ be arbitrary. Then the following statements are equivalent:*

1. *$\rho(f)$ is infinitely large.*
2. *For all $n \in I\!N$ the set $Z_n = \{x \in X : f(x) \geq n\}$ is a member of $E(\rho)$.*
3. *For all $n \in I\!N, (f \Lambda \underline{n}, \underline{n})$ belongs to ρ.*
4. *f is unbounded on each member of $E(\rho)$.*

(Compare with Result 5.7 (a) of [3]).

Proof (1) $\Rightarrow$ (2). Let $\rho(f)$ be infinitely large. Then $\rho(\underline{n}) \leq \rho(f)$ for all $n \in I\!N$. Now for an arbitrary $n \in I\!N$, in view of Theorem 4.2, $\rho(\underline{n}) \leq \rho(f)$ implies that there exists $Z \in E(\rho)$ such that $\underline{n} \leq f$ of Z. Thus $Z \subset Z_n$. Since $E(\rho)$ is a z-ultrafilter on X and Z_n is a zero set in X, it follows that Z_n belongs $E(\rho)$.

(2) $\Rightarrow$ (3). Since $Z_n = E(f\Lambda\underline{n}, \underline{n})$ for all $n \in I\!N$ and ρ is a z-congruence, the result follows.

(3) $\Rightarrow$ (2). Trivial.

(2) $\Rightarrow$ (4). Let (2) holds. Let Z be an arbitrary member of $E(\rho)$. Since $E(\rho)$ is a z-ultrafilter, $Z \cap Z_n \neq \phi$ for all $n \in I\!N$. So, for any x in $Z \cap Z_n, f(x) \geq n$, for all $n \in I\!N$. This shows that f is unbounded on Z. Consequently (4) holds.

(4) $\Rightarrow$ (1). Let (4) holds. If possible let $\rho(f)$ be not infinitely large. So there exists $n \in I\!N$ such that $\rho(f) \leq \rho(\underline{n})$. Then by Theorem 4.2 there is $Z \in E(\rho)$ such that $f \leq \underline{n}$ on Z, which contradicts our assumption. Thus $\rho(f)$ is infinitely large. □

We conclude this section with a simple but useful characterisation of real congruences.

Theorem 4.11 *A maximal congruence ρ on $C_+(X)$ is real if and only if $E(\rho)$ is closed under countable intersection.*

Proof. Let ρ be real. If possible suppose that $E(\rho)$ is not closed under countable intersection. So there exists a sequence $\{(f_n, g_n) : n \in I\!N\}$ in ρ such that the set $\cap\{E(f_n, g_n) : n \in I\!N\}$ does not belong to $E(\rho)$. Set $f = \sum_{n=1}^{\infty}(\mid f_n - g_n \mid \Lambda\underline{2}^{-n})$. Then by Weirstrass M-test it follows that $f \in C_+(X)$. Now $E(f, \underline{0}) = \cap\{E(f, g) : n \in I\!N\}$ and hence $(f, \underline{0}) \notin \rho$. Therefore $\rho(\underline{0}) < \rho(f)$, because $\underline{0} \leq f$. For any positive integer m, $f \leq 2^{-m}$ on the set $\bigcap_{n=1}^{m} E(f_n, g_n)$ which is member of $E(\rho)$. By Theorem 4.2, $\rho(f) \leq \rho(\underline{2}^{-m})$. Since m is an arbitrary positive integer, $\rho(f)$ is an infinitely small element of $C_+(X)/\rho$, whence ρ becomes hyper-real-a contradiction.

Conversely, let $E(\rho)$ be closed under countable intersection. If possible suppose that ρ is not real. Then there exists g in $C_+(X)$ such that $\rho(g)$ is infinitely large. So by Theorem 4.10, for each $n \in I\!N$ the set $Z_n = \{x \in X : n \leq g(x)\}$ is a member of $E(\rho)$. Obviously $\bigcap_{n=1}^{\infty} Z_n = \phi$, - which contradicts our hypothesis. Hence ρ is real. □

5. The realcompactification theorem

Let $W(X)$ be the collection of all maximal congruences on $C_+(X)$ and $W_R(X) = \{\rho \in W(X) : \rho \text{ is real }\}$. It is easy to verify that the collection $\{W(f, g) : f, g \in C_+(X)\}$ is a base for the closed sets of a topology on $W(X)$ where $W(f, g) = \{\rho \in W(X) : (f, g) \in \rho\}$. $W(X)$, equipped with this topology is known as the structure space of $C_+(X)$. The subspace $W_R(X)$ of $W(X)$ is called the real structure space of $C_+(X)$. It has been established in [1] that $(\eta_X, W(X))$ is the Stone-Čech compactification βX of X where $\eta_X(x) = \rho_x$ for each $x \in X$. In this section we propose to state and proof that $(\eta_X, W_R(X))$ is the Hewitt realcompactification vX of X which is the main result of this article.

In what follows we recall a definition and two results (without proof) of [2] which play a vital role to achieve our goal.

Definition 5.1 *Forany subset A of X, the set*

$$rcl\ A = \{x \in X : each\ G_\delta - set\ in X\ containing\ x\ meets\ A\}$$

is called the realclosure (or Q-closure of) A. A is called realclosed (or Q-closed) if $A = rclA$.

It is clear that every closed set in X is realclosed, while any open interval (a, b) of $I\!R$ is realclosed subset of $I\!R$ without being closed.

Theorem 5.2 *Every realclosed subset of a realcompact space is realcompact.*

Theorem 5.3 *X is realcompact if and only if it is realclosed in βX.*

Now we are in a position to state and prove our main result.

Theorem 5.4 *Let $f : X \to Y$ be a continuous map where Y is a realcompact space. There there exists continuous function $F : W_R(X) \to Y$ such that $Fo\eta_X = f$ i.e., the following diagram commutes.*

$$\begin{array}{ccc} & X \xrightarrow{f} Y \\ \eta_X & \downarrow \ \nearrow F \\ & W_R(X) \end{array}$$

In order to prove this theorem the following two lemmas are needed.

Lemma 5.5 *The subspace $W_R(X)$ of the space $W(X)$ is realcompact.*

Proof. Recall that $W(X)$ is compact and hence in particular realcompact. Thus in view of Theorem 5.2, to complete the proof it is sufficient to check that $W_R(X)$ is realclosed subset of $W(X)$.

Let us choose an element ρ_0 in $W(X) - W_R(X)$. Since ρ_0 is hyper-real, there exists $g \in C_+(X)$ such that $\rho_0(g)$ is infinitely large. Set $f_n = g \vee \underline{n}$ and $h_n = g \Lambda \underline{n}$ for each $n \in I\!N$. Then by Theorem 4.10, we get that $(h_n, \underline{n})$ belongs to ρ_0 for each $n \in I\!N$. Since $(f_n, \underline{n}) \cap E(h_{n+1}, \underline{n+1}) = \phi$ for each $n \in I\!N, (f_n \underline{n}) \notin \rho_0$ for each $n \in I\!N$. Now set $V = W(X) - \bigcup_{n=1}^{\infty} W(f_n, \underline{n})$. Then V is a G_δ-set in $W(X)$ containing ρ_0. Let ρ be an arbitrary element in $W_R(X)$. Then by Theorem 4.8, $\rho(g) \leq \rho(\underline{m})$ for some $m \in I\!N$. Also by Theorem 4.2, there is a Z in $E(\rho)$ such that $g \leq \underline{m}$ on Z and hence $Z \subset E(f_m, \underline{m})$. Consequently $(f_m, \underline{m}) \in \rho$ which implies that $\rho \in W(f_m, \underline{m})$. Thus $V \cap W_R(X) = \phi$ and hence $W_R(X)$ is realclosed in $W(X)$. □

Lemma 5.6 *Let $f : X \to Y$ be continuous, ρ be a prime z-congruence on $C_+(X)$. Then $f^*(\rho)$, defined by*

$$f^*(\rho) = \{E(h, g) : h, g \in C_+(Y), (hof, gof) \in \rho\},$$

is a prime z-filter on Y. Moreover if ρ is real maximal congruence on $C_+(X)$, then $f^*(\rho)$ has the countable intersection property.

Proof. Obviously ϕ is not a member of $f^*(\rho)$. Let Z belong to $f^*(\rho)$ and Z_1 be a zero-set in Y containing Z. Then there exists h, g, h_1, g_1 in $C_+(Y)$ such that $Z = E(h, g), Z_1 = E(h_1, g_1)$ and (hof, gof) belongs to ρ. So $E(hof, gof)$ belongs to $E(\rho)$. It can easily be verified that $E(hof, gof) \subset E(h_1of, g_1of)$ and hence, ρ being a z-congruence, (h_1of, g_1of) belongs to ρ. Consequently $Z_1 = E(h_1, g_1)$ belongs to $f^*(\rho)$.

Now suppose that Z_1, Z_2 be two arbitrary members of $f^*(\rho)$. So there are h_1, g_1, h_2, g_2 in $C_+(Y)$ such that $Z_i = E(h_i, g_i)$ and (h_iof, g_iof) are members of ρ for $i = 1, 2$. Since for any h, g in $C_+(Y), (h.g)of = (hof).(gof)$ and $(h + g)of = (hof) + (gof)$, it follows that

$$E(h_1of, g_1of) \cap E(h_2of, g_2of) = E((h_1of)^2 + (g_1of)^2 + (h_2of)^2 + (g_2of)^2,$$

$$2((h_1of).(g_1of) + (h_2of).(g_2of)))$$

$$= E((h_1^2 + g_1^2 + h_2^2 + g_2^2)of, 2(h_1.g_1 + h_2.g_2)of)$$

which is a member of $E(\rho)$. Thus

$$Z_1 \cap Z_2 = E((h_1^2 + g_1^2 + h_2^2 + g_2^2), 2(h_1.g_1 + h_2.g_2)) \in f^*(\rho).$$

This shows that $f^*(\rho)$ is a z-filter on Y.

Finally, let $Z_1 \cup Z_2$ belong to $f^*(\rho)$ where $Z_i = E(f_i, g_i); f_i, g_i \in C_+(Y), i = 1, 2$. Then since $Z_1 \cup Z_2 = E(f_1.g_2 + f_2.g_1, f_1.f_2 + g_1.g_2)$ and ρ is prime, by an argument similar to the above we can show that either $Z_1 \in f^*(\rho)$ or $Z_2 \in f^*(\rho)$. Thus $f^*(\rho)$ is a prime z-filter on Y.

To show, for a real maximal congruence ρ on $C_+(X), f^*(\rho)$ has the countable intersection property , let us take a sequence $\{E(h_n, g_n)\}$ in $f^*(\rho)$. Then for all $n \in I\!N, (h_nof, g_nof)$ belongs to ρ and hence by the Theorem 4.11, $\bigcap_{n=1}^{\infty} E(h_nof, g_nof)$ is non-empty. For any x in $\bigcap_{n=1}^{\infty} E(h_nof, g_nof), f(x) \in \bigcap_{n=1}^{\infty} E(h_n, g_n)$. Thus $f^*(\rho)$ has the countable intersection property. □

Proof of the Theorem. Let ρ be a member of $W_R(X)$. Since for a prime z-filter with countable intersection property on a realcompact space is fixed and since prime z-filter contains at most one cluster point (see 8.12 and 3.18 of [3]) it follows that there exists a unique $y \in Y$ such that $\{y\} = \cap f^*(\rho)$. For every ρ in $W_R(X)$ set $F(\rho) = y$ where $\{y\} = \cap f^*(\rho)$. This defines a map $F : W_R(X) \to Y$. For each $x \in X$ it follows that $F(\rho_x) = f(x)$ because $f(x) \in \cap f^*(\rho_x)$. Thus $F(\eta_X(x)) = f(x)\ \forall x \in X$ and hence $Fo\eta_X = f$.

To prove the continuity of the function F, choose any ρ_0 in $W_R(X)$ and any open set V in Y such that $F(\rho_0) \in V$. Then there exist $g_1, g_2 \in C_+(Y)$ such that

$$F(\rho_0) \in Y - Z(g_1) \subset Z(g_2) \subset V.$$

Clearly then $g_1.g_2 = \underline{0}$. Now $F(\rho_0)$ does not belong to $Z(g_1)$ and hence $Z(g_1) = E(g_1, \underline{0})$ does not belong to $f^*(\rho)$. Consequently $(g_1 of, \underline{0})$ does not belong to ρ_0 and this implies that the set $U = (W(X) - W(g_1 of, \underline{0})) \cap W_R(X)$ is an open neighbourhood of ρ_0 in $W_R(X)$. Now choose any ρ in U. Then $(g_1 of, \underline{0}) \notin \rho$. Since $(g_1 of).(g_2 of) = \underline{0}$ and ρ is a prime congruence on $C_+(X)$, it follows that $(g_2 of, \underline{0}) \in \rho$. Thus $Z(g_2) = E(g_2, \underline{0}) \in f^*(\rho)$ a nd hence $F(\rho) \in E(g_2, \underline{0}) = Z(g_2) \subset V$. Thus $F(U) \subset V$. Therefore the map $F : W_R(X) \to Y$ is continuous. □

Recall that the Hewitt realcompactification vX of a space X is characterised by the fact thay any continuous map of X into an arbitrary realcompact space admits a continuous extension over vX. Hence in view of the above theorem we conclude our article with the following

Corollary 5.7 *$(n_X, W_R(X))$ is the Hewitt realcompactification vX.*

References

[1] S.K. Acharyya, K.C. Chattopadhyay, and G.G. Ray. Hemirings congruences and the Stone-Čech compactification, to appear in *Simon Stevin*.

[2] R.A. Alo and H.L. Shapiro, *Normal Topological Spaces,* Cambridge University Press, Cambridge, 1974.

[3] L. Gillman and M. Jerison,*Rings of continuous Functions*, van Nostrand, 1960.

[4] L. Li, On the structures of hemirings, *Simon Stevin,* 58, 1984.

[5] M.K. Sen and S. Bandyopadhyay, Structure space of a semi-algebra over a hemiring. *Proc. Sym. on Algebra A 43-55 Second biennial conference, Allahabad Mathematical Soc., 1990.*

Gour Gopal Ray
Department of Mathematics
University of Burdwan
Burdwan 713104,W.B.
India.