On partitions, surjections, and Stirling numbers

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1 Introduction

The connection between $P(m,n)$, the number of partitions of a set containing m elements as a disjoint union of n non-empty subsets; $S(m,n)$, the number of surjections of a set of m elements onto a set of n-elements; and $St(m,n)$, the Stirling number of the second kind, given by[1]

$$St(m,n) = \frac{1}{n!}\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}r^m, \tag{1.1}$$

has long been known (see, eg, [B, C, L, T]). Indeed, their mutual relation is given by

$$\frac{1}{n!}S(m,n) = P(m,n) = St(m,n), \tag{1.2}$$

the first equality being very elementary and the second somewhat less immediate.

Our primary object in this paper is to provide an explicit formula for $St(m,n)$, and hence, by (1.2), for $P(m,n)$ and $S(m,n)$, in the case that $m > n$.

[1] Other definitions are given in the litterature; for example, $St(m,n)$ is characterized by the equation

$$x^m = \sum_{n=0}^{m} n!St(m,n)\binom{x}{n}.$$

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The form our formula takes is the following. We write $\gamma_k(m) = St(m, m-k)$, $k \geq 1$; then we represent $\gamma_k(m)$ as a linear combination of binomial coefficients $\binom{m}{r}$, $k+1 \leq r \leq 2k$. The weight a_{hk} attached to the binomial coefficient $\binom{m}{k+h+1}$ is a positive integer *independent of* m, which we call the **Stirling factor** of type (h, k). We do not calculate these Stirling factors explicitly, except for low values of k (see Figure 3) and the cases $h = 0, 1, 2, k-2, k-1$ (Theorems 4.1, 4.3, 4.4). However we give a recurrence relation expressing a_{hk}, $h \geq 1$, as a linear combination of Stirling factors $a_{h-1,j}$, $h \leq j \leq k-1$, the weight attached to the Stirling factor $a_{h-1,j}$ being the binomial coefficient $\binom{k+h}{j+h}$. It thus seems natural to represent the Stirling factors in a triangle too, to illustrate the interaction with the Pascal Triangle (see Figure 4). The recurrence relation, together with the initial condition $a_{0k} = 1$ for all $k \geq 1$, of course determines a_{hk}.

We believe that our formula, given by Theorem 4.1, should lead to useful estimates of $P(m, n)$, since the Stirling factors are always *positive* integers and should be readily estimated. We believe further that the Stirling factors a_{hk} should have interesting number-theoretical properties and interrelations. One such relation is displayed in Lemma 4.5. We hope to return to these aspects in a subsequent paper.

We draw attention to two further features of this paper. In Section 3 we prove a generalized version of a simple, but attractive, combinatorial property of binomial coefficients, often known as the **Christmas Stocking Theorem**; the reason for this choice of name is made quite clear by Figure 1. The generalization (which plays a key role in our proof of our main result, Theorem 4.1) also admits a geometric representation, which we try to convey in Figure 2. Second, we combine Theorem 4.1 with the classical recurrence relation

$$P(m, n) = nP(m-1, n) + P(m-1, n-1) \tag{1.3}$$

to obtain certain bilinear relations connecting binomial coefficients and Stirling factors (see, for example, Theorem 5.1).

It is, we think interesting to mention that this paper arose out of the (successful!) attempt to reconstruct an elementary proof of Theorem 2.6(b), a result which had been discovered by the third-named author many years ago. Of course, this elementary proof had to proceed without invoking (1.2), since the plan was to use it in a proof that $P(m, n) = St(m, n)$.

2 Classical results

All the results in this section may be found in any standard work on combinatorics (see, eg [B, C, L, T]); we collect them here for the convenience of the reader. Throughout this section m and n are non-negative integers.

Definition 2.1. *$P(m,n)$ will denote the number of partitions of a set containing m elements as a disjoint union of n non-empty subsets.*

Obviously

$$P(m,0)=0, m \geq 1; P(m,n)=0, m<n; P(n,n)=1, P(m,1)=1, m \geq 1. \quad (2.1)$$

Further, by ordering the elements of the set A, where A has m elements, and considering partitions of the first $(m-1)$ elements of A, we readily establish the recurrence relation

$$P(m,n)=nP(m-1,n)+P(m-1,n-1), m>n \geq 2. \quad (2.2)$$

It is easy to see that $P(m,n)$ is entirely determined by (2.1), (2.2).

Definition 2.2. *Let the set A have m elements and the set B have n elements. Then $S(m,n)$ will denote the number of* **surjections** *of A onto B.*

Plainly we have

Theorem 2.1. $S(m,n)=n!P(m,n)$.

For a surjection $f: A \rightarrow B$ may be regarded as a *labeled* partition of A into n non-empty subsets $f^{-1}b$, $b \in B$, where the subset $f^{-1}b$ is labeled by b. Since there are $n!$ ways of assigning the labels, Theorem 2.1 follows.

Definition 2.3. *$St(m,n)$ will denote the* **Stirling number of the second kind**

$$St(m,n)=\frac{1}{n!}\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}r^{m}. \quad (2.3)$$

We then have

Theorem 2.4. $St(m,n)=P(m,n)$.

Corollary 2.5. $S(m,n)=n!St(m,n)$.

Theorem 2.4 may be proved using the inclusion-exclusion principle; an alternative proof simply consists of verifying that $St(m,n)$ satisfies properties (2.1), (2.2) adduced for $P(m,n)$.

Since calculations of $St(m,n)$ imply, via Theorem 2.4 and Corollary 2.5, results on $P(m,n)$ and $S(m,n)$, we may henceforth state all our results for $St(m,n)$. Classically, we have

Theorem 2.6. (a) $St(m,n)=0$ *if* $m<n$; (b) $St(n,n)=1$.

Remark. We need Theorem 2.6, of course, for the second proof of Theorem 2.4 referred to above.

We give a proof of Theorem 2.6 since it expoits a strategy which we adopt again in proving our main result in Section 4. We observe first from (2.3) that

$$St(0,n)=\begin{cases} 0^0=1 & \text{if } n=0 \\ \frac{1}{n!}(-1)^n(1-1)^n=0 & \text{if } n \geq 1 \end{cases} \quad (2.4)$$

Thus we may prove Theorem 2.6 by induction on n, starting with $n=0$. To prove (a) we assume that, for a given $n \geq 1$, $St(m, n-1) = 0$ for all $m < n-1$. Thus, expanding $(r+1)^{m-1}$ by the binomial theorem, we conclude that

$$\sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{n-1}{r} (r+1)^{m-1} = 0, \quad 1 \leq m < n. \tag{2.5}$$

Set $s = r+1$, so that

$$\sum_{s=1}^{n} (-1)^{n-s} \binom{n-1}{s-1} s^{m-1} = 0, \quad 1 \leq m < n. \tag{2.6}$$

Since $\binom{n}{s} = \frac{n}{s}\binom{n-1}{s-1}$, we infer from (2.6) that

$$\sum_{s=1}^{n} (-1)^{n-s} \binom{n}{s} s^{m} = 0, \quad 1 \leq m < n.$$

We may adjoin $s = 0$, since $m \geq 1$, and replace s by r, to obtain

$$\sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} r^{m} = 0, \quad 1 \leq m < n. \tag{2.7}$$

Since we already know, by (2.4), that $St(0, n) = 0$, $n \geq 1$, it follows that (2.7) establishes the inductive step in the proof of (a).

The proof of (b) is now very similar. Thus we assume that, for a given $n \geq 1$, $St(n-1, n-1) = 1$. In view of (a), this implies that

$$\sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{n-1}{r} (r+1)^{n-1} = (n-1)!$$

Set $s = r+1$ and exploit again the identity $\binom{n}{s} = \frac{n}{s}\binom{n-1}{s-1}$ to infer that

$$\sum_{s=1}^{n} (-1)^{n-s} \binom{n}{s} s^{n} = n!$$

We may adjoin $s = 0$, since $n \geq 1$, and thereby complete the inductive step in the proof of (b).

One immediate consequence of Theorem 2.4 which we may include here is

Theorem 2.7. $St(n+1, n) = \binom{n+1}{2}$.

Proof. It is plain that $P(n+1, n) = \binom{n+1}{2}$, since a partition of a set containing $(n+1)$ elements into n non-empty subsets is equivalent to a choice of one subset containing 2 elements.

Of course, Theorem 2.7 could also be proved using a variant of our proof of Theorem 2.6.

3 A combinatorial lemma

In this section we prove a combinatorial result which may be interpreted 'geometrically' within the Pascal Triangle. We will use this result in an essential way in the proof of our main theorem.

We start with the familiar result

$$\sum_{r=1}^{n} r(r+1)\dots(r+s-1) = \frac{n(n+1)\dots(n+s)}{s+1}. \tag{3.1}$$

Now consider $A = \sum_{r=s}^{n} r(r-1)\dots(r-s+1)$, $1 \le s \le n$. By replacing the variable r by $R = r-s+1$ (and then writing r for R) we see that

$$A = \sum_{r=1}^{n-s+1} (r+s-1)(r+s-2)\dots r = \frac{(n-s+1)(n-s+2)\dots(n+1)}{s+1},$$

by (3.1). Thus

$$\sum_{r=s}^{n} r(r-1)\dots(r-s+1) = \frac{(n+1)(n)\dots(n-s+2)(n-s+1)}{s+1}, \quad 1 \le s \le n. \tag{3.2}$$

We now enunciate the lemma

Lemma 3.1. $\sum_{r=s}^{n} \binom{r}{j}\binom{r-j}{s-j} = \binom{n+1}{s+1}\binom{s}{j}$, $j \le s \le n$.

Remark. If $j = s$, then Lemma 3.1 asserts that

$$\sum_{r=s}^{n} \binom{r}{s} = \binom{n+1}{s+1}. \tag{3.3}$$

This is sometimes known as the Christmas Stocking Theorem and admits a very elementary proof. Its name (suggested, we believe, by our late colleague Dave Logothetti) is easily understood when one locates the binomial coefficients entering into the statement in the Pascal Triangle (see Figure 1).

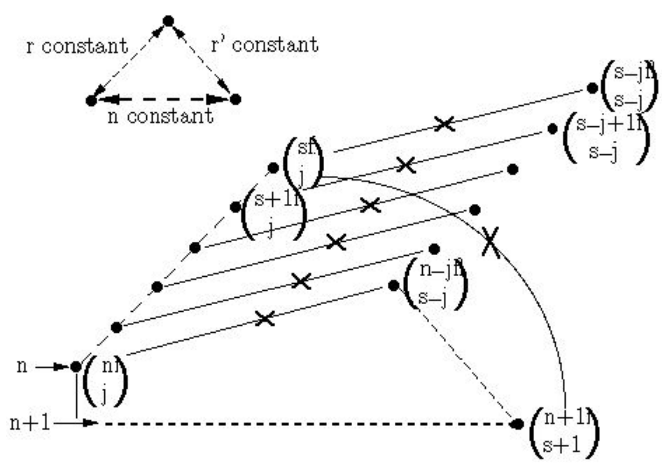
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

The case $s = 2, n = 5$ of the Christmas Stocking Theorem (3.3)
Figure 1.

Proof of Lemma 3.1. Since the case $j = s = 0$ is trivial, we may assume $s \geq 1$. Then

$$\begin{aligned}\sum_{r=s}^{n} \binom{r}{j}\binom{r-j}{s-j} &= \sum_{r=s}^{n} \frac{r(r-1)\dots(r-j+1)(r-j)\dots(r-s+1)}{j!(s-j)!} \\ &= \frac{(n+1)(n)\dots(n-s+1)}{(s+1)j!(s-j)!}, \quad \text{by (3.2)} \\ &= \frac{(n+1)(n)\dots(n-s+1)}{(s+1)!}\frac{s!}{j!(s-j)!} \\ &= \binom{n+1}{s+1}\binom{s}{j}.\end{aligned}$$

Figure 2 provides a geometrical picture of Lemma 3.1.



The geometrical picture (for the case $j < \frac{s}{2}$) associated with the formula

$$\sum_{r=s}^{n} \binom{r}{j}\binom{r-j}{s-j} = \binom{n+1}{s+1}\binom{s}{j}$$

Note : The binomial coefficient $\binom{n}{r}$ may also be written $\binom{n}{r,r'}$, with $r + r' = n$.
Figure 2

4 The main theorem

In this section we show how to calculate the Stirling numbers $St(m, n) = P(m, n)$ for $m > n \geq 1$. We will change notation and consider

$$\gamma_k(m) = P(m, m-k), \quad 1 \leq k \leq m-1. \tag{4.1}$$

We will express $\gamma_k(m)$ as a linear combination of consecutive entries in row m of the Pascal Triangle, the weights being certain positive integers which are independent of m; we call these weights the **Stirling factors** a_{hk}, $0 \leq h \leq k-1$. Precisely we will prove

Theorem 4.1. *We may express $\gamma_k(m)$, $1 \le k \le m-1$, as a linear combination*

$$\gamma_k(m) = \sum_{h=0}^{k-1} a_{hk} \binom{m}{k+h+1}$$

where the Stirling factors a_{hk} are positive integers, independent of m, given inductively by the rule

$$a_{0k} = 1, \quad a_{hk} = \sum_{j=1}^{k-h} \binom{k+h}{j} a_{h-1,k-j}, \quad 1 \le h \le k-1. \tag{4.2}$$

Proof. We find it convenient, in the proof, to work with

$$g_k(n) = \gamma_k(n+k) = P(n+k, n).$$

We first prove a lemma.

Lemma 4.2.

$$g_{k+1}(n) = \binom{n+k+1}{k+2} + \sum_{r=1}^{n}\sum_{j=1}^{k} \binom{r+k}{j} g_{k-j+1}(r-1).$$

Proof of Lemma 4.2. Let[2] $S = \sum_{r=0}^{n-1} (-1)^{n-r-1} \frac{(r+1)^{n+k}}{r!(n-r-1)!}$. Setting $s = r+1$ and adjoining $s = 0$, we see that

$$S = \sum_{s=0}^{n} (-1)^{n-s} \frac{s^{n+k+1}}{s!(n-s)!} = g_{k+1}(n).$$

On the other hand, expanding $(r+1)^{n+k}$ by the binomial theorem, and applying Theorem 2.6, we have

$$\begin{aligned} S = g_{k+1}(n-1) \quad &+ \quad (n+k)g_k(n-1) + \binom{n+k}{2} g_{k-1}(n-1) + \dots \\ &+ \quad \binom{n+k}{k} g_1(n-1) + \binom{n+k}{k+1}. \end{aligned}$$

Thus

$$g_{k+1}(n) - g_{k+1}(n-1) = \binom{n+k}{k+1} + \sum_{j=1}^{k} \binom{n+k}{j} g_{k-j+1}(n-1). \tag{4.3}$$

Adding up (4.3) for $n = 1, 2, \dots, N$, noting that $g_k(0) = 0$ for all $k \ge 1$, and finally replacing N by n, we infer that

$$g_{k+1}(n) = \sum_{r=1}^{n} \binom{r+k}{k+1} + \sum_{r=1}^{n}\sum_{j=1}^{k} \binom{r+k}{j} g_{k-j+1}(r-1). \tag{4.4}$$

[2]Compare the proof of Theorem 2.6.

But $\sum_{r=1}^{n}\binom{r+k}{k+1}=\sum_{r=k+1}^{n+k}\binom{r}{k+1}=\binom{n+k+1}{k+2}$, by (3.3), and the lemma is proved.

We return to the proof of Theorem 4.1. We reformulate the theorem in terms of $g_k(n)$ as asserting that, for $k\ge 1$,

$$g_k(n)=\sum_{h=0}^{k-1}a_{hk}\binom{n+k}{k+h+1}, \tag{4.5}$$

where the weights a_{hk} are given by (4.2); and we prove the theorem in this form by induction on k.

We note first that (4.5) holds if $k=1$ by Theorem 2.7. The inductive hypothesis allows us to write

$$g_{k-j+1}(r-1)=\sum_{\ell=0}^{k-j}a_{\ell,k-j+1}\binom{r+k-j}{k-j+\ell+2},\quad 1\le j\le k,$$

with $a_{\ell,k-j+1}$ independent of r, whence, by Lemma 4.2,

$$g_{k+1}(n)=\binom{n+k+1}{k+2}+\sum_{r=1}^{n}\sum_{j=1}^{k}\sum_{\ell=0}^{k-j}\binom{r+k}{j}\binom{r+k-j}{k-j+\ell+2}a_{\ell,k-j+1}. \tag{4.6}$$

We now calculate $\sum_{r=1}^{n}\binom{r+k}{j}\binom{r+k-j}{k-j+\ell+2}$, using Lemma 3.1. For

$$\begin{aligned}\sum_{r=1}^{n}\binom{r+k}{j}\binom{r+k-j}{k-j+\ell+2} &= \sum_{r=\ell+2}^{n}\binom{r+k}{j}\binom{r+k-j}{k-j+\ell+2},\\ &\quad \text{by removing zero terms}\\ &= \sum_{r=k+\ell+2}^{n+k}\binom{r}{j}\binom{r-j}{k-j+\ell+2}\\ &= \binom{n+k+1}{k+\ell+3}\binom{k+\ell+2}{j},\ \text{by Lemma 3.1.}\end{aligned}$$

Thus, by (4.6),

$$g_{k+1}(n)=\binom{n+k+1}{k+2}+\sum_{j=1}^{k}\sum_{\ell=0}^{k-j}\binom{n+k+1}{k+\ell+3}\binom{k+\ell+2}{j}a_{\ell,k-j+1}. \tag{4.7}$$

Formula (4.7) shows that, as required,

$$g_{k+1}(n)=\sum_{h=0}^{k}a_{h,k+1}\binom{n+k+1}{k+h+2}, \tag{4.8}$$

where $a_{0,k+1} = 1$, and each $a_{h,k+1}$ is independent of n and satisfies, for $h \geq 1$,

$$\begin{aligned} a_{h,k+1} &= \sum_{j=1}^{k} \binom{k+h+1}{j} a_{h-1,k-j+1} \\ &= \sum_{j=1}^{k-h-1} \binom{k+h+1}{j} a_{h-1,k-j+1}, \text{ removing zero terms.} \end{aligned}$$

Thus (4.5) is established, along with the recurrence relation (4.2) for determining the weights a_{hk}. As already explained, this is equivalent to Theorem 4.1. A table of values of a_{hk} is given in Figure 3.

h	0	1	2	3	4	5	6	7	8	9
k↓										
1	1									
2	1	3								
3	1	10	15							
4	1	25	105	105						
5	1	56	490	1260	945					
6	1	119	1918	9450	17325	10395				
7	1	246	6825	56980	190575	270270	135135			
8	1	501	22935	302995	1636635	4099095	4729725	2027025		
9	1	1012	74316	1487200	12122110	47507460	94594500	91891800	34459425	
10	1	2035	235092	6914908	84131350	466876410	1422280860	2343240900	1964187225	654729075

Table of Stirling Factors a_{hk} (see Theorem 4.1)
Figure 3

Notice that the recurrence relation (4.2), determining the Stirling factors a_{hk}, may be re-expressed in the rather more attractive form

$$a_{hk} = \sum_{j=h}^{k-1} \binom{k+h}{j+h} a_{h-1,j}. \tag{4.9}$$

It seems natural to display the Stirling factors a_{hk}, $0 \leq h \leq k-1$, in a triangle. If we left-justify both the Pascal Triangle and the Stirling Factor Triangle, then the entries in the triangles stand in the following relation to each other.

First, we see from Theorem 4.1 that the **Stirling number** $\gamma_k(n)$ is a linear combination of the entries $\binom{n}{r}$ in the n^{th} (horizontal) row of the **Pascal Triangle**, the entries in question running consecutively from $r = k+1$ to $r = 2k$, and the weight of the entry $\binom{n}{k+h+1}$ being the **Stirling factor** a_{hk}.

Second, we see from (4.2) that the **Stirling factor** a_{hk} is a linear combination of the entries $a_{h-1,j}$ in the $(h-1)^{st}$ (vertical) column of the **Stirling Factor Triangle**, the entries in question running consecutively from $j = h$ to $j = k-1$, and the weight of the entry $a_{h-1,j}$ being the **binomial coefficient** $\left(\begin{array}{c} k+h \\ j+h \end{array} \right)$. (See Figure 4).

$$\binom{0}{0}$$

$$\binom{1}{0}\binom{1}{1}$$

$$\binom{2}{0}\binom{2}{1}\binom{2}{2}$$

$$\vdots$$

$$\binom{n}{0}\binom{n}{1}\binom{n}{2} \quad \cdot \quad \cdot \quad \cdot \quad \underbrace{\binom{n}{k+1}\binom{n}{k+2}\cdots\binom{n}{2k}}_{\text{Pascal Triangle}} \quad \cdot \quad \cdot \quad \cdot \quad \binom{n}{n-1}\binom{n}{n}$$

$$P(n, n-k) = St(n, n-k) = \gamma_k(n) = \sum_{h=0}^{k-1} a_{hk} \binom{n}{k+h+1}, 1 \le k \le n-1$$

$$\begin{array}{lllllllll}
a_{01} \\
a_{02} & a_{12} \\
a_{03} & a_{13} & a_{23} \\
\vdots & \vdots & \vdots \\
a_{0h} & a_{1h} & a_{2h} & \cdots & a_{h-1,h} \\
 & & & & \vdots \\
 & & & & a_{h-1,k-1} \\
a_{0k} & a_{1k} & a_{2k} & \cdots & a_{h-1,k} & & a_{hk} & \cdots & a_{k-1,k} \\
 & & & & & & \uparrow
\end{array}$$

Stirling Factor Triangle

$$a_{hk} = \sum_{j=1}^{k-h} \binom{k+h}{j} a_{h-1,k-j} = \sum_{j=h}^{k-1} \binom{k+h}{j+h} a_{h-1,j}, \quad 1 \le h \le k-1, \quad a_{0k} = 1$$

The interaction of the Pascal Triangle and the Stirling Factor Triangle
Figure 4

In general, it does not appear profitable to seek explicit formulae for the Stirling factors a_{hk}, although there are clearly some interesting divisibility properties underlying their definitions. However, the extreme cases $h = 1$, $h = k-1$ are easily calculated. Thus

Theorem 4.3. $a_{1k} = 2^{k+1} - (k+3)$; $a_{k-1,k} = \dfrac{(2k)!}{2^k k!}$.

Proof. By (4.2),

$$a_{1k} = \sum_{j=1}^{k-1} \binom{k+1}{j} = \sum_{j=0}^{k+1} \binom{k+1}{j} - 1 - (k+1) - 1 = 2^{k+1} - k - 3.$$

Again, by (4.2), $a_{k-1,k} = (2k-1)a_{k-2,k-1}$, $k \geq 2$. Since $a_{01} = 1$, this yields

$$a_{k-1,k} = (2k-1)(2k-3)\ldots 3 = \frac{(2k)!}{2^k k!};$$

this formula holds, of course, also for $k = 1$.

We may use Theorems 4.1 and 4.3 to calculate a_{2k} and $a_{k-2,k}$. Thus

Theorem 4.4.

$$a_{2k} = \frac{1}{2}3^{k+2} - (k+5)2^{k+1} + \frac{1}{2}(k^2 + 7k + 13); \quad a_{k-2,k} = \frac{k-1}{3}\frac{(2k)!}{2^k k!}.$$

Proof. By (4.9), $a_{2k} = \sum_{j=2}^{k-1} \binom{k+2}{j+2} a_{1j} = \sum_{j=2}^{k-1} \binom{k+2}{j+2}(2^{j+1} - j - 3)$, by Theorem 4.3. We complete the calculation of a_{2k} by an elementary argument, using the identities

$$(1+x)^{k+2} = \sum_{j=0}^{k+2} \binom{k+2}{j} x^j, \quad (k+2)(1+x)^{k+1} = \sum_{j=1}^{k+2} j \binom{k+2}{j} x^{j-1}.$$

To calculate $a_{k-2,k}$, we exploit Lemma 4.5 below. For if $\frac{a_{k-2,k}}{a_{k-1,k}} = \frac{k-1}{3}$, then Theorem 4.3 immediately yields the given value of $a_{k-2,k}$.

Lemma 4.5. $\frac{a_{k-2,k}}{a_{k-1,k}} = \frac{k-1}{3}$, $k \geq 2$.

Proof of Lemma. We argue by induction on k. If $k = 2$, the result certainly holds. Suppose that $\frac{a_{k-2,k}}{a_{k-1,k}} = \frac{k-1}{3}$, for some $k \geq 2$. Now by (4.9)

$$\begin{aligned}
a_{k-1,k+1} &= \binom{2k}{2} a_{k-2,k-1} + 2k\, a_{k-2,k} \\
&= k(2k-1)a_{k-2,k-1} + 2k\, a_{k-2,k} \\
&= k\, a_{k-1,k} + \frac{2k(k-1)}{3} a_{k-1,k}, \text{ using the proof of Theorem 4.3} \\
&\qquad\qquad \text{and the inductive hypothesis} \\
&= \frac{k}{3}(2k+1)a_{k-1,k} \\
&= \frac{k}{3} a_{k,k+1}, \text{ again by the proof of Theorem 4.3.}
\end{aligned}$$

This establishes the inductive hypothesis, the lemma, and, with it, Theorem 4.4.

Lemma 4.5 expresses one of the many number-theoretical patterns which, we feel sure, reside in the Stirling factor triangle. We plan to study these patterns further. We remark that Lemma 4.5 came to light in a scrutiny of the values of a_{hk} obtained by the use of **Mathematica**^{TM} and displayed in the table of Figure 3.

5 Supplementary results

We revert to the recurrence relation (2.2),

$$P(m,n) = nP(m-1,n) + P(m-1,n-1).$$

If we again set $\gamma_k(m) = P(m, m-k)$, then this relation reads

$$\gamma_k(m) = (m-k)\gamma_{k-1}(m-1) + \gamma_k(m-1). \tag{5.1}$$

Applying Theorem 4.1, we deduce that

$$\sum_{h=0}^{k-1} a_{hk} \binom{m}{k+h+1} = (m-k)\sum_{h=0}^{k-2} a_{h,k-1} \binom{m-1}{k+h} + \sum_{h=0}^{k-1} a_{hk} \binom{m-1}{k+h+1}. \tag{5.2}$$

Theorem 5.1. $\displaystyle\sum_{h=0}^{k-1} a_{hk} \binom{n+k-1}{k+h} = n\sum_{h=0}^{k-2} a_{h,k-1} \binom{n+k-1}{k+h}.$

Proof. We write $n = m-k$ and derive Theorem 5.1 from (5.2) by means of the Pascal Identity in the form $\displaystyle\binom{n+k}{k+h+1} - \binom{n+k-1}{k+h+1} = \binom{n+k-1}{k+h}$.

Remark. We regard Theorem 5.1 as providing a family of identities, indexed by $k \geq 2$, satisfied by the binomial coefficients in row $(n+k-1)$. We may replace $k-1$ by k in the theorem, obtaining for each $k \geq 1$, the identity

$$\sum_{h=0}^{k} a_{h,k+1} \binom{n+k}{k+h+1} = n\sum_{h=0}^{k-1} a_{hk} \binom{n+k}{k+h+1}. \tag{5.3}$$

We may now re-express (5.3), using Theorem 4.3, in the following way.

Theorem 5.2. $\displaystyle\sum_{h=0}^{k-1} (na_{hk} - a_{h,k+1}) \binom{n+k}{k+h+1} = \frac{(2k+2)!}{2^{k+1}(k+1)!} \binom{n+k}{2k+1}.$

Notice that, in applying this theorem, n and k may be chosen independently. Thus we infer a family of linear relations in a given row of the Pascal Triangle.

Corollary 5.3. *If $n < k+1$, then*

$$\sum_{h=0}^{n-1} (na_{hk} - a_{h,k+1}) \binom{n+k}{k+h+1} = 0.$$

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