A Basis for the non-Archimedian Holomorphic Theta Functions

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In this paper we introduce an analytic version of the concept of a theta group on a non-archimedean analytic torus.We construct a basis for a vector space of theta functions similar with the basis for the global sections of a ample line bundle such as given in [2].

Notations: k is a complete non-archimedean valued field. We assume k to be algebraically closed.

1 Analytic tori and 1-cocycles

Let $T = G/\Lambda$ be an analytic torus ; $G = (k^*)^g$ and $\Lambda \subset G$ is a lattice.
Let A be the group of nowhere vanishing holomorphic functions on G and let H be the character group of G. The lattice Λ acts on A in a canonical way: $\alpha^\gamma(x) = \alpha(\gamma x)$. Each 1-cocycle $\xi \in \mathcal{Z}^1(\Lambda, A)$ has a canonical decomposition of the following form:

$$\xi_\gamma(x) = c(\gamma).p(\gamma, \sigma(\gamma)).\sigma(\gamma)(x) \qquad \text{with}$$

(1) $c \in \mathrm{Hom}(\Lambda, k^*)$;
(2) $\sigma \in \mathrm{Hom}(\Lambda, H)$ such that $\sigma(\gamma)(\delta) = \sigma(\delta)(\gamma)$ for all $\gamma, \delta \in \Lambda$;
(3) $p : \Lambda \times H \to k^*$ a bihomomorphism such that $p(\gamma, u)^2 = u(\gamma)$ for all $\gamma \in \Lambda$ and $u \in H$.
(We may assume that $p(\gamma, \sigma(\delta)) = p(\delta, \sigma(\gamma))$ for all $\gamma, \delta \in \Lambda$.)

We will always assume that ξ is non-degenerate and positive i.e. σ is injective and $|\sigma(\gamma)(\gamma)| < 1$ for all $1 \neq \gamma \in \Lambda$. The existence of such a cocycle implies that T is an abelian variety, (see [1]).

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We will always assume that $\mathrm{char}(k) \not| [H : \sigma(\Lambda)]$.

Let $\hat{T}$ be the dual variety of T. ($\hat{T} = \mathrm{Hom}(\Lambda, T)/\hat{\Lambda}$ with $\hat{\Lambda} = \{\gamma \mapsto u(\gamma) | u \in H\}$). The cocycle ξ induces an isogeny $\lambda_{\bar{\xi}} : T \to \hat{T}$ defined by the lift $\lambda_\xi : G \to \mathrm{Hom}(\Lambda, k^*)$ with $\lambda_\xi(x)(\gamma) = \sigma(\gamma)(x)$.

One of the main objects in this paper is the group

$$\mathrm{Ker}\lambda_{\bar{\xi}} = \{\bar{x} | \exists u_x \quad \text{such that} \quad \sigma(\gamma)(x) = u_x(\gamma) \quad \text{for all} \quad \gamma \in \Lambda\}.$$

Let $\mathrm{Ker}\lambda_{\bar{\xi}}^\circ = \{x \in G | \bar{x} \in \mathrm{Ker}\lambda_{\bar{\xi}}\}$.

Proposition 1.1
i) $u_{xy} = u_x.u_y \quad$ for all $\quad x, y \in \mathrm{Ker}\lambda_{\bar{\xi}}^\circ$;
ii) $u_\gamma = \sigma(\gamma) \quad$ for all $\quad \gamma \in \Lambda$;
iii) $u_x = 1 \iff x \quad$ has finite order $\quad (\iff x \in \mathrm{Ker}\lambda_\xi)$;
iv) for all $\quad u \in H \quad$ there exists an $\quad x \in \mathrm{Ker}\lambda_{\bar{\xi}}^\circ \quad$ such that $\quad u = u_x$.
Proof For (i) and (ii) it suffices to verify the assertions on Λ. For (iii) we use the facts that H is torsion free and $u_{x^n} = u_x^n$. Since σ is injective there exists a free basis $\gamma_1, \cdots, \gamma_g$ for Λ and $u_1, \cdots, u_g$ for H such that $\sigma(\gamma_i) = u_i^{a_i}$ for all $i = 1, \cdots, g$. Let $\rho_i \in G$ such that $\rho_i^{a_i} = \gamma_i$ and let $x_i \in G$ such that $u_i(x_j) = u_j(\rho_i)$ for all $i, j = 1, \cdots, g$.
We have $\sigma(\gamma_i)(x_j) = u_i(x_j)^{a_i}$. Hence $x_j \in \mathrm{Ker}\lambda_{\bar{\xi}}^\circ$ and $u_{x_j} = u_j$. This proves (iv). ∎

Define $e : \mathrm{Ker}\lambda_{\bar{\xi}} \times \mathrm{Ker}\lambda_{\bar{\xi}} \to k^*$ by $e(\bar{x}, \bar{y}) = \dfrac{u_x(y)}{u_y(x)}$.

Then e is a non-degenerate antisymmetric bi-homomorphism, see [1]. As a consequence $\mathrm{Ker}\lambda_{\bar{\xi}}$ has a decomposition $\mathrm{Ker}\lambda_{\bar{\xi}} = K_1 + K_2$ such that K_1 and K_2 are isotropic with respect to e and e induces an isomorphism $K_1 \to K_2^*$ (the dual group of K_2).

2 Theta groups

Let $\mathcal{L}(\xi)$ be the vectorspace of holomorphic theta functions with type ξ.
Elements of $\mathcal{L}(\xi)$ satisfy a functional equation $h(z) = \xi_\gamma(z).h(\gamma z), (\gamma \in \Lambda)$. A special element of $\mathcal{L}(\xi)$ is the function h_T defined by the series $h_T(z) = \sum_{\gamma \in \Lambda} \xi_\gamma(z)$.
The dimension of $\mathcal{L}(\xi)$ is given by the index $[H : \sigma(\Lambda)]$, see [1].
Let $\mathcal{M}(T)$ be the field of meromorphic functions on T. Elements of $\mathcal{M}(T)$ can be regarded as Λ-invariant meromorphic functions on G.
Definition: The theta group associated with ξ is defined as

$$\mathcal{G}(\xi) = \{(\bar{x}, f) | \quad \bar{x} \in \mathrm{Ker}\lambda_{\bar{\xi}}, f \in \mathcal{M}(T), \quad \mathrm{div}(f) = \mathrm{div}(\frac{h_T(xz)}{h_T(z)})\}$$

Multiplication is defined by: $(\bar{x}, f).(\bar{y}, g) = (\bar{x}\bar{y}, g(xz)f(z))$.
Remark: $(\bar{x}, f)^{-1} = (\bar{x}^{-1}, f(x^{-1}z)^{-1})$ **Proposition 2.1** If ξ and ξ' are equivalent cocycles in $\mathcal{Z}^1(\Lambda, A)$, (i.e.: $\bar{\xi} = \bar{\xi}'$ in $H^1(\Lambda, A)$), then $\mathcal{G}(\xi)$ and $\mathcal{G}(\xi')$ are isomorphic.

Proof Let $h'_T = \sum_{\gamma \in \Lambda} \xi'_\gamma(z)$ be the theta function associated with ξ'. Since ξ and ξ' are equivalent we have $\lambda_{\bar{\xi}} = \lambda_{\bar{\xi'}}$ and hence $\mathrm{Ker}\lambda_{\bar{\xi}} = \mathrm{Ker}\lambda_{\bar{\xi'}}$. The isomorphism $\alpha : \mathcal{G}(\xi) \to \mathcal{G}(\xi')$ is given by

$$\alpha(\bar{x}, f) = (\bar{x}, f.\frac{h_T(z)}{h_T(xz)}.\frac{h'_T(xz)}{h'_T(z)}). \quad \blacksquare$$

Proposition 2.2 The sequence

$$1 \longrightarrow k^* \xrightarrow{\alpha} \mathcal{G}(\xi) \xrightarrow{\beta} \mathrm{Ker}(\lambda_{\bar{\xi}}) \longrightarrow 1$$

with $\alpha(\lambda) = (\bar{1}, \lambda)$ and $\beta(\bar{x}, f) = \bar{x}$ is exact.

Proof If $x \in \mathrm{Ker}\lambda^\circ_{\bar{\xi}}$ then $f_x = u_x.\frac{h_T(xz)}{h_T(z)}$ is Λ-invariant and $(\bar{x}, f_x)$ is an element of $\mathcal{G}(\xi)$. Furthermore $(\bar{1}, f) \in \mathcal{G}(\xi)$ if and only if $\mathrm{div}(f) = 0$. This implies that f is constant. $\blacksquare$

If one identifies k^* with its image in $\mathcal{G}(\xi)$ then it is easy to verify that k^* is the center of $\mathcal{G}(\xi)$. Furthermore, if $(\bar{x}, f)$ and $(\bar{y}, g)$ are elements of $\mathcal{G}(\xi)$ then the commutator $[(\bar{x}, f), (\bar{y}, g)]$ is equal to $e(\bar{y}, \bar{x})$.

Let K be a subgroup of $\mathrm{Ker}(\lambda_{\bar{\xi}})$.

Definition A subgroup $\mathcal{K} \subset \mathcal{G}(\xi)$ is a *level subgroup* over K if $\beta(\mathcal{K}) = K$ and if $\mathcal{K} \cap k^* = 1$ i.e., $\mathcal{K}$ is isomorphic to K.

Proposition 2.3. Let $K \subset \mathrm{Ker}(\lambda_{\bar{\xi}})$ be a subgroup.

a) A level subgroup $\mathcal{K} \subset \mathcal{G}(\xi)$ over K exists if and only if K is isotropic with respect to e.

b) If $\mathcal{K}$ and $\mathcal{K}'$ are level subgroups over K then there exists a homomorphism $\rho \in \mathrm{Hom}(K, k^*)$ such that $\mathcal{K}' = \{(\bar{x}, \rho(\bar{x})f) | (\bar{x}, f) \in \mathcal{K}\}$.

Proof a) A level subgroup $\mathcal{K}$ over K is isomorphic to K and hence commutative. It follows that for all $(\bar{x}, f)$ and $(\bar{y}, g)$ in $\mathcal{K}$

$$e(\bar{x}, \bar{y}) = [(\bar{y}, g), (\bar{x}, f)] = 1.$$

So K is isotropic.

Conversely, assume that K is isotropic.

Let $\phi : T \to S$ be an isogeny with $\mathrm{Ker}(\phi) = K$. Then S is an analytic torus, say $S = G_S/\Lambda_S$. Let $\psi : G \to G_S$ be a lifting of ϕ. There exists a 1-cocycle $\xi_S \in \mathcal{Z}^1(\Lambda_S, A_S)$ such that $\psi^*(\xi_S) = \xi$; i.e. $\xi_\gamma(x) = \xi_{S,\psi(\gamma)}(\psi(x))$, see[3].

Let $h_S = \sum_{\delta \in \Lambda_S} \xi_{S,\delta}$ be the corresponding theta function in $\mathcal{L}(\xi_S)$. Then $h_S \circ \psi$ is a theta function in $\mathcal{L}(\xi_T)$ and the set

$$\mathcal{K} = \{(\bar{x}, \frac{h_S \circ \psi(z)}{h_S \circ \psi(xz)}.\frac{h_T(xz)}{h_T(z)} | \bar{x} \in K\}$$

is a level subgroup over K.

b) Let $\mathcal{K}$ and $\mathcal{K}'$ be level subgroups over K. For each $\bar{x} \in K$ there exist unique elements $(\bar{x}, f)$ and $(\bar{x}, f')$ in $\mathcal{K}$ and $\mathcal{K}'$ respectively. It follows that $f = \rho(\bar{x})f'$ with $\rho(\bar{x}) \in k^*$. It is easy to verify that ρ is a homomorphism. $\blacksquare$

3 The vectorspace $\mathcal{L}(\xi)$.

The vectorspace $\mathcal{L}(\xi)$ has dimension equal to the index $[H : \sigma(\Lambda)]$. The theta group $\mathcal{G}(\xi)$ acts on $\mathcal{L}(\xi)$ in the following way

$$h^{(\bar{a},f)}(z) = h(az).f(z).\frac{h_T(z)}{h_T(az)} \; ; h \in \mathcal{L}(\xi),\ (\bar{a},f) \in \mathcal{G}(\xi).$$

A straightforward verification shows that $\mathcal{L}(\xi)$ is a $\mathcal{G}(\xi)$-module.
Let $\phi_i : T \to S_i$ be an isogeny with $\mathrm{Ker}\phi_i = K_i, (i = 1, 2)$.
Let S_i be the torus G_i/Λ_i and let $\psi_i : G \to G_i$ be a lifting of ϕ_i. Since K_i is isotropic there exists a 1-cocycle in $\mathcal{Z}^1(\Lambda_i, A_i)$ such that $\psi_i^*(\xi_i) = \xi$. Let $\mathcal{L}(\xi_i)$ be the corresponding space of theta functions.
Lemma 3.1. $\dim(\mathcal{L}(\xi_i)) = 1$.
Proof We have the following commutative diagram:

$$\begin{array}{ccc} T & \xrightarrow{\phi} & S_i \\ \lambda_{\bar{\xi}}\downarrow & & \downarrow\lambda_{\bar{\xi}_i} \\ \hat{T} & \underset{\hat{\phi}}{\longleftarrow} & \hat{S}_i \end{array}$$

($\hat{\phi}$ is the dual of ϕ)
Hence

$$\dim(\mathcal{L}(\xi_i))^2 = \frac{\#\mathrm{Ker}\lambda_{\bar{\xi}}}{\#\mathrm{Ker}\phi.\#\mathrm{Ker}\hat{\phi}} = \frac{\#K_1.\#K_2}{\#K_2.\#\mathrm{Ker}\hat{\phi}}.$$

It is easy to prove that $\#\mathrm{Ker}\phi = \#\mathrm{Ker}\hat{\phi}\ \ (= \#K_2)$.
Hence $\dim(\mathcal{L}(\xi_i))^2 = 1$. ∎

It follows that $\mathcal{L}(\xi_i)$ is generated by the theta function $h_i = \sum_{\delta\in\lambda_i} \xi_{i,\delta}$.
Let $\mathcal{K}_i$ be the level subgroup over K_i such as in 2.3. For each $\bar{x} \in \mathcal{K}_i$ we have $(\bar{x}, g_{\bar{x}}) \in \mathcal{K}_i$ with $g_{\bar{x}} = \dfrac{h_i \circ \psi_i(z)}{h_i \circ \psi_i(xz)}.\dfrac{h_T(xz)}{h_T(z)}$.
It is easy to see that $(h_i \circ \psi_i)^{(\bar{x},g_{\bar{x}})} = h_i \circ \psi_i$ for each $\bar{x} \in K_i$.
If $\bar{a} \in K_1$ then define $h_{\bar{a}} = (h_2 \circ \psi_2)^{(\bar{a},g_{\bar{a}})}$.

Theorem 3.2 The set $\{h_{\bar{a}}|\bar{a} \in K_1\}$ is a basis for $\mathcal{L}(\xi)$.
Proof We only have to prove that the functions are linearly independent.
For each $\bar{a} \in K_1$ and $\bar{b} \in K_2$ we have

$$\begin{aligned} h_{\bar{a}}^{(\bar{b},g_{\bar{b}})} &= (h_2 \circ \psi_2)^{(\bar{b},g_{\bar{b}}).(\bar{a},g_{\bar{a}})} && (1) \\ &= e(\bar{a},\bar{b}).(h_2 \circ \psi_2)^{(\bar{a},g_{\bar{a}})} = e(\bar{a},\bar{b}).h_{\bar{a}} && (2) \end{aligned}$$

Hence $h_{\bar{a}}$ is an eigenvector with respect to the action of $\mathcal{K}_2$. A standard argument shows that the functions are linearly independent. (Use the fact that e is non degenerate.)

Theorem 3.3 $\mathcal{L}(\xi)$ is an irreducible $\mathcal{G}(\xi)$ representation.

Proof Let $\mathcal{L} \subset \mathcal{L}(\xi)$ be a $\mathcal{G}(\xi)$-invariant subspace. Since char$(k) \not| \# K_i$ there exists an eigenvector $h \in \mathcal{L}$ with respect to the action of $\mathcal{K}_2$. It follows from the previous theorem that h is a multiple of some $h_{\bar{a}}$, $(\bar{a} \in K_1)$. Hence each ${h_{\bar{a}'}}^{(\bar{a}', g_{\bar{a}'})} = h_{\bar{a}\bar{a}'}$ is an element of $\mathcal{L}$ and consequently $\mathcal{L} = \mathcal{L}(\xi)$.

Remark One can prove that $\mathcal{L}(\xi)$ is the only irruducible representation of $\mathcal{G}(\xi)$ on which k^* acts by multiplication.

References

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