Non-Archimedian GP-Spaces

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Abstract

We study non-archimedean locally convex spaces in which every limited set is compactoid. In particular, we are interested in spaces of continuous functions.

1 Preliminaries

Throughout this paper K is a non-archimedean valued field that is complete for the metric induced by the non-trivial valuation | . |. Also, E,F are Hausdorff locally convex spaces over K.

A subset B of E is called compactoid if for every zero-neighbourhood U in E there exists a finite set $S \subset E$ such that $B \subset coS + U$, where coS is the absolutely convex hull of S.

Obviously every compactoid set is bounded, and spaces in which all the bounded subsets are compactoid have been studied in [5] and [6].

An other interesting subclass of the class of the bounded subsets of E consists of the limited sets (Definition 2.1). It turns out that every compactoid subset is limited and therefore it is quite natural to study the spaces E in which every limited set is compactoid. We call them Gelfand-Philips spaces (GP-spaces) following Lindström and Schlumprecht who studied such spaces in the complex case (see [10]).

The non-archimedean situation is however completely different from the classical one (Remark 2.5). In fact, in our case there are "much more" GP-spaces (Theorem 2.8). In particular - and this is the main objective of this paper- we show that most of the interesting non-archimedean functions spaces are GP-spaces.

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For unexplained terms, notations and background we refer to [15] (locally convex spaces), [16] (normed spaces) and [4] (tensor products and nuclearity).

2 Limited sets and GP-spaces

Definition 2.1 (Compare [10])
A bounded subset B of E is called limited in E, if every equicontinuous $\sigma(E', E)$-null sequence in E' converges to zero uniformly on B.

Using the natural identification of the $\sigma(E', E)$-null sequences in E' with the continuous linear maps from E to c_0 ([2], Lemma 2.2) along with the form of the compactoid subsets of c_0 ([11], Proposition 2.1), we obtain:

Lemma 2.2 *A bounded subset B of E is limited in E if and only if for each continuous linear map T from E to c_0, T(B) is compactoid in c_0.*

From this Lemma we easily derive,

Proposition 2.3 .
i) Every compactoid subset of E is limited in E.
ii) If B is limited in E and $T \in L(E, F)$, then T(B) is limited in F (where L(E,F) denotes the vector space of all continuous linear maps from E to F).
iii) If B is limited in E and $D \subset B$, then D is limited in E.
iv) Let M be a subspace of E and $B \subset M$. If B is limited in M then B is limited in E . The converse is also true when M is complemented or dense in E (For an example showing that the converse is not true in general, see Remark 2.9) .

It follows from Lemma 2.2 that if every continuous linear map from E to c_0 is compact, then every bounded subset of E is limited. In particular, if the valuation on K is dense, we have

Corollary 2.4 *If the valuation on K is dense then the unit ball of l^∞ is limited (non-compactoid) in l^∞.*

Remark 2.5 Corollary 2.4 shows that, for densely valued fields, the behaviour of limited sets in non-archimedean analysis is in sharp contrast with the one in locally convex spaces over the real or complex field. For this difference compare e.g. [1], Proposition, property 6, [8], Theorem 1 and [9], Proposition 1) with our results.

We'll see in Theorem 2.8.iii) that this difference is even more striking when the valuation on K is discrete.

Definition 2.6 Compare [10])
A locally convex space E is called a Gelfand-Philips space (GP-space in short) if every limited set in E is compactoid.

The following is easily seen:

Proposition 2.7 .

i) A subspace of a GP-space is a GP-space.

ii) The product of a family of GP-spaces is a GP-space.

Theorem 2.8 .

i) Every locally convex space E of countable type is a GP-space.

ii) Every Banach space E with a base is a GP-space.

iii) If the valuation on K is discrete then every locally convex space over K is a GP-space.

PROOF

i) From Lemma 2.2 it follows that c_0 (and hence every normed space of countable type) is a GP-space. Then use the fact that E can be considered as a subspace of $\prod_{p\in\mathcal{P}} E_p$, where $\mathcal{P}$ is a family of seminorms determining the topology of E and for each $p \in \mathcal{P}$, E_p is the normed space $E/kerp$. Now all the E_p are of countable type. Then apply Proposition 2.7

ii) Let $A \subset E$ be limited. We can assume that A is absolutely convex. It suffices to prove that every countable subset B of A is compactoid. Let [B] stand for the closed linear hull of B. Then ([16] Corollary 3.18) [B] is complemented in E and so, by Proposition 2.3.iv) we have that B is limited in [B]. By i) B is compactoid in $[B]$ and hence in E.

iii) Again use the fact that $E \subset \prod_{p\in\mathcal{P}} E_p$ where now each of the spaces E_p has a base ([16], Theorem 5.16). Then apply ii) and Proposition 2.7

Remark 2.9 Property iv) of Proposition 2.3 is not true in general. For example, let the valuation on K be dense, take $E = l^\infty$, $M = c_0$ and B the unit ball in c_0. Then, apply 2.4, 2.3.i) and 2.8.i).

3 Spaces of continuous functions

Let X be a Hausdorff zero-dimensional topological space. We consider the following K-valued function spaces:

PC(X): The space of all continuous functions $f : X \longrightarrow K$ for which f(X) is precompact, endowed with the toplogy τ_u of uniform convergence.

C(X): The space of all continuous functions $X \longrightarrow K$ endowed with the compact open topology τ_c.

$BC(X)$: The space of all bounded continuous functions $X \longrightarrow K$, endowed with the uniform topology τ_u or with the strict topology τ_β. This last one is the topology generated by the seminorms $p_\phi(f) = sup_{x\in X} \mid \phi(x).(f(x) \mid$, where $\phi : X \longrightarrow K$ is a bounded function vanishing at infinity.

Since $PC(X)$ is a Banach space with a base ([16] Theorem 3.4) we obtain inmediately from 2.8.ii),

Theorem 3.1 *PC(X) is a GP-space.*

We now tackle the GP-property for $C(X)$ and $BC(X)$.

Lemma 3.2 *Let $\mathcal{K}$ be a compact subset of X. Then, for every clopen set G in $\mathcal{K}$ there exists a clopen set U_G in X such that $G = U_G \cap \mathcal{K}$.*

PROOF

Let τ_X be the original topology on X and $\tau_{\mathcal{K}}$ the trace of τ_X on $\mathcal{K}$.

Let $G \subset \mathcal{K}$ be $\tau_{\mathcal{K}}$-clopen. Clearly, there exists $U \subset X$, τ_X-open, such that $G = \mathcal{K} \cap U$. Also, for each $a \in G \subset U$, there exists a τ_X-clopen set W_a in X with $a \in W_a \subset U$. Then use a compactness argument.

Theorem 3.3 (Compare [12], Theorem 3.3) *For a set $\mathcal{F} \subset C(X)$, the following properties are equivalent:*

i) $\mathcal{F}$ is compactoid in C(X).

ii) For every compact set $\mathcal{K} \subset X$ the set $\mathcal{F} \mid \mathcal{K}$ is compactoid in $C(\mathcal{K})$ (where $\mathcal{F} \mid \mathcal{K}$ is the set of the restrictions $f \mid \mathcal{K}$ of f to $\mathcal{K}$ with $f \in \mathcal{F}$).

PROOF

i) $\Rightarrow$ *ii)*: This follows directly from the fact that, for each compact set $\mathcal{K} \subset X$, the restriction map

$$C(X) \longrightarrow C(\mathcal{K}) : f \longrightarrow f \mid \mathcal{K}$$

is linear and continuous.

ii) $\Rightarrow$ *i)*: Let U be a zero-neighbourhood in $C(X)$. We can assume that U has the form

$$U = \{f \in C(X) : sup_{x\in X} \mid f(x) \mid \leq \epsilon\},\ \epsilon > 0,\ \mathcal{K}\ compact\ subset\ of\ X.$$

We have to find $f_1, \ldots, f_r \in C(X)$ such that

$$\mathcal{F} \subset co\{f_1, \ldots, f_r\} + U. \tag{1}$$

Put $U_{\mathcal{K}} = \{g \in C(\mathcal{K}) : sup_{x\in\mathcal{K}} \mid g(x) \mid \leq \epsilon\}$. Then, since $\mathcal{F} \mid \mathcal{K}$ is compactoid in $C(\mathcal{K})$, there exist $g_1, \ldots, g_r \in C(\mathcal{K})$ such that

$$\mathcal{F} \mid \mathcal{K} \subset co(g_1, \ldots, g_r) + U_{\mathcal{K}}. \tag{2}$$

Fix $m \in \{1, \ldots, r\}$ and put $V = \{e \in K :\mid e \mid \leq \epsilon\}$. Since $g_m(\mathcal{K})$ is compact in K, there are $e_m^1, \ldots, e_m^s \in K$ such that the sets $e_m^1 + V, \ldots, e_m^s + V$ are disjoint and

$$g_m(\mathcal{K}) \subset (e_m^1 + V) \cup \ldots \cup (e_m^s + V).$$

Then $\{\mathcal{K}_m^1, \ldots, \mathcal{K}_m^s\}$, where $\mathcal{K}_m^i = \{x \in \mathcal{K} : g_m(x) \in e_m^i + V\}$ $(i = 1, \ldots, s)$, constitutes a partition of $\mathcal{K}$ consisting of $\tau_{\mathcal{K}}$-clopen subsets of $\mathcal{K}$. Hence, for each $m \in \{1, \ldots, r\}$ the locally constant function $g'_m : \mathcal{K} \longrightarrow K$ defined by $g'_m(x) = e_m^i$ for $x \in \mathcal{K}_m^i$ is continuous and it has the property $sup_{x\in\mathcal{K}} \mid g_m(x) - g'_m(x) \mid \leq \epsilon$. So (2) can be changed into

$$\mathcal{F} \mid \mathcal{K} \subset co(g'_1, \ldots, g'_r) + U_{\mathcal{K}}.$$

By lemma 3.2, each of the functions g'_m has a locally constant continuous extension $f_m : X \longrightarrow K$ $(m = 1, \ldots, r)$. Then, $f_1, \ldots, f_r$ satisfy (1) and we are done.

Corollary 3.4 *C(X) is a GP-space.*

PROOF
Let $\mathcal{F} \subset C(X)$ be a limited set. Then (Proposition 2.3.ii)) for each compact set $\mathcal{K} \subset X$, $\mathcal{F} \mid \mathcal{K}$ is limited in $C(\mathcal{K})$ and hence compactoid in $C(\mathcal{K})$ (Theorem 3.1). Now apply Theorem 3.3.

Corollary 3.5 *$BC(X), \tau_\beta$ is a GP-space .*

PROOF
Let $\mathcal{F} \subset BC(X)$ be a τ_β-limited set. Since τ_β is finer than τ_c ([7], Proposition 2.10) we obtain from Proposition 2.3.ii) that $\mathcal{F}$ is τ_c-limited in $BC(X)$. By Propsition 2.7.i) and Corollary 3.4 we have that $\mathcal{F}$ is compactoid in $BC(X), \tau_c$. Now apply Corollary 2.9.a) and Proposition 2.11 of [7].

The picture changes completely when we endow $BC(X)$ with the uniform topology τ_u. We have:

Theorem 3.6 *If the valuation on K is dense (Compare 2.8.iii)), then $BC(X), \tau_u$ is a GP-space if and only if X is pseudocompact.*

PROOF
If X is pseudocompact one verifies that $BC(X) = PC(X)$. Then apply Theorem 3.1.
If X is not pseudocompact, then $BC(X), \tau_u$ contains a subspace which is linearly homeomorphic to l^∞ (see [14], proof of Corollary 2.7). Then apply Proposition 2.7.i) and Corollary 2.4.

In [4] (resp. [3]) the nuclearity of the locally convex space $C(X), \tau_c$ (resp. $BC(X)$, τ_β) is characterized. Combining those results with Corollaries 3.4 and 3.5 we obtain:

Theorem 3.7 *The following are equivalent:*
i) $C(X), \tau_c$ is nuclear.
ii) $BC(X), \tau_\beta$ is nuclear.
iii) Every τ_c-bounded subset of C(X) is limited.
iv) Every τ_β-bounded subset of BC(X) is limited.

We now consider tha case where the continuous functions have their values in a polar complete locally convex Hausdorff space E. We define the function spaces $PC(X, E), \tau_u$; $C(X, E), \tau_c$; $BC(X; E), \tau_u$ and $BC(X, E), \tau_\beta$ in the canonical way and we then have:

Theorem 3.8 *.*
i) PC(X,E) is a GP-space if and only if E is a GP-space.
ii) C(X,E) is a GP-space if and only if E is a GP-space.
iii) BC(X,E),τ_β is a GP-space if and only if E is a GP-space.
iv) If the valuation on K is dense, then $BC(X, E), \tau_u$ is a GP-space if and only if X is pseudocompact and E is a GP-space.

PROOF

The proof of ii), iii), iv) is essentially the same as in the K-valued case.

For the proof of i) one needs [13] Theorem 1.3 and the following result.

Theorem 3.9 *Let E and F be complete, polar, locally convex Hausdorff spaces. Then* $E\hat{\otimes}F$*, the completion of the tensor product for its canonical topology, is a GP-space if and only if E and F are GP-spaces.*

PROOF

The consecutive steps are:

i) If E is quasicomplete, then $(E'_c)' = E$, where E'_c is the dual E' of E endowed with the topology τ_{cp} of uniform convergence on the compactoid subsets of E.

ii) If E is quasicomplete. Let $H \subset L(E'_c, F)$ be such that $H(U^o)$ is compactoid in F for all zero-neighbourhoods U in E and $H^*(V^o)$ is compactoid in E for all zero-neighbourhoods V in F. Then, H is compactoid in $L_\epsilon(E'_c, F)$, where the ϵ means that we consider on $L(E'_c, F)$ the topology of uniform convergence on the equicontinuous subsets of E'.

iii) If E and F are GP-spaces, E quasicomplete, then $L_\epsilon(E'_c, F)$ is also a GP-space. (As a consequence, $E \otimes F$ is a GP-space if and only if E and F are GP-spaces).

iv) If E and F are complete, then so is $L_\epsilon(E'_c, F)$. The Theorem is then a direct consequence of this result.

The proofs of ii) and iii) are similar to the archimedean case (see [10]) and are therefore omitted. The proof of iv) is standard. So let us prove i):

Since E is quasicomplete and by [15], Theorem 5.12, it follows that τ_{cp} is the topology of uniform convergence on the sets $A \subset E$ which are absolutely convex, compactoid, edged and $\sigma(E, E')$-complete. Since the family of these sets form a special covering of E (see [15], Definition 7.3), the conclusion follows from [15], Proposition 7.4.

Note that i) is not true in general.

Indeed, take $E = c_{oo}$ and let $x_1, x_2 \ldots$ be a non-convergent Cauchy sequence in E. Then, the map $T : E' \longrightarrow K : f \longrightarrow lim_n f(x_n)$ is an element of $(E'_c)'$. But T cannot be represented by an element of E.

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