A Dialogue Between Two Lifting Theorems

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En conmemoración de los 20 años del Postgrado en Matemática
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Han pasado 20 años desde que formé parte de la primera generación de estudiantes del Postgrado en Matemática de la Facultad de Ciencias de la UCV. Por entonces, la "casa que vence a las sombras" me había ofrecido la oportunidad de volver a estudiar, enseñar y respirar el aire vivificante de una universidad autónoma y democráticamente cogobernada, lo que era imposible en el Cono Sur ensombrecido por las dictaduras. Hace más de una década, al retornar al Uruguay para colaborar en la reconstrucción de una enseñanza devastada, afirmé en la renuncia a mi cargo en la UCV: "pase lo que pase, ésta será para siempre mi Universidad." Hoy quiero agregar que, cerca o lejos, siempre me he sentido trabajando en el Grupo de Teoría de Operadores de la UCV. Lo que sigue se inscribe en esa labor.

Abstract. The relation between the lifting theorems due to Nagy-Foias and Cotlar-Sadosky is discussed.

PRESENTATION.

The Nagy-Foias commutant lifting theorem is a basic result in Operator Theory and its applications to interpolation problems. Its scope is shown in a fundamental book due to Foias and Frazho where we can read that "the work on the general framework of the commutant lifting theorem continued to grow mainly in Romania, the U.S.A. and Venezuela." [FF, p. viii]

Now, the "Southamerican" contribution to the subject stems from the purpose of understanding the relations between the Nagy-Foias theorem and the Cotlar-Sadosky theorem on "weakly positive" matrices of measures.

The aim of this note is to recall some aspects of a "dialogue" between those two theorems that ends by showing that they can be seen as alternative ways of describing the same facts: see below, theorems (4) and (7).

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THE COTLAR-SADOSKY THEOREM.

We shall use the following notation $e_n(t) = e^{int}, n \in \mathbb{Z}$ and $t \in \mathbb{R}$, $\mathcal{P}$ is the space of trigonometric polynomials, i.e. of finite sums $\sum a_n e_n$, with $n \in \mathbb{Z}$ and $a_n \in \mathbb{C}$, $\mathcal{P}_+ = \{\sum a_n e_n \in \mathcal{P} : a_n = 0 \text{ if } n < 0\}$, $\mathcal{P}_- = \{\sum a_n e_n \in \mathcal{P} : a_n = 0 \text{ if } n \geq 0\}$; $\mathbb{T}$ denotes the unit circle on the complex plane $\mathbb{C}$, $C(\mathbb{T})$ is the Banach space of complex continuous functions on $\mathbb{T}$ and $M(\mathbb{T})$ its dual, i.e., the space of complex Radon measures on $\mathbb{T}$; for any $p \geq 1$, $H^p = \{f \in L^p \equiv L^p(\mathbb{T}) : \hat{f}(n) = 0 \text{ if } n < 0\}$, where $\hat{f}$ is the Fourier transform of f.

If $\mu = \{\mu_{jk}\}_{j,k=1,2}$ is a matrix with entries in $M(\mathbb{T})$ and $f = (f_1, f_2) \in C(\mathbb{T}) \times C(\mathbb{T})$, we set

$$\langle \mu f, f \rangle = \sum \{ \int_{\mathbb{T}} f_j \bar{f}_k \, d\mu_{jk} : j,k = 1,2\}.$$

Then $\langle \mu f, f \rangle \geq 0, \forall f = (f_1, f_2) \in C(\mathbb{T}) \times C(\mathbb{T})$, iff μ is a positive matrix measure, i.e., $\{\mu_{jk}(\Delta)\}$ is a positive matrix for any Borel set $\Delta \subset \mathbb{T}$, and the Cotlar-Sadosky theorem [CS. 1] can be stated as follows.

(1) Theorem. *If the matrix measure $\mu = \{\mu_{jk}\}_{j,k=1,2}$ is such that $\langle \mu f, f \rangle \geq 0$, $\forall f = (f_1, f_2) \in \mathcal{P}_+ \times \mathcal{P}_-$, there exists a positive matrix measure $\sigma = \{\sigma_{jk}\}_{j,k=1,2}$ such that $\langle \sigma f, f \rangle = \langle \mu f, f \rangle$, $\forall f \in \mathcal{P}_+ \times \mathcal{P}_-$.*

The above statement implies that

$$\sigma_{11} = \mu_{11}, \ \sigma_{22} = \mu_{22}, \ \sigma_{12} = \bar{\sigma}_{21} = \mu_{11} + h\,dt,$$

where dt is the Lebesgue measure in $\mathbb{T}$ and $h \in H^1$.

The matrix μ such that $\langle \mu f, f \rangle \geq 0$ for every $f \in \mathcal{P}_+ \times \mathcal{P}_-$ is called "weakly positive" and the theorem says that the weakly positive form defined by μ on $\mathcal{P}_+ \times \mathcal{P}_-$ can be "lifted" (or, more precisely, extended) to the positive form defined by σ on $C(\mathbb{T}) \times C(\mathbb{T})$.

THE NAGY-FOIAS THEOREM IMPLIES THE COTLAR-SADOSKY THEOREM

When theorem (1) was proved in 1979, Cotlar said that it was related to the abstract version of Sarason's generalized interpolation theorem [S.1], i.e., the famous Nagy-Foias commutant lifting theorem proved in 1968 ([NF.1]; see also [NF.2] and [FF]).

In order to recall its statement we fix the following notation. If G, H are Hilbert spaces, $\mathcal{L}(G, H)$ is the set of bounded linear operators from G to H and $\mathcal{L}(G) = \mathcal{L}(G, G)$; if K is a closed subspace of G, P_K denotes the orthogonal projection of G onto K, i_K the injection of K in G and $G \,\theta\, K$ the orthogonal complement of K in G. Also, $\bigvee$ means "closed linear span of". Unless otherwise stated, all spaces are Hilbert spaces and all subspaces are closed subspaces.

If $X \in \mathcal{L}(E_1, E_2)$ and E_j is a subspace of the space $G_j, j = 1,2$, then $B \in \mathcal{L}(G_1, G_2)$ is a *lifting* of X if $P_{E_2}B = XP_{E_1}$.Nagy's dilation theorem ([NF. 2], [FF]) says that if $T \in \mathcal{L}(E)$ is a contraction there exists an essentially unique unitary operator $U \in \mathcal{L}(F)$ such that $E \subset T$, $T^n = P_E U^n|_E$ for every $n \geq 0$ and $F = \vee\{U^n E : n \in \mathbb{Z}\}$; U is called the *minimal unitary dilation* of T; set $G = \vee\{U^n E : n \geq 0\}$ and $W = U_{|G}$, then $W \in \mathcal{L}(G)$ is the essentially unique *minimal isometric dilation* of $T : W$ is an isometry that lifts T, $P_E W = TP_E$, and $G = \vee\{W^n E : n \geq 0\}$. Then:

(2) Theorem. *For $j = 1,2$ let $T_j \in \mathcal{L}(E_j)$ be a contraction in a Hilbert space, $W_j \in \mathcal{L}(G_j)$ its minimal isometric dilation and $U_j \in \mathcal{L}(F_j)$ its minimal unitary dilation. If $X \in \mathcal{L}(E_1, E_2)$ and $XT_1 = T_2X$, then:*
i) $\exists B \in \mathcal{L}(G_1, G_2)$ such that $BW_1 = W_2B$, $P_{E_2}B = XP_{E_1}$, $||B|| = ||X||$;
ii) $\exists Y \in \mathcal{L}(F_1, F_2)$ such that $YU_1 = U_2Y$, $P_{E_2}Y_{|E_1} = X$, $||Y|| = ||X||$;

In fact, (1) can be proved by means of (2) in the way we now sketch. Let the shift S be given by $(Sf)(z) \equiv zf(z)$. Set $F_j = L^2(\mu_{jj})$, U_j the shift in $F_j, j = 1,2$, E_1 (E_2) the closure of $\mathcal{P}_+$ $(\mathcal{P}_-)$ in F_1 (F_2), $T_1 = U_{1|E_1}$ and $T_2 = P_{E_2}U_{2|E_2}$. Define $X \in \mathcal{L}(E_1, E_2)$ by

$$\langle Xf_1, f_2\rangle = \int_{\mathbb{T}} f_1 \bar{f}_2 \, d\mu_{12}, \ \forall (f_1, f_2) \in \mathcal{P}_+ \times \mathcal{P}_-.$$

Then U_j is the minimal unitary dilation of $T_j, j = 1,2$, $||X|| \leq 1$ and $XT_1 = T_2X$. Any Y as in (2ii) is given by the multiplication by a function $u = Ye_0$ so $\langle Yf_1, f_2\rangle = \int_{\mathbb{T}} f_1 \bar{f}_2 u \, d\mu_{22}$, $\forall (f_1, f_2) \in \mathcal{P} \times \mathcal{P}$. Since $||Y|| = ||X|| \leq 1$, the matrix measure σ given by $\sigma_{11} = \mu_{11}, \sigma_{22} = \mu_{22}, \sigma_{12} = \bar{\sigma}_{21} = u d\mu_{22}$ is as stated.
Remark We obtained the function u because any operator Y that intertwines the shifts, ie., such that $YS_1 = S_2Y$, is a multiplication. In this way, the commutant lifting theorem extends Sarason's method and gives all the solutions of several interpolation problems [FF].

THE EXTENDED COTLAR-SADOSKY THEOREM IMPLIES THE NAGY-FOIAS THEOREM

The following [CS.2] is an extension of theorem (1).

(3) Theorem. *For $j = 1,2$ let V_j be a vector space, L_j a subspace and $\tau_j : V_j \to V_l$ a linear isomorphism such that $\tau_1 L_1 \subset L_1$ and $\tau_2^{-1} L_2 \subset L_2$, $\alpha_j : V_j \times V_j \to \mathbb{C}$ is a positive form such that $\alpha_j(\tau_j v, \tau_j w) \equiv \alpha_j(v, w)$, and $\beta' : L_1 \times L_2 \to \mathbb{C}$ a sesquilinear form such that $\beta'(\tau_1 w_1, w_2) = \beta'(w_1, \tau_2^{-1} w_2)$ and $|\beta'(w_1, w_2)|^2 \leq \alpha_1(w_1, w_1)\alpha_2(w_2, w_2)$, $\forall (w_1, w_2) \in L_1 \times L_2$. Then β' can be extended to a sesquilinear form $\beta : V_1 \times V_2 \to \mathbb{C}$ such that $\beta(\tau_1 v_1, \tau_2 v_2) = \beta(v_1, v_2)$ and $|\beta(v_1, v_2)|^2 \leq \alpha_1(v_1, v_1)\alpha_2(v_2, v_2)$, $\forall (v_1, v_2) \in V_1 \times V_2$.*

Set $V_j = L^2(\mu_{jj}),\ \tau_j$ the shift in $V_j,\ \alpha_j(v,w) \equiv \int_{\mathbb{T}} v\bar{w}d\mu_{jj},\ L_1 = \mathcal{P}_+,\ L_2 = \mathcal{P}_-$ and $\beta' \equiv \int_{\mathbb{T}} v\bar{w}d\mu_{12}$; apply (3), then β is given by an operator that intertwines the shifts and (1) follows as above.

When $\beta'(\tau_1 w_1, w_2) \equiv \beta'(w_1, \tau_2^{-1} w_2)$ it is said that β' is a generalized Hankel form, and when $\beta(\tau_1 v_1, \tau_2 v_2) \equiv \beta(v_1, v_2)$ it is said that β is a generalized Toeplitz form; thus, (3) is a result concerning the extension of Hankel forms to Toeplitz forms.

Theorem (1) was presented as a property of a class of "modified Toeplitz kernels" [CS.1]. That result was extended to vector valued "generalized Toeplitz kernels" in [AC], where an extension of the famous Naimark dilation for Toeplitz kernels was proved by the method of unitary extensions of isometries. The dilation theorem for generalized Toeplitz kernels gives a proof of the Nagy-Foias theorem ([A.1]; see also [FF], VII.8) so the last and (an extension of) theorem (1) are in fact closely related.

But the story of the dialogue between these two lifting theorems is much longer. For example, theorem (3) was first proved as a consequence of the Nagy-Foias theorem and then an independent proof was given by the method of unitary extensions of isometries ([CS.3]), a method by means of which a direct proof of the Nagy-Foias theorem can be given ([A.2]; see also [S.2] and [F]).

We shall now show that theorem (3) implies (2). With notations as before, assume $||X|| = 1$ and set $V_j = F_j,\ \tau_j = U_j,\ L_1 = G_1,\ L_2 = G_2' := \vee\{U_2^n E_2 : n \leq 0\},\ \alpha_j$ the scalar product in F_j; let $\beta' : L_1 \times L_2 \to \mathbb{C}$ be given by $\beta'(w_1, w_2) = \langle XP_{E_1} w_1, w_2\rangle$. Thus

$$\beta'(U_1 w_1, w_2) \equiv \langle XP_{E_1} W_1 w_1, w_2\rangle \equiv \langle T_2 X P_{E_1} w_1, w_2\rangle = \beta'(w_1, U_2^{-1} w_2)$$

and $|\beta'(w_1, w_2)|^2 \leq \langle w_1, w_1\rangle\langle w_2, w_2\rangle\ \forall (w_1, w_2) \in L_1 \times L_2$. Then (3) says that there exists an extension β of β' such that $||\beta|| \leq 1$ and $\beta(U_1 v_1, U_2 v_2) = \beta(v_1, v_2)$. Consequently, there exists $Y \in \mathcal{L}(F_1, F_2)$ such that $\beta(v_1, v_2) \equiv \langle Y v_1, v_2\rangle$ and that (2.ii) holds. Moreover, $P_{G_2'} Y_{|G_1} = XP_{E_1}$; since $G_2'\theta E_2 = F_2 \theta G_2$, we see that $YG_1 \subset G_2$; setting $B = Y_{|G_1}$, (2.i) follows.

THE FIXED POINT PROOF OF AN EXTENDED NAGY-FOIAS THEOREM

As an illustration of the approach to lifting problems developed in [AADM.1,2] and [G], and related with [TV], we shall sketch the proof of a particular case of the results obtained by means of a fixed point theorem.

It is said that $W \in \mathcal{L}(G)$ is an expansive operator if $||v|| \leq ||Wv||$ for every $v \in G$. The following is a slightly extended version of the Nagy-Foias theorem.

(4) Theorem. *Let $W_1 \in \mathcal{L}(G_1)$ be an expansive lifting of $T_1 \in \mathcal{L}(E_1)$ and $W_2 \in \mathcal{L}(G_2)$ be a contracting lifting of $T_2 \in \mathcal{L}(E_2)$; if $X \in \mathcal{L}(E_1, E_2)$ and $XT_1 = T_2 X$ there exists $B \in \mathcal{L}(G_1, G_2)$ such that $BW_1 = W_2 B$, $P_{E_2} B = XP_{E_1}$ and $||B|| = ||X||$.*

The theorem can be proved in two steps which we now sketch. We may assume $||X|| = 1$.

Assertion (i) Set $\beta = \{B \in \mathcal{L}(G_1, G_2) : P_{E_2}B = XP_{E_1}, ||B|| = ||X||\}$; for any $B \in \beta$ there exists $B^\sharp \in \beta$ such that $B^\sharp W_1 = W_2 B$.

Set $X' = XP_{E_1} \in \mathcal{L}(G_1, G_2)$; then $X' \in \beta$ and $T_2 X' = X' W_1$. Thus, $B \in \beta$ iff $B = X' + K(I - X'^* X')^{1/2}$ with K a contraction in $\mathcal{L}(G_1, G_2 \theta E_2)$. With obvious notation, $B^\sharp W_1 = W_2 B$ iff $K^\sharp (I - X'^* X')^{1/2} W_1 = P_{G_2\theta E_2} W_2 B$. Now,

$$\begin{aligned} ||P_{G_2\theta E_2} W_2 Bw||^2 &= ||W_2 Bw||^2 - ||T_2 X' w||^2 \leq ||w||^2 - ||X' W_1 w||^2 \\ &\leq ||W_1 w||^2 - ||X' W_1 w||^2 = ||(I - X'^* X')^{1/2} W_1 w||^2 \end{aligned}$$

for every $w \in G_1$; let L be the closure of $(I - X'^* X')^{1/2} W_1 G_1$; a unique contraction $K^\sharp \in \mathcal{L}(G_1, G_2\theta E_2)$ is defined by $K^\sharp = K^\sharp P_L$ and $K^\sharp (I - X'^* X')^{1/2} W_1 w \equiv P_{G_2\theta E_2} W_2 Bw$. Assertion (i) follows.

Assertion (ii) Set $\Sigma = \{K \in \mathcal{L}(G_1, G_2\theta E_2) : ||K|| \leq 1\}$; the map $\lambda : \Sigma \to \Sigma$ given by $\lambda(K) \equiv K^\sharp$ has a fixed point.

With the operator topology in $\mathcal{L}(G_1, G_2\theta E_2)$, Σ is compact and λ is continuous: if $K_t \to K$ in Σ then, for every $w \in G_1$ and $w \in G_2\theta E_2$,

$$\begin{aligned} \langle \lambda(K_t)[(I - X'^* X')^{1/2} W_1 w], v\rangle &= \langle W_2[X' + K_t(I - X'^* X')^{1/2}]w, x\rangle \\ &\to \langle W_2[X' + K(I - X'^* X')^{1/2}]w, x\rangle \\ &= \langle \lambda(K)[(I - X'^* X')^{1/2} W_1 w], v\rangle \end{aligned}$$

so $\lambda(K_t) \to \lambda(K)$. Thus, (ii) follows from the Schauder-Tychonov fixed point theorem [DS].

Clearly, if $\lambda(K) = K, B = X' + K(I - X'^* X')^{1/2}$ is as in (II.1).

Remark The lifting problem can have no solution: set ([FF], p.100) $E_1 = E_2 = \mathbb{C},\ T_1 = T_2 = 0,\ X = 1,\ G_1 = G_2 = \mathbb{C}^2,\ W_1 = [w_{jk}^{(1)}]$ with $w_{11} = w_{12} = w_{21} = 0,\ w_{22} = 1,\ W_2 = [w_{jk}^{(2)}]$ with $w_{11} = w_{12} = w_{22} = 0,\ w_{21} = 1$. Then an operator B as in (II.1) does not exist. Note that W_2 is a contractive lifting of T_2 and that W_1 is a lifting of T_1 but W_1 is not expansive.

(5) Corollary. *For $j = 1, 2$ let $S_j \in \mathcal{L}(E_j)$ be a contraction with minimal isometric dilation $V_j \in \mathcal{L}(G_j)$ such that $P_{E_j} R_j = R_j P_{E_j}$. If R_1 is expansive, R_2 is contractive and $X \in \mathcal{L}(E_1, E_2)$ is such that $XS_1 R_{1|E_1} = R_2 S_2 X$, then, there exists $B \in \mathcal{L}(G_1, G_2)$ such that $BV_1 R_1 = R_2 V_2 B,\ P_{E_2} B = XP_{E_1}$ and $||B|| = ||X||$.*

Proof. Set $W_1 = V_1 R_1,\ T_1 = S_1 R_{1|E_1}, W_2 = R_2 V_2, T_2 = R_2 S_2$. Then W_1 is expansive, W_2 is contractive, $P_{E_1} W_1 = T_1 P_{E_1}, P_{E_2} W_2 = T_2 P_{E_2}$ and $XT_1 = T_2 X$. The result follows from (4).

The corollary above was suggested by the following result due to Sebestyén [Se].

(6) Theorem. *Let $S \in \mathcal{L}(E)$ be a contraction with minimal isometric dilation $V \in \mathcal{L}(G)$ and $R \in \mathcal{L}(G)$ a contraction that commutes with the orthogonal projection P_n of G onto $\vee\{V^j E : 0 \leq j \leq n\}$ for $n = 0, 1, \ldots$ If $X \in \mathcal{L}(E)$ satisfies $XS = RSX$ there exists $B \in \mathcal{L}(G)$ such that $BV = RVB$, $P_{E_2}B = xP_{E_1}$ and $||B|| = ||X||$.*

Since $P_0 = P_e$, (6) is a particular case of (5).

A REFORMULATION A LA COTLAR-SADOSKY OF THE EXTENDED NAGY-FOIAS THEOREM

As we shall see, the following result is not only quite similar but also equivalent to the extended Nagy-Foias theorem (4) and it gives an extension of the Cotlar-Sadosky theorem (7).

(6) Theorem. *Let E_j be a Hilbert space and $T_j \in \mathcal{L}(E_j), j = 1, 2$ and $\gamma : E_1 \times E_2 \to \mathbb{C}$ a sesquilinear bounded form such that $\gamma(T_1 e_1, e_2) \equiv \gamma(e_1, T_2 e_2)$. If $W_1 \in \mathcal{L}(G_1)$ is an expansive lifting of T_1 and $W_2 \in \mathcal{L}(G_2)$ a contractive extension of T_2, there exists a sesquilinear bounded extension $\lambda : G_1 \times G_2 \to \mathbb{C}$ of γ such that:*

$$\lambda(W_1 g_1, g_2) \equiv \lambda(g_1, W_2 g_2) \quad (1)$$
$$\lambda(g_1, e_2) = \gamma(P_{E_1} g_1, e_2) \text{ for every } g_1 \in G_1 \text{ and } e_2 \in E_2 \quad (2)$$
$$||\lambda|| = ||\gamma|| \quad (3)$$

Assertion (i) *Theorem 4 implies theorem 7.*

Let $X \in \mathcal{L}(E_1, E_2)$ be such that $\gamma(e_1, e_2) \equiv \langle Xe_1, e_2\rangle$, then $XT_1 = T_2^* X$ and W_2^* is a contractive lifting of T_2^*, so there exists $B \in \mathcal{L}(G_1, G_2)$ such that $BW_1 = W_2 B$, $P_{E_2} B = XP_{E_1}$ and $||B|| = ||X||$. Setting $\lambda(g_1, g_2) \equiv \langle Bg_1, g_2\rangle$ the result follows.

Assertion (ii) *Theorem 7 implies theorem 4.*

Set $\gamma(e_1, e_2) \equiv \langle Xe_1, e_2\rangle$, then $\gamma(T_1 e_1, e_2) \equiv \gamma(e_1, T_2^* e_2)$ and W_2^* is a contractive extension of T_2^*, so there exists λ as in (7); let $B \in \mathcal{L}(G_1, G_2)$ be such that $\lambda(g_1, g_2) \equiv \langle Bg_1, g_2\rangle$. Then $||B|| = ||\lambda|| = ||\gamma|| = ||X||$; also,

$$\langle BW_1 g_1, g_2\rangle \equiv \lambda(W_1 g_1, g_2) \equiv \lambda(g_1, W_2^* g_2) \equiv \langle Bg_1, W_2^* g_2\rangle,$$

so $BW_1 = W_2 B$; finally,

$$\langle P_{E_2} Bg_1, e_2\rangle_{E_2} \equiv \lambda(g_1, e_2) \equiv \gamma(P_{E_1} g_1, e_2) \equiv \langle XP_{E_1} g_1, e_2\rangle$$

so $P_{E_2} B = XP_{E_1}$.

Assertion (iii) *Theorem 7 implies theorem 3.*

For $j = 1, 2$ let F_j be the Hilbert space generated by the vector space V_j and the positive form α_j: there exists a linear operator $\pi_j : V_j \to F_j$ such that $\pi_j(V_j)$ is dense in F_j and $\langle \pi_j v, \pi_j v' \rangle = \alpha_j(v, v')$ for every $v, v' \in V_j$. Let $U_j \in \mathcal{L}(F_j)$ be the unitary operator given by $U_j \pi_j = \pi_j \tau_j$ and G_j be the closure in F_j of $\pi_j L_j$. Let γ be a sesquilinear form $\gamma : G_1 \times G_2 \to \mathbb{C}$ such that $\gamma(U_1 g_1, g_2) \equiv \gamma(g_1, U_2^{-1} g_2)$ and $||\gamma|| \leq 1$ is defined by setting $\gamma(\pi_1 v_1, \pi_2 v_2) = \beta'(v_1, v_2)$ for every $(v_1, v_2) \in L_1 \times L_2$.

Since $U_2^{-1} \in \mathcal{L}(F_2)$ is a contractive extension of $U_2^{-1}{}_{|G_2}$, there exists a sesquilinear form $\lambda : G_1 \times F_2 \to \mathbb{C}$ that extends γ and is such that $||\lambda|| \leq 1$ and that $\lambda(U_1 g_1, U_2 f_2) = \lambda(g_1, f_2)$ holds for every $(g_1, f_2) \in G_1 \times F_2$.

Now set $F_1' = \vee \{U_1^{-n} G_1 : n \geq 0\}$ and extend λ to a sesquilinear form $\lambda_1 : F_1 \times F_2 \to \mathbb{C}$ by setting, for any $n \geq 0$ and $(g_1, f_2) \in G_1 \times F_2$, $\lambda_1(U_1^{-n} g_1, f_2) = \lambda(g_1, U_2^n f_2)$, then $||\lambda_1|| = ||\lambda||$ and $\lambda_1(f_1, f_2) \equiv \lambda_1(U_1 f_1, U_2 f_2)$.

Setting $\beta(v_1, v_2) = \lambda_1(P_{F_1'} \pi_1 v_1, \pi_2 v_2)$ for every $(v_1, v_2) \in V_1 \times V_2$, the result follows.

Final Remark.

Summing up, as Cotlar anticipated, theorems 1 and 2 are in fact very closely related.

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