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Rank of Matrices and the Pexider Equation

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Abstract. The paper [1] gives a characterization of those linear operators which preserve the rank of matrices over $\mathbb{R}$. In this paper we characterize the operators of type (1) having this property, without making the linearity assumption. For matrices from $M_{n,m}$ and $\min\{n,m\} \geq 3$ the operator must be linear and of the form from [1]. If $1 \leq \min\{n,m\} \leq 2$, then the operator may be nonlinear.

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Introduction

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{N}$ the set of positive integers. Let $m, n \in \mathbb{N}$ be constants. Let $M_{m,n}$ be the set of $m \times n$ real matrices, i.e. $M \in \mathbb{R}^{m \times n}$ and $M_n = M_{n,n}$, where $m, n \in \mathbb{N}$.

First of all let us introduce

Definition 1. *We say that an operator*

$$F = [f_i], \quad \text{where } f_i : \mathbb{R} \longrightarrow \mathbb{R} \text{ for } i = 1, 2, \ldots, m, \tag{1}$$

preserves the rank of matrices from $M_{m,n}$ if for every matrix $A \in M_{m,n}$ the equation

$$\operatorname{rank}(A) = \operatorname{rank}(F(A)) \tag{2}$$

holds, where the matrix $F(A) := [f_i(a_{i,j})]$ for $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

The problem of the form of operators defined on the space of matrices over $\mathbb{R}$ into itself was studied by many authors under assumption that the operator is linear (see references of [1]).

In this paper, using the Pexider type additive functional equation, we obtain without a linearity assumption some results for the operator of the form (1). We prove, that if $\min\{m,n\} \geq 3$, then a rank preserving operator must be linear and of the form from paper [1]. If the cases $1 \leq \min\{m,n\} \leq 2$ the operator may be nonlinear.

Main results

Remark 1. An operator F of the form (1) preserves the rank of matrices from M_1 if and only if

$$f_1(x) = 0 \iff x = 0.$$

We prove the following

Lemma 1. *Let an operator F of the form* (1) *be an operator preserving the rank of matrices from M_n for $n \geq 2$. Then the equivalence*

$$x = 0 \iff f_i(x) = 0 \quad \text{for} \quad i = 1, 2, \ldots, n \tag{3}$$

is true.

Proof. For the matrix $B_1 \in M_n$ with all entries equal to zero, $\operatorname{rank}(B_1) = 0$. If the operator F is an operator preserving the rank of matrices, then $\operatorname{rank}(F(B_1)) = 0$ and it follows that $f_i(x) = 0$ for $i = 1, 2, \ldots, n$, i.e.

$$x = 0 \implies f_i(x) = 0 \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{4}$$

Consider the matrix

$$B_2 = \operatorname{diag}(x, x, \ldots, x),$$

where $x \in \mathbb{R}$ and $x \neq 0$. Using implication (4) we obtain

$$F(B_2) = \operatorname{diag}(f_1(x), f_2(x), \ldots, f_n(x))$$

for arbitrary $x \in \mathbb{R}$.

Let us observe that $\operatorname{rank}(B_2) = n$. If the operator F is an operator preserving the rank of matrices from M_n, then $\operatorname{rank}(F(B_2)) = n$ and $\det(F(B_2)) \neq 0$, i.e. $f_1(x) \cdot f_2(x) \cdots f_n(x) \neq 0$. Then $f_i(x) \neq 0$ for $i = 1, 2, \ldots, n$.

We obtain the implication

$$x \neq 0 \implies f_i(x) \neq 0 \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Then the equivalent implication

$$f_i(x) = 0 \implies x = 0 \quad \text{for} \quad i = 1, 2, \ldots, n \tag{5}$$

holds.

From (4) and (5) the equivalence (3) is true. □

We prove

Theorem 1. *An operator F of the form* (1) *is an operator preserving the rank of matrices from M_2 if and only if there exist constants $c_i \neq 0$, $i = 1, 2$, such that*

$$f_i(x) = c_i \cdot g(x), \quad x \in \mathbb{R}. \tag{6}$$

Then the function g is an injective solution of the multiplicative functional Cauchy equation

$$g(x \cdot y) = g(x) \cdot g(y) \quad \text{for} \quad x, y \in \mathbb{R}, \tag{7}$$

such that

$$g(x) = 0 \quad \Longleftrightarrow \quad x = 0. \tag{8}$$

Proof. Let F be an operator preserving the rank of matrices from M_2 and $c_i = f_i(1)$ for $i = 1, 2$. From Lemma 1 we obtain $c_i \neq 0$ for $i = 1, 2$.

First consider the matrix

$$B_3 = \begin{bmatrix} x & 1 \\ x & 1 \end{bmatrix}$$

for arbitrary $x \in \mathbb{R}$. Then $\operatorname{rank}(F(B_3)) = 1$ and $\det(F(B_3)) = 0$. Now

$$f_1(x) \cdot c_2 = c_1 \cdot f_2(x) \quad \text{for} \quad x \in \mathbb{R}. \tag{9}$$

Define new functions

$$g_i(x) = \frac{f_i(x)}{c_i} \quad \text{for} \quad x \in \mathbb{R}$$

for $i = 1, 2$.

Then from (9) we define the function

$$g(x) := g_1(x) = g_2(x) \quad \text{for} \quad x \in \mathbb{R}. \tag{10}$$

Next consider the matrix

$$B_4 = \begin{bmatrix} 1 & x \\ y & x \cdot y \end{bmatrix}$$

for arbitrary $x, y \in \mathbb{R}$. Then $\operatorname{rank}(F(B_4)) = 1$ and $\det(F(B_4)) = 0$. Now

$$c_1 \cdot f_2(x \cdot y) = f_1(x) \cdot f_2(y).$$

Dividing both sides by $c_1 \cdot c_2 \neq 0$ and using the definition (10) we obtain

$$g_2(x \cdot y) = g_1(x) \cdot g_2(y) \quad \text{for} \quad x, y \in \mathbb{R}. \tag{11}$$

From (10) and (11) the function g satisfies the multiplicative functional Cauchy equation (7).

Consider the matrix

$$B_5 = \begin{bmatrix} x & 1 \\ y & 1 \end{bmatrix}$$

for arbitrary $x, y \in \mathbb{R}$, $x \neq y$. Then $\det(B_5) \neq 0$ and $\operatorname{rank}(B_5) = 2$. Then $\operatorname{rank}(F(B_5)) = 2$ and $\det(F(B_5)) \neq 0$, i.e. $f_1(x) \cdot c_2 - f_2(y) \cdot c_1 \neq 0$, then from (10)

$$g(x) \neq g(y) \quad \text{for} \quad x \neq y, \quad x, y \in \mathbb{R}.$$

This means that the function g is injective. Condition (8) follows from Lemma 1.

Let us assume that an operator F is of the form (1), with functions f_i for $i = 1, 2$. The injective function g fulfils the multiplicative functional Cauchy equation (7) and the coefficients $c_i \neq 0$ for $i = 1, 2$. We prove that F is an operator preserving the rank of matrices from M_2.

Let us consider an arbitrary matrix $H \in M_2$ of the form

$$H = \begin{bmatrix} u & v \\ w & z \end{bmatrix},$$

where $u, v, w, z \in \mathbb{R}$.

Consider three possible cases:

1° $\operatorname{rank}(H) = 2$:

Then $\det(H) \neq 0$ and $u \cdot z - v \cdot w \neq 0$, i.e. $u \cdot z \neq v \cdot w$. From injectivity of the function g we obtain that $g(u \cdot z) \neq g(v \cdot w)$. The function g fulfils the multiplicative functional Cauchy equation (7), so we obtain $g(u) \cdot g(z) \neq g(v) \cdot g(w)$. Multiplying both sides of the above relation by $c_1 \cdot c_2 \neq 0$ we obtain

$$c_1 \cdot g(u) \cdot c_2 \cdot g(z) \neq c_1 \cdot g(v) \cdot c_2 \cdot g(w)$$

and from definition (3)

$$f_1(u) \cdot f_2(z) - f_1(v) \cdot f_2(w) \neq 0.$$

Then $\det(F(H)) \neq 0$ and $\operatorname{rank}(F(H)) = 2$.

2° $\operatorname{rank}(H) = 1$:

Then $\det(H) = 0$ and $u \cdot z - v \cdot w = 0$, i.e. $u \cdot z = v \cdot w$. From the injectivity of the function g we obtain that $g(u \cdot z) = g(v \cdot w)$. The function g fulfils the multiplicative functional Cauchy equation (4), so we obtain $g(u) \cdot g(z) = g(v) \cdot g(w)$. Multiplying both sides of the above relation by $c_1 \cdot c_2 \neq 0$ we obtain

$$c_1 \cdot g(u) \cdot c_2 \cdot g(z) = c_1 \cdot g(v) \cdot c_2 \cdot g(w)$$

and from definition (3)

$$f_1(u) \cdot f_2(z) - f_1(v) \cdot f_2(w) = 0.$$

Thus $\det(F(H)) = 0$. The rank of the matrix H is one, so at least one of the entries u, v, w, z is nonzero. Then from injectivity of the function g at least one of the numbers $g(u), g(v), g(w), g(z)$ is nonzero. Multiplying these numbers by nonzero coefficients c_1, c_1, c_2, c_2, respectively, we obtain that at least one of the entries $f_1(u), f_1(v), f_2(w), f_2(z)$ of the matrix $F(H)$ is nonzero and has $\operatorname{rank}(F(H)) = 1$.

3° $\operatorname{rank}(H) = 0$:

Then $u = v = w = z = 0$. From Lemma 1 and definition (6) we obtain that $f_1(u) = f_1(v) = f_2(w) = f_2(z) = 0$ and $\operatorname{rank}(F(H)) = 0$. □

A few simple numerical examples will illustrate the role of the injectivity of the function g in Theorem 1.

Example 1. The operator F defined by formulae $f_1(x) = x^3$, $f_2(x) = 2x^3$ for $x \in \mathbb{R}$ fulfils the assumptions of Theorem 1. It is the nonlinear operator of the form (1) preserving the rank of matrices from M_2.

Without the assumption of injectivity on the function g Theorem 1 is not true.

Example 2. Let $g(x) = x^2$, and $x \in \mathbb{R}$ a non-injective solution of the multiplicative functional Cauchy equation (7). Let $f_1(x) = f_2(x) = x^2$ and consider the matrices

$$B_6 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad F(B_6) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Observe that $\det(B_6) = 2$ and $\operatorname{rank}(B_6) = 2$, $\det(F(B_6)) = 0$ and $\operatorname{rank}(F(B_6))=1$. The operator F is not an operator preserving the rank of matrices from M_2.

Let us observe that the result obtained in Theorem 1 for $n = 2$ is not true for $n = 3$. Consider the following

Example 3. Let $g(x) = x^3$, and $x \in \mathbb{R}$ an injective solution of the functional multiplicative Cauchy equation (7). Let $f_1(x) = f_3(x) = x^3$, $f_2(x) = 2x^3$ and consider the matrices

$$B_7 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \quad \text{and} \quad F(B_7) = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 8 & 1 & 1 \end{bmatrix}.$$

Observe that $\det(B_7)=0$ and $\operatorname{rank}(B_7)=2$, $\det(F(B_7))=12$ and $\operatorname{rank}(F(B_7))=3$. The operator F is not an operator preserving the rank of matrices from M_3.

For $n = 3$ we prove

Theorem 2. *An operator F of the form* (1) *is an operator preserving the rank of matrices from M_3 if and only if there exist constants $c_i \neq 0$, $i = 1, 2, 3$ such that*

$$f_i(x) = c_i \cdot x, \quad x \in \mathbb{R}. \tag{12}$$

Proof. Let F be an operator preserving the rank of matrices from M_3. From Lemma 1 it follows that the equivalence (3) holds. Let the constants $c_i := f_i(1) \neq 0$ for $i = 1, 2, 3$.

First consider the matrices

$$B_8 = \begin{bmatrix} x & 1 & 0 \\ y & 0 & 1 \\ x+y & 1 & 1 \end{bmatrix} \quad \text{and} \quad F(B_8) = \begin{bmatrix} f_1(x) & c_1 & 0 \\ f_2(y) & 0 & c_2 \\ f_3(x+y) & c_3 & c_3 \end{bmatrix}$$

for arbitrary $x, y \in \mathbb{R}$. Then

$$\det(F(B_8)) = c_1 \cdot c_2 \cdot f_3(x+y) - f_1(x) \cdot c_2 \cdot c_3 - f_2(y) \cdot c_1 \cdot c_3.$$

Because $\operatorname{rank}(B_8) = 2$, we have $\operatorname{rank}(F(B_8)) = 2$ and $\det(F(B_8)) = 0$, so the following equation is satisfied:

$$c_1 \cdot c_2 \cdot f_3(x+y) = f_1(x) \cdot c_2 \cdot c_3 + f_2(y) \cdot c_1 \cdot c_3$$

for all $x, y \in \mathbb{R}$. From the above and $c_1 \cdot c_2 \cdot c_3 \neq 0$ we obtain

$$\frac{f_3(x+y)}{c_3} = \frac{f_1(x)}{c_1} + \frac{f_2(y)}{c_2}$$

for $x, y \in \mathbb{R}$. We define new functions

$$h_i(x) := \frac{1}{c_i} \cdot f_i(x) \quad \text{for} \quad x \in \mathbb{R}, \tag{13}$$

where $i = 1, 2, 3$. In other words, the Pexider type additive functional equation

$$h_3(x+y) = h_1(x) + h_2(y) \quad \text{for } x, y \in \mathbb{R}$$

is fulfilled by h_i, $i = 1, 2, 3$.

By [2, Theorem 1, p. 317] the functions

$$h_1(x) = h(x) + \alpha_1, \quad h_2(x) = h(x) + \alpha_2, \quad h_3(x) = h(x) + \alpha_1 + \alpha_2$$

are the solution, where h is an additive function with constants $\alpha_1, \alpha_2 \in \mathbb{R}$. For an additive function h it follows $h(0) = 0$ and from (3) we obtain $\alpha_1 = \alpha_2 = 0$. Then $h_1 = h_2 = h_3 = h$ and the additive functional Cauchy equation

$$h(x+y) = h(x) + h(y) \quad \text{for all} \quad x, y \in \mathbb{R} \tag{14}$$

is fulfilled.

For any square matrix

$$B_9 = \begin{bmatrix} 1 & x & 0 \\ y & x \cdot y & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad F(B_9) = \begin{bmatrix} c_1 & f_1(x) & 0 \\ f_2(y) & f_2(x \cdot y) & 0 \\ 0 & 0 & c_3 \end{bmatrix},$$

where $x, y \in \mathbb{R}$, we obtain $\operatorname{rank}(B_9) = 2$ and also $\operatorname{rank}(F(B_9)) = 2$. Thus $\det(F(B_9)) = c_3 \cdot [c_1 \cdot f_2(x \cdot y) - f_1(x) \cdot f_2(y)] = 0$. From the above and $c_1 \cdot c_2 \cdot c_3 \neq 0$, together with $h_1 = h_2 = h$, the function h satisfies the multiplicative functional Cauchy equation

$$h(x \cdot y) = h(x) \cdot h(y) \quad \text{for all} \quad x, y \in \mathbb{R}. \tag{15}$$

Now, from [2; Theorem 1, p. 356], it follows that the only functions satisfying simultaneously (14) and (15), i.e. the additive and the multiplicative functional Cauchy equations are $h = 0$ and $h = \mathrm{id}$, where id denotes the identity function on $\mathbb{R}$. Since $h(1) = 1$, we see that in our case $h(x) = x$ for $x \in \mathbb{R}$.

From definition (11) we obtain for $i = 1, 2, 3$ that

$$f_i(x) = c_i \cdot h_i(x) = c_i \cdot h(x) = c_i \cdot x \quad \text{for all} \quad x \in \mathbb{R},$$

where $c_i = f_i(1) \neq 0$ and the functions f_i, $i = 1, 2, 3$, are of the form (13).

From the properties of determinants it follows that the operators F the form (1) preserve the rank of matrices from M_3. □

We prove a theorem which describes all operators of the form (1) preserving the rank of real matrices from M_n for $n \geq 3$.

Theorem 3. *An operator F of the form* (1) *preserves the rank of matrices from M_n for $n \geq 3$ if and only if $f_i(x) = c_i \cdot x$, $i = 1, 2, \ldots, n$, where $c_i \neq 0$ are constants.*

Proof. Assume that the operator F preserves the rank of any matrix H from M_n, where $n \geq 3$.

For $n = 3$ the assertion was proved in Theorem 2.

For $n > 3$ consider the matrices $D_i \in M_n$ with a minor of degree 3 of the form

$$H_i = \begin{bmatrix} a_{i,i} & a_{i,i+1} & a_{i,i+2} \\ a_{i+1,i} & a_{i+1,i+1} & a_{i+1,i+2} \\ a_{i+2,i} & a_{i+2,i+1} & a_{i+2,i+2} \end{bmatrix}$$

for $i = 1, 2, \ldots, n-2$ and other entries equal to zero. Observe that $\operatorname{rank}(D_i) = \operatorname{rank}(H_i)$. For an operator F preserving the rank of matrices from M_n it follows that $\operatorname{rank}(F(D_i)) = \operatorname{rank}(F_i(H_i))$, where $F_i = [f_i, f_{i+1}, f_{i+2}]$ for $i = 1, 2, \ldots, n-2$. From Theorem 2 we obtain that there exist constants $c_k \neq 0$ such that $f_k(x) = c_k \cdot x$ for $k = i, i+1, i+2$, where $k = 1, 2, \ldots, n-2$. Then the operator F is of the form (1).

From the properties of determinants it follows that the operators of the form (1) preserve the rank of matrices from M_n for $n \geq 3$. □

Remark 2. If we define an operator $F(A) := [f_j(a_{i,j})]$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ for matrices $A \in M_{m,n}$, then results analogous to those in Theorems 1, 2 and 3 are valid.

Corollary. *If $m > n$, then results analogous to those in Theorems* 1, 2 *and* 3 *are valid for matrices from $M_{m,n}$.*

Remark 3. The results in Theorems 1, 2 and 3 were obtained without assumptions on the operator F (i.e. continuity, measurability or others).

In the paper [1] a similar problem and result are presented. Let $F : M_{m,n} \longrightarrow M_{m,n}$ be a linear operator. By [1, Theorem 3.1] we obtain that $\operatorname{rank}(F(A)) = \operatorname{rank}(A)$ for all $A \in M_{m,n}$ if and only if there exist invertible matrices $M \in M_m$ and $N \in M_n$ such that $F(A) = MAN$ or if $m = n$ then $F(A) = MA^tN$, where A^t denotes transposition of the matrix A.

Let us observe that using Remark 2 in case $\min\{m, n\} \geq 3$ the result obtained for operators of the form (1) is the same as that in Theorem 3. The M is the diagonal matrix $M = \operatorname{diag}(c_1, c_2, \dots, c_m)$ and $N = I_n$, where $I_n \in M_n$ is the unit matrix.

In case $n = 2$ we obtain a better result: operators F of the form (1) preserving the rank of matrices from M_2 may be linear or nonlinear (see Example 1).

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