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Disjoint Faces of Complementary Dimension

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Abstract. In this short note, we show that if P is a d-polytope which is not the simplex, then for all $0 < k < d$, we can find a k-face of P and a $(d-k)$-face of P which are disjoint. This statement generalizes a result of Miller and Helm [1], who proved it for the case $k = 1$.

1. A unique property of the simplex

In this note, we prove the following theorem, of which the $k = 1$ case was previously proved by Helm and Miller [1] in a commutative algebra context.

Theorem 1. *Suppose P is a d-polytope which is not the simplex. Then for all $0 < k < d$, we can find a k-face of P and a $(d-k)$-face of P which are disjoint. Equivalently, if P is a d-polytope for which all k-faces of P intersect all $(d-k)$-faces of P, then P must be the simplex.*

Proof. The proof is by induction on d. For $d = 2$, the only case to check is $k = 1$, where it is clear that any polygon with more than three sides has a pair of disjoint edges.

Next, suppose $d > 2$. Since the statement is symmetric in k and $d-k$, we can assume $k < d-1$. Since P is not the simplex, it cannot be both simplicial and simple. Suppose P is not simplicial. Let F be a facet of P which is not a simplex. By induction, we can find a k-face G and a $(d-1-k)$-face H of F which are disjoint. These are of course also faces of P. By elementary manipulation of face-defining linear functionals or any number of other methods, we can find a face R of dimension $d-k$ which contains H but which is not contained in F. This face R will be disjoint from G, since $G \subset F$, $R \cap F = H$, and so $R \cap G = H \cap G = \varnothing$.

Now, suppose P is not simple. As before, since the statement is symmetric in k and $d-k$, we can assume $k > 1$. Take some vertex figure P/v which is not a simplex. By induction, we can find disjoint faces in P/v of dimensions k and $d-k-1$. These correspond to faces G and H of P, of respective dimension $k+1$ and $d-k$, which intersect only in v. However, taking any facet of G not containing v, we obtain a face of dimension k which is disjoint from H, completing the proof. □

Given Theorem 1, the natural question to ask is the following.

Question 1. *What can we say about k-faces which intersect all $(d-k)$-faces of a d-polytope, or about polytopes which have such faces?*

For $k = d-1$, for instance, the polytope P must be a pyramid over the face in question. However, even for the case $k = 1$ and $d = 3$, there are many polytopes which have such a k-face: for instance, we can start with the simplex and a distinguished edge, and repeatedly add points beyond a 2-face containing the edge in question.

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References

[1] Helm, D.; Miller, E.: *Bass numbers of semigroup-graded local cohomology.* Pac. J. Math. **209** (2003), 41–66.

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