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Construction of Non-Wythoffian Perfect 4-Polytopes

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Abstract. A polytope is perfect if its shape cannot be changed without changing the action of its symmetry group on its face-lattice. There was a conjecture by which perfect 4-polytopes formed a rather limited class of Wythoffian polytopes. It was disproved in a preceding paper of the author by showing that this class is much more wide. In the present paper we go even further by giving a construction that provides non-Wythoffian perfect 4-polytopes. The construction is based on including the copies of a suitable 3-polytope into the facets of a facet-transitive 4-polytope in a symmetry-preserving way.

Keywords: nodal polytope, perfect polytope, regular polytope, semi-nodal polytope, Wythoff's construction

1. Introduction

The notion of a perfect polytope was introduced by S. A. Robertson [11], as a generalization of regular polytopes. Intuitively, a polytope is perfect if it cannot be deformed to a polytope of different shape without altering its symmetry properties. Perfect polytopes are completely known in dimension 2 and 3. Namely, the perfect 2-polytopes coincide with the regular (convex) polygons, and the class of perfect 3-polytopes includes the Platonic solids, the cuboctahedron and icosidodecahedron (these are called "quasi-regular polyhedra" by Coxeter [2]) along with their polars, the rhombic dodecahedron and rhombic triacontahedron discovered by Kepler.

In dimension 4, and from there on, however, the classification problem of perfect polytopes is still open. There was a conjecture by Rostami that looked promising to solve the problem for 4-polytopes. It is formulated as follows ([10], p. 370): "*Any perfect 4-polytope P is either a square $Q \,\square\, Q$ or $Q \diamond Q$ of some regular polygon Q, or, for some irreducible finite reflection group W with fundamental domain D, either P or its polar P^**

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is the convex hull of the orbit under W of a vertex of D". Loosely speaking, it states that any perfect 4-polytope (non-prime or prime) can be obtained by Wythoff's construction from a suitable reflection group W (reducible or irreducible, respectively) such that the initial point of the construction is a vertex of the fundamental domain of W.

Both this conjecture and an attempt [9] to confirm it proved to be false by showing that there are classes of perfect 4-polytopes P of which the conjecture cannot give an account [6]. Namely, neither P nor its polar P^* can be obtained by the above construction but only in a way that

- the initial point is *not a vertex* of the fundamental domain of a reflection group, or
- the group itself is *not a reflection group*, but some proper subgroup of such a group.

Nevertheless, these polytopes (or their polars) can still be obtained by Wythoff's construction, hence they are called *Wythoffian* perfect polytopes (respectively, polars of Wythoffian perfect polytopes).

In the present contribution we go even further. In fact, it is shown that there exist even perfect polytopes for which there is no way of obtaining by Wythoff's construction. Thus it is a natural continuation of the preceding paper [6] of the author.

2. Preliminaries

Here we briefly summarize the necessary tools that are most important for the rest of the paper. For further details the reader is referred to [6] and the references therein.

By a (*convex*) *n-polytope* P we mean the intersection of finitely many closed half-spaces in a Euclidean space, which is bounded and n-dimensional. A *proper face* F of P is the non-empty intersection of P with a supporting hyperplane H, where a supporting hyperplane of P in $\mathbb{E}^n$ is an affine $(n-1)$-plane H such that $H \cap P \neq \emptyset$ and P lies in one of the closed half-spaces bounded by H. A proper face of dimension $0, 1, k$ and $n-1$ is called a *vertex*, *edge*, *k-face* and *facet*, respectively. The proper faces of P along with $\emptyset$ and P (the *improper faces* of dimension -1 and n, respectively) form a lattice $F(P)$ under inclusion, the *face-lattice* of P. The *f-vector* of P is the n-tuple $f(P) = (f_0(P), f_1(P), \dots, f_{n-1}(P))$, where $f_i(P)$ $(i=0,\dots,n-1)$ denotes the number of i-faces of P.

We say that the polytopes P and Q are *combinatorially equivalent* if and only if there is a lattice isomorphism $\lambda : F(P) \to F(Q)$. If $F(P)$ can be mapped to $F(Q)$ by an order-reversing bijection the polytopes P and Q are said to be *duals* of each other. As a special case of duality, we define the *polar* of P as $P^* = \{\mathbf{y} \in \mathbb{E}^\mathbf{n} : \forall \mathbf{x} \in \mathrm{P}, \langle \mathbf{x}, \mathbf{y} \rangle \leq 1\}$, provided that the origin coincides with the centroid of P (recall that the term *reciprocal* is also used for P^* defined this way [2]).

By a *symmetry transformation* of an n-polytope P we mean an isometry of $\mathbb{E}^n$ keeping P setwise fixed. The group $G(P)$ of all symmetry transformations of P is called the *symmetry group* of P. Note that $G(P) = G(P^*)$.

The action of $G(P)$ on P induces an action $G(P) \times F(P) \to F(P)$ on $F(P)$. The *orbit vector* of an n-polytope P is $\theta(P) = \big(\theta_0, \dots, \theta_{n-1}\big)$, where θ_i is the number of orbits of i-faces of P, for each $i = 0, \dots, n-1$, under the action of $G(P)$.

Following Robertson [5, 11], we define an equivalence relation on the set of all n-polytopes as follows. Two n-polytopes P and Q are *symmetry equivalent* if and only if there exists an isometry φ of $\mathbb{E}^n$ and a face-lattice isomorphism $\lambda : F(P) \to F(Q)$ such that for each $g \in G(P)$ and each $A \in F(P)$, $\lambda\big(g(A)\big) = (\varphi g \varphi^{-1})\big(\lambda(A)\big)$. Each symmetry equivalence class is called a *symmetry type*.

A polytope P is said to be *perfect* if and only if all polytopes symmetry equivalent to P are similar to P.

We recall the following notions and theorem from [6].

Let G be a finite group of isometries of $\mathbb{E}^n$. Then the *symmetry scaffolding* of G is the union of the fixed point sets of all transformations in G and is denoted by $\operatorname{scaf} G$. Here we prefer using the same term (and notation) for the intersection of this set with the unit sphere $\mathbb{S}^{n-1}$ (centered at the origin); however, when the distinction is important, the attribute *spherical* will be used for the latter.

Likewise, it is often useful to replace a polytope with its spherical variant in the following sense. For a given n-polytope P, take a unit sphere $\mathbb{S}^{n-1}$ centered at the centroid of P. Then project P radially to $\mathbb{S}^{n-1}$. The image of the set of facets of P under this projection forms a tessellation of $\mathbb{S}^{n-1}$, which we shall refer to as the *spherical image* of P.

For a given group G and a point A in $\operatorname{scaf} G$, the *fixed point set* of A is defined as the set $\operatorname{fix}_A = \{x \in \mathbb{E}^n : g(x) = x,\ \forall g \in G_A\}$, where G_A is the stabilizer of A in G. Then $\dim(\operatorname{fix}_A)$, the dimension of fix_A, is called the *degree of freedom* of A. A point in the spherical symmetry scaffolding of G is called a *node* in exactly the case it has zero degree of freedom. A vertex of a polytope P is called *nodal* if in the spherical image of P it coincides with a node in $\operatorname{scaf} G(P)$. A *nodal polytope* is a polytope whose vertices are all nodal.

Theorem 2.1. *Every vertex-transitive nodal polytope is perfect.*

Proof. [6], p. 245.

3. A construction for non-Wythoffian perfect 4-polytopes

First we recall *Wythoff's construction* [1, 2] (the following formulation is taken almost literally from [4], p. 3):

Construction 3.1. *Form the convex hull of the orbit of a suitable point for one of the finite reflection groups or for the rotatory subgroup of such a group.*

Definition 3.2. *A polytope which can be obtained by Wythoff's construction is called Wythoffian. A polytope P such that neither P nor its polar P^* can be obtained by Wythoff's construction is called non-Wythoffian.*

Various known types of Wythoffian perfect 4-polytopes are reviewed in [6]. The main point of the present paper is the following

Construction 3.3. *Take a 4-polytope $P^{(4)}$ such that the following conditions hold:*
(1) *$P^{(4)}$ is a facet-transitive polytope,*
(2) *the stabilizer of a facet of $P^{(4)}$ in $G(P^{(4)})$ is isomorphic to the symmetry group of a regular 3-polytope.*

Then, include a copy of a 3-polytope $P^{(3)}$ in each of the facets of $P^{(4)}$ with the following conditions:

(3) *$P^{(3)}$ is a perfect polytope,*

(4) *the vertex set of $P^{(3)}$ decomposes to two transitivity classes under the action of this stabilizer,*

(5) *for each included copy, the vertices in the one class are located on the boundary of the facet it is included in and coincide with nodes,*

(6) *each copy is stabilized by the stabilizer of the including facet.*

Finally, take the convex hull of the union of the vertex sets of all copies of $P^{(3)}$ included in $P^{(4)}$.

In what follows we apply this construction starting from various kinds of polytopes.

4. Applying the construction

First we take regular 4-polytopes for $P^{(4)}$ in Construction 3.3. Note that in this case condition (2) is fulfilled automatically. Furthermore, it is easily checked that for each regular polytope $P^{(4)}$, the perfect polyhedron $P^{(3)}$ allowed by conditions (1-6) is unique (recall that there are only 9 types of perfect 3-polytopes). Namely, this is

- cube in the case of regular 5-cell, 16-cell and 600-cell,
- rhombic dodecahedron in the case of hyper-cube and regular 24-cell,
- rhombic triacontahedron in the case of regular 120-cell.

It is found that these polyhedra form the one type of facets of our new polytopes P. We shall denote this type by F_3. Besides, closer investigation shows that in all but the case of 16-cell there are 2 other types of facets.

To find the other two types of facets in question, first we note that the union of the vertex sets of all included 3-polytopes $P^{(3)}$ form the set of vertices of P. Conditions (1-6) imply that this set decomposes to 2 transitivity classes under the action of the symmetry group of $P^{(4)}$. Moreover, an easy consequence of the symmetry properties determined by our six conditions is that each $P^{(3)}$ is situated within the cells of $P^{(4)}$ so that

(1) the vertices lying on the boundary of the cells coincide just with the centroids of 2-faces of $P^{(4)}$.

(2) each vertex in the other class is in the relative interior of a line segment connecting a facet centre and a vertex of the same facet of $P^{(4)}$, at a fixed distance from the end points.

We denote by V_B and V_I these two classes, respectively.

Now vertices from V_I form the vertex sets of the second type of facets of P. Such a facet is just the cell of the polar of $P^{(4)}$ and, when considering the spherical image, its centre coincides with a vertex of $P^{(4)}$. It will be called a facet of type F_0.

A facet of the third type is obtained as follows. Take the r rhombic 2-faces of the included copies of $P^{(3)}$ around an edge E of $P^{(4)}$, where $P^{(4)}$ is taken as a regular polytope of type $\{p, q, r\}$ (we use the well-known *Schläfli symbol* of a regular polytope [2]). These quadrilaterals are connected by their opposite vertices and the whole figure is stabilized by the stabilizer of E in $G(P^{(4)})$. Now the facet in question is the convex hull of this figure.

Since the fixed point of its stabilizer is located on a 1-face of $P^{(4)}$, we shall call it a facet of type F_1. It is in fact a truncated r-gonal dipyramid. (Its shape for $P^{(4)} = \{4,3,3\}$ is shown in Figure 1.)

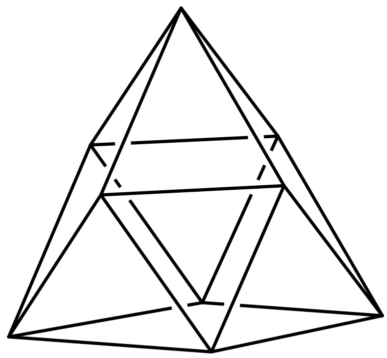

Figure 1

The truncating planes are perpendicular to the original vertex vectors starting from its centroid and are located at a depth that any two of them truncating adjacent vertices are meeting just on the edge between these vertices. Such a truncated figure is bounded by facets of 3 distinct types, and it is found that P has exactly the same types of 2-faces as well.

The following facts are checked. All but in the case of $P^{(4)} = \{3,3,4\}$ the symmetry group is preserved by the construction, i.e. $G(P) = G(P^{(4)})$. Furthermore, each type of faces of dimension 2 and 3 forms a transitivity class as well under the action of this group. On the other hand, both the set of vertices and the set of edges decomposes to two transitivity classes. Thus, for five types of P the orbit vector is $(2,2,3,3)$.

In the exceptional case the facets of type F_0 and F_3 are cubes alike, while the facets of type F_1 are cuboctahedra. The cubes belonging to the two different types are equivalent under the action of a symmetry group larger than $[3,3,4]$, which is $[3,4,3]$. Thus we obtain a simple truncation of the regular 24-cell. In Coxeter's notation ([3], p. 575) this is the uniform polytope $t_1\{3,4,3\} = t_{0,2}\beta_4$, thus it is Wythoffian.

On the other hand, the five types with orbit vector $\theta = (2,2,3,3)$ are non-Wythoffian, since both the first and the last entry in θ is greater than 1.

We denote our new polytopes by $P_4(i,j,k)$, where $P = A, B, F$ *or* H; $i,j,k \in \{0,1,2,3\}$. As for the numerals in the bracket, we recall the following facts. It is well known that a tetrahedral cell in the barycentric subdivision of a regular polytope $\{p,q,r\}$ (in the spherical image) serves as a fundamental domain for the symmetry group of the polytope. On the other hand, the Coxeter graph of such a reflection group has nodes which represent these vertices, (or equivalently, the opposite mirror walls of the tetrahedron). By a convention the Coxeter graph is taken in the form $\bullet\!-\!\!\!-\!\!\!-\!\bullet\!-\!\!\!-\!\!\!-\!\bullet\underset{p}{-\!\!\!-\!\!\!-}\bullet$, and the nodes are numbered from left to right from 0 to 3 (here the label of the last edge may coincide either with p or r depending on which of the two possible regular polytopes having the same symmetry group is considered). Now the numerals in the bracket of a symbol $P_4(i,j,k)$ indicate the position of the centroid of a typical facet of our perfect polytope in the fundamental domain (considered in the spherical image.)

SYMBOL	NUMBER OF FACETS OF TYPE F_0	F_1	F_3	f-VECTOR
$A_4(0,1,3)$	5	10	5	$(30,90,80,20)$
$B_4(0,2,3)$	16	32	8	$(88,288,256,56)$
$F_4(0,1,3)$	24	96	24	$(240,864,768,144)$
$H_4(0,2,3)$	600	1200	120	$(3120,10800,9600,1920)$
$H_4(0,1,3)$	120	720	600	$(3600,10800,8640,1440)$

Table 1

Some numerical data of the face-lattice for the five non-Wythoffian types are summarized in Table 1.

To see perfectness, first recall that the vertices in the class V_B are nodes. Furthermore, this set occupies a whole equivalence class of points determined by the action of the symmetry group $[p,q,r]$. Hence V_B cannot be displaced from its location without changing the action of the symmetry group in question on it. On the other hand, consider the vertices of a copy of $P^{(3)}$ belonging to the class V_I. Although they have one degree of freedom in the symmetry scaffolding of $[p,q,r]$, they cannot be moved away from their location either, since they are vertices of a perfect 3-polytope, the latter being fixed by their vertices of the other type. Thus our polytope P is perfect.

To sum up our results:

Proposition 4.1. *The 4-polytope obtained from a regular polytope of type $\{p,q,r\}$ by Construction 3.3 is perfect. Moreover, it is non-Wythoffian, except for the case of $\{3,3,4\}$, when it is Wythoffian. Its symmetry group is $[p,q,r]$, and the non-Wythoffian types all have the orbit vector $\theta=(2,2,3,3)$.*

Now we apply Construction 3.3 using a non-regular facet-transitive polytope for $P^{(4)}$. This is the perfect 10-cell $\mathrm{t}_{1,2}\,\alpha_4$ bounded by ten Archimedean truncated tetrahedra. It is described in [3] as a uniform polytope, and in [6] as a perfect polytope. Its face-structure is shown in Figure 2 (identical vertices are labelled by the same number). We recall that its f-vector is: $(30,60,40,10)$. Since there are several perfect polytopes that can be derived from this 10-cell, we shall use the simpler symbol X (Roman ten) for its notation.

Choose a cube for $P^{(3)}$. The conditions of Construction 3.3 imply that the copies of the cube must be included in the cells of our starting polytope in the way that four alternating vertices of a cube coincide with the centroids of the hexagonal faces. Then the other four vertices of the same cube are in the interior of the cell it is included in. By the same convention as above, we shall denote the two classes of vertices arranged this way by V_B and V_I, respectively. The union $V_B \cup V_I$ forms just the set of vertices of our new non-Wythoffian perfect polytope. For this reason, it will be denoted by $X(H,I)$, where the letter H and I refers to the centroids of the hexagonal faces and to the vertices in the interior of the cells of X, respectively.

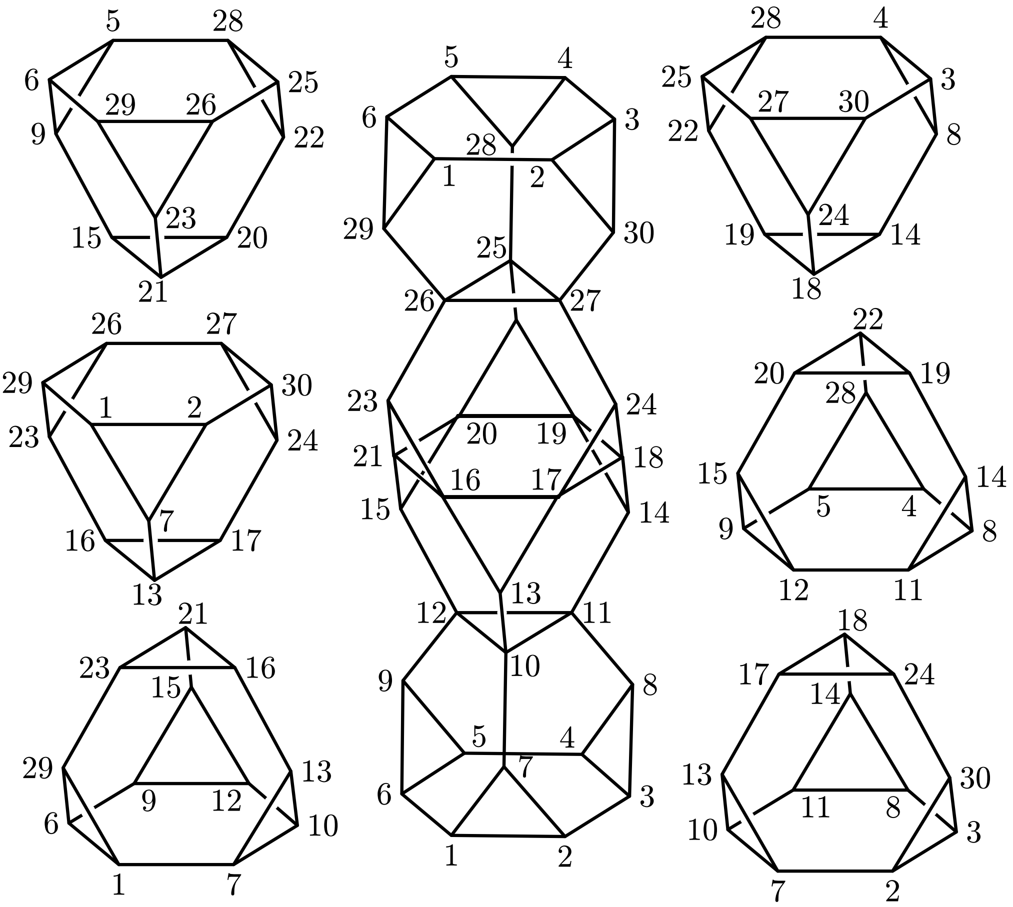

Figure 2

Here, just as above, it is found that facets of the one type of $\mathrm{X}(H, I)$ are just the cubes used in the construction. Let this type be denoted by F_3. A facet, the type of which will be denoted by F_0, can be found as follows. We start from an observation that follows from the construction of the perfect 10-cell (cf. [6], Section 3.3). Namely, the stabilizer of each vertex in its symmetry group is isomorphic to the group $D_{2d} \cong [4, 2^+]$. Choose a vertex V and take the 4 cells that are incident to V. Consider the four vertices belonging to V_I in the vicinity of V, one from each cell. They are contained in a hyperplane that is perpendicular to the straight line connecting the centroid of the 10-cell and V (a simple consequence of that this hyperplane is stabilized by the stabilizer of V). The convex hull of these 4 vertices is a tetrahedron, namely, a *tetragonal disphenoid*, i.e. its facets are isosceles triangles (a consequence of the symmetry determined by its stabilizer, also confirmed by a simple calculation). If the cubes are unit cubes, the length of the base and of the lateral side of such a triangle is $\sqrt{2}/2$ and $\sqrt{6}/2$, respectively.

We note that, in contrast to the preceding case, here some simple calculations in 4-space are necessary in order to find the exact shape of two types of facets. To this end, we determined the coordinates of the vertices of the 10-cell. These are given in Table 2

$\mathbf{v}_1(1,1,0,-1,-1)$	$\mathbf{v}_2(1,0,1,-1,-1)$	$\mathbf{v}_3(1,-1,1,0,-1)$	$\mathbf{v}_4(1,-1,0,1,-1)$
$\mathbf{v}_5(1,0,-1,1,-1)$	$\mathbf{v}_6(1,1,-1,0,-1)$	$\mathbf{v}_7(0,1,1,-1,-1)$	$\mathbf{v}_8(0,-1,1,1,-1)$
$\mathbf{v}_9(0,1,-1,1,-1)$	$\mathbf{v}_{10}(-1,1,1,0,-1)$	$\mathbf{v}_{11}(-1,0,1,1,-1)$	$\mathbf{v}_{12}(-1,1,0,1,-1)$
$\mathbf{v}_{13}(-1,1,1,-1,0)$	$\mathbf{v}_{14}(-1,-1,1,1,0)$	$\mathbf{v}_{15}(-1,1,-1,1,0)$	$\mathbf{v}_{16}(-1,1,0,-1,1)$
$\mathbf{v}_{17}(-1,0,1,-1,1)$	$\mathbf{v}_{18}(-1,-1,1,0,1)$	$\mathbf{v}_{19}(-1,-1,0,1,1)$	$\mathbf{v}_{20}(-1,0,-1,1,1)$
$\mathbf{v}_{21}(-1,1,-1,0,1)$	$\mathbf{v}_{22}(0,-1,-1,1,1)$	$\mathbf{v}_{23}(0,1,-1,-1,1)$	$\mathbf{v}_{24}(0,-1,1,-1,1)$
$\mathbf{v}_{25}(1,-1,-1,0,1)$	$\mathbf{v}_{26}(1,0,-1,-1,1)$	$\mathbf{v}_{27}(1,-1,0,-1,1)$	$\mathbf{v}_{28}(1,-1,-1,1,0)$
$\mathbf{v}_{29}(1,1,-1,-1,0)$	$\mathbf{v}_{30}(1,-1,1,-1,0)$		

Table 2

(the subscripts of the vertex vectors are equal to the labels of the corresponding vertices in Figure 2). The starting data were the coordinates of a typical vertex (in our notation $\mathbf{v}_1$) as were given by Coxeter in [3], p. 574 (for simplicity of the coordinates, we work in the hyperplane $x_1 + x_2 + x_3 + x_4 + x_5 = 0$ of $\mathbb{E}^5$). Note that in this setting the cubes in question are indeed unit cubes.

To find the facets of the third type of our polytope, first observe that each edge of the 10-cell is a common edge of one triangular and two hexagonal faces. Choose an edge and consider the two hexagons having this edge in common. The centroids of these hexagons are the end points of the one diameter of a square face of a cube located in the cell that the hexagons belong to. Take the triangular face incident to the chosen edge and consider the disphenoid edge of length $\sqrt{2}/2$ that is obtained above and is passing through the triangular face. Now it is checked that this edge is parallel to the square face in question (actually, it is parallel to the one diameter of the square). Taking the convex hull of this square face and this edge just the desired facet is obtained. Its stabilizer in $G(P)$ is equal to the stabilizer of the chosen edge and is isomorphic to the group $C_{2v} \cong [2]$. Hence, in accordance with our convention above, it will be called a facet of type F_1. Note that the symmetry group of this facet is the same as its stabilizer. Its shape is shown in Figure 3.

We note that all vertices of degree four of this facet F belong to the same transitivity class under the action of $G(P)$ (namely, this is V_I). It follows that all the six triangular faces are isosceles triangles. Two of these triangles, having smaller apex angle, are such that each forms a common face of F and an adjacent disphenoid facet. The four other triangles, having larger apex angle, are shared with facets of the same type as F.

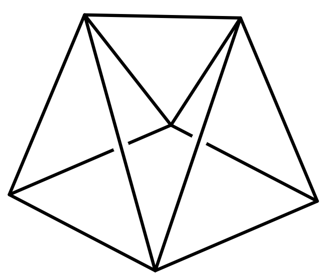

Figure 3

(As an interesting coincidence, we note also, that a 3-polytope symmetry equivalent to that in Figure 3 happens to occur on the cover of Handbook of Discrete and Computational Geometry [7] as redrawn from Figure 13.1.2, page 246 [8] of the book.)

It is also observed, that the length of the edges of P characterizes the transitivity classes of edges as well: there are altogether 3 such classes, namely, those of edges with length $\sqrt{2}/2$, 1 and $\sqrt{6}/2$, respectively (in our scale). Let us call the *degree* of an edge E the number of facets having E in common. Then, it is found that the degree of edges of these types is 6, 4 and 3, respectively (observe that in the shortest edges there are alternately 3 disphenoids and 3 facets of type F_1 meeting). By a simple calculation, the f-vector $f=(60,260,300,100)$ is obtained (note that the number of facets of type F_0, F_1 and F_3 is 30, 60 and 10, respectively; cf. the f-vector of the 10-cell). It is directly seen that our polytope is non-Wythoffian.

We note as well, that similarly as above, the symmetry group of $P^{(4)}$ is preserved through the construction. Hence the symmetry group of our new polytope (in Coxeter's notation) is isomorphic to $[[3,3,3]]$, or, in another approach, to the semi-direct product $[3,3,3]\rtimes\langle\rho\rangle$ [6]. Having this symmetry group, the orbit vector is found to be $\theta=(2,3,3,3)$.

Finally, the proof of perfectness of P is quite analogous to the former arguments. Note that here the convex hull of V_B is $\mathrm{t}_{0,3}\,\alpha_4$, a Wythoffian perfect polytope given in [3, 6].

To sum up, we have

Proposition 4.2. *The 4-polytope* $\mathrm{X}(H,I)$ *obtained from the perfect 10-cell* $\mathrm{t}_{1,2}\,\alpha_4$ *by Construction 3.3 is perfect and non-Wythoffian. It has the f-vector* $f=(60,260,300,100)$ *and orbit vector* $\theta=(2,3,3,3)$. *Its symmetry group is* $[[3,3,3]]\cong[3,3,3]\rtimes\langle\rho\rangle$.

We remark that a further perfect non-Wythoffian polytope, closely related to $\mathrm{X}(H,I)$, can be obtained if a perfect 48-cell is chosen for $P^{(4)}$. This latter is bounded by 48 Archimedean truncated cubes (ATC) [6]. In this case $P^{(3)}$ is a rhombic dodecahedron. Its vertices of degree 4 are positioned in the centres of the octagonal faces of the ATC cell, and its vertices of degree 3 are in the interior of the cell.

5. The existence of semi-nodal perfect polytopes

The construction in the former section gives a positive answer to a problem raised earlier by the author ([6], Problem 4.3). The problem is as follows: "*Does there exist a semi-nodal perfect polytope?*" Recall that a polytope P is semi-nodal if and only if both P and P^* has vertices which are not nodal. Now it can be seen that our polytope $X(H,I)$ is semi-nodal. For, its vertices of type I are indeed nodal (in the spherical image each of them is located in a scaffolding arc consisting of points with one degree of freedom). On the other hand, we have seen that the stabilizer of its facet of type F_1 is isomorphic to $C_{2v}\cong[2]$. But the fixed point set of this group is a straight line. Hence a vertex of the polar of $X(H,I)$ corresponding to such a facet has one degree of fredom. Thus we have

Theorem 5.1. *There exists semi-nodal perfect polytope.*

Observe that the existence of a semi-nodal perfect 4-polytope represents an even greater conceptual distance from the polytopes allowed by Rostami's conjecture. For, these latter not only Wythoffian but nodal as well.

We remark that work of constructing and describing further semi-nodal perfect 4-polytopes is in progress. This will be the topic of a subsequent paper.

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