Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 42 (2001), No. 2, 439-442.

Polygons with Hidden Vertices

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Abstract. Given a point M in Euclidean 3-space we show that there exists a polygon without self-intersection and not containing M such that viewed from M each vertex of the polygon is hidden behind an edge of the polygon. As an application, we construct a toric 4-variety which has peculiar compactification properties.

Let $P_1P_2\cdots P_m$ be a polygon $\mathcal{P}$ without self-intersection in Euclidean space E^n (so $\mathcal{P} = [P_1,P_2]\cup[P_2,P_3]\cup\cdots\cup[P_{m-1},P_m]$ where $[P_i,P_{i+1}]$ is the line segment with end points $P_i,P_{i+1},\ i=1,\dots,m-1$). Given a point M not on $\mathcal{P}$ we say that viewing from M the vertex P_j is hidden behind $[P_{i,P i+1}]$ if $[M,P_j]\cap[P_i,P_{i+1}]$ is relative interior to $[P_i,P_{i+1}]$.

Problem 1. Given M, does there exist a polygon $\mathcal{P} = P_1P_2\cdots P_m$ such that viewing from M each vertex P_j is hidden behind some edge $[P_i,P_{i+1}]$ of $\mathcal{P}$?

For $n=1$ one can "see" from M either P_1 or P_m , hence Problem 1 has no solution. For $n=2$ we shall prove (Theorem 1) that also no solution exists. In Theorem 2 we present a solution for $n=3$ and $m=14$. It also provides an example for $n>3$.

The following question remains open:

Problem 2. What is the minimal number m for which, in case $n=3$, Problem 1 has a solution?

M must lie in the interior of *conv* $\mathcal{P}$ (convex hull), as one can see as follows: If this is not so, consider the point U of *conv* $\mathcal{P}$ next to M, and let H be a supporting hyperplane of *conv* $\mathcal{P}$ in U. If $M=U$ we apply Theorem 1' below to $H\cap$ *conv* $\mathcal{P}$. If $M\neq U$ we choose H perpendicular to the line joining M and U. Now we can "see" from M the vertices of *conv* $\mathcal{P}$ lying in $H\cap$ *conv* $\mathcal{P}$.

As a consequence, $\mathcal{P}$ must have at least four vertices which are not coplanar. Since each of these vertices is hidden behind an edge, it is readily seen that four further vertices exist. So if m_{min} is the least possible m we obtain

$$8 \leq m_{min} \leq 14.$$

We conjecture that m_{min} is closer to 14 than to 8.

Theorem 1. *Problem 1 has no solution for $n = 2$.*

Proof. Suppose $\mathcal{P} = P_1 P_2 \cdots P_m$ does solve Problem 1 for $n = 2$. Let κ be the largest circle with center M whose interior is not intersected by $\mathcal{P}$. Since $\mathcal{P}$ is a closed set, κ contains a point P of $\mathcal{P}$. If P were a vertex of $\mathcal{P}$ we could "see" this vertex from M. So P lies in the relative interior of an edge $[P_i, P_{i+1}]$. The edge behind which P_i is hidden, has one end point, P_r say, inside the triangle MPP_i (Figure 1).

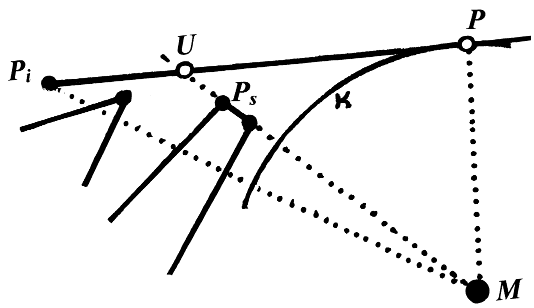

Figure 1.

Among all vertices of $\mathcal{P}$ in this triangle there is one, P_s say, such that the line through M and P_s intersects $[P_i, P]$ in a point U for which inside the triangle MUP there is no further vertex of $\mathcal{P}$. There may be more such vertices on $[M, P_s]$; the one closest to M can be "seen" from M, a contradiction. The proof applies without change to the following more general statement:

Theorem 1'. *Theorem 1 remains true if the polygon is replaced by a finite set of non-crossing line segments, their end points, and further points.*

Theorem 2. *Problem 1 has a solution for $n = 3$ and $m = 14$.*

Proof. Let M lie in the origin of a Cartesian coordinate system. We present the coordinates of each P_i and express it as a positive linear combination of the end points of some edge of $\mathcal{P}$ (Figure 2):

$$\begin{array}{llll} P_1 = (0, 10, 10) & = \frac{5}{16}P_{10} + \frac{15}{16}P_{11} & P_8 = (0, 15, 15) & = \frac{15}{32}P_{10} + \frac{45}{32}P_{11} \\ P_2 = (-10, 0, -10) & = \frac{5}{16}P_5 + \frac{15}{16}P_6 & P_9 = (-12, -3, -9) & = \frac{6}{5}P_2 + \frac{3}{10}P_3 \\ P_3 = (0, -10, 10) & = \frac{15}{16}P_{10} + \frac{5}{16}P_{11} & P_{10} = (0, -16, 8) & = \frac{4}{15}P_{13} + \frac{4}{3}P_{14} \end{array}$$

$$
\begin{array}{llll}
P_4 = (10, 0, -10) = \frac{15}{16}P_5 + \frac{5}{16}P_6 & & P_{11} = (0, 16, 8) = \frac{4}{3}P_{12} + \frac{4}{15}P_{13} \\
P_5 = (16, 0, -8) = \frac{4}{3}P_7 + \frac{4}{15}P_8 & & P_{12} = (-3, 12, 9) = \frac{6}{5}P_1 + \frac{3}{10}P_2 \\
P_6 = (-16, 0, -8) = \frac{4}{15}P_8 + \frac{4}{3}P_9 & & P_{13} = (15, 0, -15) = \frac{45}{32}P_5 + \frac{15}{32}P_6 \\
P_7 = (12, -3, -9) = \frac{3}{10}P_3 + \frac{6}{5}P_4 & & P_{14} = (-3, -12, 9) = \frac{3}{10}P_2 + \frac{6}{5}P_3
\end{array}
$$

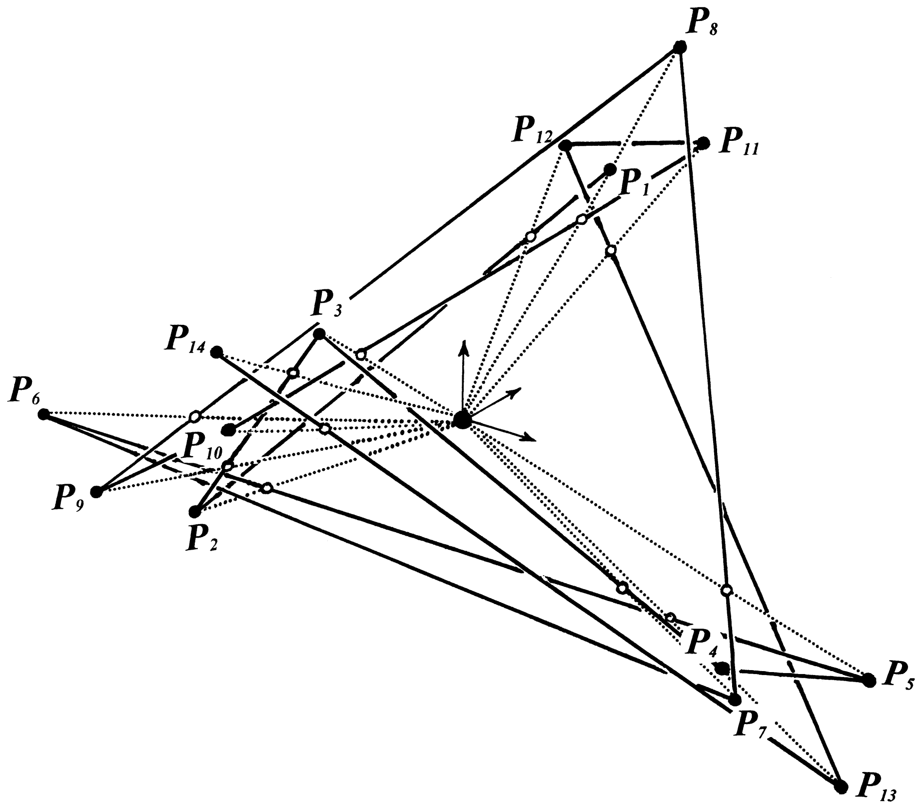

Figure 2.

Let Σ be a fan in $\mathbf{R}^n$, that is, a system of finitely many rational convex polyhydral cones with apex 0 such that if a cone is in Σ also its faces lie in Σ, and such that the intersection of two cones is always a common face of the cones (conditions of a cell complex). To Σ an n-dimensional algebraic variety X_Σ is assigned, called a *toric variety* (see, for example, [1], [2], [3]). We remind in the following facts:

(a) X_Σ is complete (compact) if Σ covers all of $\mathbf{R}^n$.
(b) If ρ is a k-dimensional cone of Σ, then to the star of ρ in Σ there corresponds an $(n-k)$-dimensional subvariety $X_{\Sigma/\rho}$ of X_Σ which is again toric. If Σ/ρ is isomorphic to the fan $\{\{0\}\}$ we obtain an algebraic torus $(K\backslash\{0\})^{n-k}$ if K is the field under consideration.

We say $X_{\Sigma'}$ is a *special partial completion* of X_Σ if Σ is a subfan of Σ' and if Σ, Σ' have the same 1-cones. By the *positive hull* $\operatorname{pos} X$ of a set X we mean the set of all linear combinations of elements of X with nonnegative coefficients. $|\Sigma|$ denotes the set of all points contained in some cone of Σ.

Theorem 3. *There exists a* 4*-dimensional toric variety* X_Σ *with* $\operatorname{pos}|\Sigma| = \mathbf{R}^4$ *and a* 3*-dimensional algebraic torus* $X_{\Sigma/\rho}$ (ρ *a* 1*-cone*) *in it such that for each special partial completion* $X_{\Sigma'}$ *of* X_Σ *we have* $X_{\Sigma'/\rho} = X_{\Sigma/\rho}$.

Remark 1. As a consequence of Theorem 3 we find: X_Σ cannot be completed (compactified) without increasing the number of 1-cones in Σ. This, however, can also be shown by using simpler fans, for example one easily constructed from a polyhedron presented in [4].

Remark 2. Theorem 3 and Remark 1 have no analogues for three-dimensional toric varieties.

Proof of Theorem 3. We construct a fan Σ as follows. Let the polygon $\mathcal{P}$ lie in the hyperplane $\{x_4 = 1\}$ of $\mathbf{R}^4$. We set $P_i' := (P_i, 1), i = 1, \ldots, 14$, and $M' := (M, 1)$. Let $\operatorname{pos}\{P_i'\}$, $i = 1, \ldots, 14$, $\operatorname{pos}\{M'\}$, and $\operatorname{pos}\{-M'\}$ be the 1-dimensional cones of Σ, and let $\operatorname{pos}[P_i', P_{i+1}']$, $i = 1, \ldots, 14$, be the 2-dimensional cones of Σ. No 3- and 4-dimensional cones are assumed to belong to Σ.

If we look for a fan Σ' which contains Σ and has the same 1-cones as Σ has, we see that a new cone cannot contain both, $\operatorname{pos}\{M'\}$ and $\operatorname{pos}\{-M'\}$ (it would not have 0 as apex), also not $\operatorname{pos}\{M'\}$ and $\operatorname{pos}\{P_i'\}$, by construction of $\mathcal{P}$. So choosing $\rho = \operatorname{pos}\{M'\}$ we obtain the theorem.

References

[1] Ewald, G.: *Combinatorial Convexity and Algebraic Geometry.* Graduate Texts in Math., Springer, New York, Berlin, Heidelberg 1996.

[2] Fulton, W.: *Introduction to Toric Varieties.* Ann. Math. Studies **131**, Princeton University Press, Princeton 1993.

[3] Oda, T.: *Convex Bodies and Algebraic Geometry. An Introduction to the Theory of Toric Varieties.* Springer, Berlin, Heidelberg, New York 1988.

[4] Schönhardt, E.: *Über die Zerlegung von Dreieckspolyedern in Tetraeder.* Math. Ann. **98** (1928), 310–312.

Received March 5, 2000; revised version July 24, 2000