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Relatively Free *-Bands

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Abstract. A *-band is an algebra consisting of a band (idempotent semigroup) on which an involution * is defined satisfying an extra condition; in summary

$$(xy)^* = y^*x^*, \quad x^{**} = x, \quad x = xx^*x, \quad (xy)z = x(yz), \quad x^2 = x.$$

The lattice of all *-band varieties was determined by Adair who also provided a basis for the identities of each variety. Another system of bases was devised by Petrich. Defining certain operators on the free involutorial semigroup F on a nonempty set X, we construct a system of fully invariant congruences on F which is in bijection with the set of all proper *-band varieties, with the exception of normal *-band varieties which require a different treatment. The proof of this result is based on those evoked above and is broken into a long sequence of lemmas.

MSC 2000: 20M05, 20M07

1. Introduction and summary

Varieties of bands (idempotent semigroups) have attracted considerable attention and with good reason. Their lattice has been determined representing a prime example of mathematical ingenuity and success. The varieties themselves admit copious characterizations and their members convenient structure theorems. For example, free objects in these varieties have been described in several ways exhibiting a relatively transparent structure.

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As a variant of a general semigroup and inspired by ring theory, emerged a *-semigroup, which is an algebra consisting of a semigroup on which an involution is defined. As a natural special case, we have a *-band as a band and a *-semigroup with the involution satisfying an additional "regularity" condition. All together, we have a mapping $x \to x^*$ of a semigroup S into itself satisfying the following axioms:

$$(xy)^* = y^*x^*, \qquad x^{**} = x, \qquad x = xx^*x, \qquad x^2 = x. \tag{1}$$

The lattice of *-band varieties was determined by Adair [1] based on the approach of Fennemore [2] in describing the lattice of band varieties. Adair also provided a system of bases for identities for each *-band variety. As proved in [5], the system of identities devised in [4] for join irreducible band varieties can be used also for *-band varieties. As a consequence, we have that *-band varieties admit bases for their identities consisting of star-free identities. Even though *-band varieties (as well as varieties of band monoids) are treated as an afterthought of band varieties, the deliberations involved turn out quite nontrivial and intrinsically interesting.

Yamada [8] characterized finitely generated free *-bands using an inductive process. The purpose of this paper is to construct free objects in every *-band variety on any nonempty set X by solving the corresponding word problem on the free involutorial semigroup. Section 2 contains a minimum of needed notation and terminology. For varieties different from $\mathcal{B}$, the variety of all *-bands, and varieties of normal *-bands, our treatment consists of three stages. In the first stage, Section 3, we introduce operators h_n and i_n on the free involutorial semigroup F on X and establish some of their properties. In the second stage, Section 4, we use these operators to introduce operators γ_{t_n} with $t \in \{h, i\}$ and $n \geq 3$ and prove a number of their properties which culminate in the fact that they induce fully invariant *-band congruences on F. The third stage, Section 5, consists in identifying the *-band varieties which correspond to these congruences. In Section 6 the main theorem provides free objects for all nontrivial *-band varieties.

2. Notation and terminology

For these, we follow the standard texts in semigroups and [4] with the following supplements.

If Y is a set, $\mid Y \mid$ stands for the cardinality of Y. We fix a nonempty set X and consider a bijection $x \to x^*$ of X onto a disjoint copy X^* of X. Let $I = X \cup X^*$ and F be the free semigroup on I, which consists of all nonempty words over the alphabet I. We may view F as an involutory semigroup by defining

$$(x^*)^* = x, \qquad (y_1 \, \dots \, y_n)^* = y_n^* \, \dots \, y_1^*$$

for all $x \in X$, $n \geq 2$ and $y_1, \dots, y_n \in I$. We denote by F^1 the free monoid on I, obtained by adjoining the empty word 1 to F.

For emphasis, we refer to a homomorphism of *-semigroups as a *-homomorphism and the induced congruence as a *-congruence. In this terminology, the terms homomorphism and congruence refer to multiplication alone. We denote

$End(F)$ – the set of all *-endomorphisms of F.

For an identity $u = v$ on *-bands, we denote by $[u = v]$ the variety of *-bands determined by $u = v$. We omit the covering identities (1).

In the treatment of relatively free bands in [6], we have used a number of invariants and operators on the set of words over a nonempty set X, $c(w)$ for content of w, $\sharp(w)$ for cardinality of $c(w)$, $s(w)$, $\sigma(w)$, $\varepsilon(w)$, $e(w)$. In the context of *-bands when we are concerned with $I = X \cup X^*$ rather than with X alone, we must make suitable adjustments in the definitions of these functions relative to the occurrence of some starred letters. It is natural then to denote the new function by p^* if it represents a modification of the previously used symbol p relative to X. For typographical, as well as esthetical, reasons we shall use the same notation p as for X but in the context of *-bands. This should cause no confusion as to the meaning of the symbol p since it will be precisely defined and the original ones relative to X will not be used.

Now let $w \in F$; then

$c(w)$ – the set of all letters $x \in X$ such that either x or x^* occurs in w, $c(1) = \emptyset$ (in [3], the notation $c_X(w)$ is used),

$\sharp(w) = |\, c(w)\, |$,

u – a *prefix* of w if $w = uv$ for some $v \in F^1$,

$\overline{w}$ – the word obtained from w by reversing the order of letters, that is if $w = x_1x_2 \ldots x_n$, then $\overline{w} = x_n \ldots x_2x_1$, $\overline{1} = 1$,

$s(w)$ and $\sigma(w)$ – there is a unique factorization $w = uyv$ with $u, v \in F^1$, $y \in I$ and $c(u) \subset c(uy) = c(w)$; we write $s(w) = u$ and $\sigma(w) = y$ (in [3], the notation $s_X(w)$ and $\sigma_X(w)$ is used),

$e(w)$ – the left-right dual of $s(w)$,

$\varepsilon(w)$ – the left-right dual of $\sigma(w)$.

For any operator t on F, set $t(1) = 1$ thereby extending it to F^1, and define operators $\overline{t}$ and t^* on F by

$$\overline{t}(w) = \overline{t(\overline{w})}, \qquad t^*(w) = (t(w^*))^*.$$

For any $w \in F$, we have $\overline{\overline{w}} = w^{**} = w$ and thus $\overline{\overline{t}} = t^{**} = t$. We shall need the following simple result.

Lemma 2.1. *We have $\overline{s} = s^* = e$, $\overline{\sigma} = \sigma^* = \varepsilon$ and $\sigma(w) = \varepsilon(\overline{w})$, $\varepsilon(w) = \sigma(\overline{w})$ for all $x \in F$.*

Proof. Let $w \in F$. Then there exist $u, v \in F^1$ and $y \in I$ such that $w = uyv$ and $c(v) \subset c(yv) = c(w)$. Hence

$$\overline{s}(w) = \overline{s(\overline{w})} = \overline{s(\overline{v}\,\overline{y}\,\overline{u})} = \overline{\overline{v}} = v = e(w),$$

$$\overline{\sigma}(w) = \overline{\sigma(\overline{w})} = \overline{\sigma(\overline{v}\,\overline{y}\,\overline{u})} = \overline{\overline{y}} = y = \varepsilon(w)$$

and the proof for star is the same.

The last assertions of the lemma are obvious. □

Besides the results in Lemma 2.1, we shall see in the next section that $\overline{t} = t^*$ for the operators t which are of central importance in our deliberations. We thus could either use $\overline{t}$ or t^* throughout; our choice is $\overline{t}$ for typographical reasons.

3. Operators h_n and i_n

This section consists of lemmas which will be used in the next section. We start by introducing the needed notation. Let $w \in F$.

If $w = yz$ with $y \in I$ and $z \in F^1$, we write $h_2(w) = y$. The operator i_2 is defined on F inductively on $\sharp(w)$ by the formula

$$i_2(w) = i_2 s(w)\, \sigma(w).$$

Hence $i_2(w)$ is the word obtained from w retaining only the first occurrence of each letter regarding x and x^* as the same letter.

The next set of operators is also defined on F inductively: for $t \in \{h, i\}$ and $n > 2$, let

$$t_n(w) = t_n s(w)\, \sigma(w)\, \overline{t_{n-1}}(w). \tag{2}$$

It is important to note that this formula harbors two inductions: one is on n and the other is on $\sharp(w)$. For $\sharp s(w) = \sharp(w) - 1$ unless $w = 1$. The proofs will generally be by (the main) induction on n and occasionally, for the first and/or the inductive step, also by induction on $\sharp(w)$.

In several proofs by induction, the following notation will come in handy:

$$i_1(w) = 1 \quad (w \in F), \qquad\qquad \chi_n = h_{n+1} \quad (n \geq 1). \tag{3}$$

This device will make it possible to start the induction process at $n = 1$. The case $\chi_1 = h_2$ is generally easy to check while the instance i_1 usually holds trivially. Observe that the inductive formula (2) remains valid for operators χ_n and i_n for $n > 1$.

In this section $t \in \{\chi, i\}$.

Lemma 3.1. *For $n \geq 2$ and $w \in F$, we have*

$$\overline{t_n}(w) = t_{n-1}(w)\, \varepsilon(w)\, \overline{t_n}e(w).$$

Proof. Direct application of definitions yields

$$\begin{aligned}\overline{t_n}(w) &= \overline{t_n(\overline{w})} = \overline{t_n s(\overline{w})\, \sigma(\overline{w})\, \overline{t_{n-1}}(\overline{w})} = \overline{\overline{t_{n-1}}(\overline{w})}\, \sigma(\overline{w})\, \overline{t_n s(\overline{w})} \\ &= t_{n-1}(w)\, \overline{\sigma}(w)\, \overline{t_n}\overline{s(\overline{w})} = t_{n-1}(w)\, \varepsilon(w)\, \overline{t_n}e(w).\end{aligned}$$ □

Lemma 3.2. *For $n \geq 2$, we have $ct_n = c$, $st_n = t_n s$, $\sigma t_n = \sigma$.*

Proof. We use induction on n. Let $n \geq 2$ and assume that the lemma holds for t_m whenever $2 \leq m < n$. We use a secondary induction on $\sharp(w)$, where $w \in F^1$.

All equalities hold trivially for $w = 1$. Let $w \in F$ and assume the equalities hold for t_n and $z \in F^1$ whenever $\sharp(z) < \sharp(w)$. Recall formula (2). By induction hypothesis on $\sharp(w)$, we have

$$ct_n s(w) = cs(w) \tag{4}$$

and thus

$$c(t_n s(w)\,\sigma(w)) = c(s(w)\,\sigma(w)) = c(w). \tag{5}$$

If $n = 2$, then $c\overline{t_{n-1}}(w) \subseteq c(w)$ follows from the definition of t_1. Otherwise, we get

$$c\overline{t_{n-1}}(w) = c(\overline{t_{n-1}(\overline{w})}) = ct_{n-1}(\overline{w}) = c(\overline{w}) = c(w)$$

by induction on n. Either way, it follows that $ct_n(w) = c(w)$ and so $ct_n = c$. From (4) and (5) we deduce that $st_n(w) = t_n s(w)$ and $\sigma t_n(w) = \sigma(w)$. □

Lemma 3.3. *For $n \geq 2$ and $w = uyv$ where $u, v \in F^1$, $y \in I$ and $c(y) \cap c(u) = \emptyset$, we have $t_n(w) \in t_n(u)\,yF^1$.*

Proof. The argument is by induction on $\sharp(w)$. Assume that the lemma holds for all $q \in F$ such that $1 \leq \sharp(q) < \sharp(w)$. If $u = s(w)$, then $y = \sigma(w)$ and so

$$t_n(w) = t_n s(w)\,\sigma(w)\,\overline{t_{n-1}}(w) \in t_n(u)\,yF^1.$$

Assume that $u \neq s(w)$. Since $\sharp(u) < \sharp(w)$, we may write $s(w) = uyr$ for some $r \in F^1$. By our induction hypothesis, we have $t_n s(w) = t_n(u)\,yz$ for some $z \in F^1$ and thus

$$t_n(w) = t_n s(w)\,\sigma(w)\,\overline{t_{n-1}}(w) = t_n(u)\,yz\,\sigma(w)\,\overline{t_{n-1}}(w) \in t_n(u)\,yF^1.$$ □

Lemma 3.4. *For $n \geq 1$, we have $\overline{t_n} = t_n^*$.*

Proof. We use induction on n. Let $w \in F$.

Let $n = 1$. The case $t = i$ being trivial, we assume that $t = \chi$. If $w = zy$ with $y \in I$ and $z \in F^1$, then

$$\overline{\chi_1}(w) = \overline{\chi_1(\overline{w})} = \chi_1(y\overline{z}) = y = (y^*)^* = (\chi_1(y^* z^*))^* = (\chi_1(w^*))^* = \chi_1^*(w).$$

Let $n > 1$ and assume that the result holds for $n - 1$.

We use secondary induction on $\sharp(w)$. Assume that $\sharp(w) > 0$ and $\overline{t_n}(z) = t_n^*(z)$ whenever $\sharp(z) < \sharp(w)$. Hence

$$\begin{aligned}
\overline{t_n}(w) &= t_{n-1}(w)\,\varepsilon(w)\,\overline{t_n}e(w) && \text{by Lemma 3.1}\\
&= t_{n-1}(w)\,\varepsilon(w)\,t_n^* e(w) && \text{by induction on } \sharp(w)\\
&= t_{n-1}(w)\,\sigma^*(w)\,t_n^* s^*(w) && \text{by Lemma 2.1}\\
&= t_{n-1}(w)\,\sigma^*(w)\,t_n^*((s(w^*))^*) = (t_{n-1}^*(w^*))^*\,(\sigma(w^*))^*\,(t_n s(w^*))^* \\
&= (t_n s(w^*)\,\sigma(w^*)\,t_{n-1}^*(w^*))^* \\
&= (t_n s(w^*)\,\sigma(w^*)\,\overline{t_{n-1}}(w^*))^* && \text{by induction on } n\\
&= (t_n(w^*))^* = t_n^*(w).
\end{aligned}$$ □

Lemma 3.5. *For $n \geq 1$, we have $\chi_1 = \chi_1\chi_n = \chi_1\overline{\chi_{n+1}}$.*

Proof. Since

$$\overline{\chi_{n+1}}(w) = \chi_n(w)\,\varepsilon(w)\,\overline{\chi_{n+1}}e(w)$$

by Lemma 3.1, it is enough to show that $\chi_1\chi_n(w) = \chi_1(w)$ for all $n \geq 1$ and $w \in F$. This follows from Lemma 3.3 by taking $u = 1$. □

Lemma 3.6. *For $m \geq n \geq 1$ and $u, v, w \in F^1$, we have*

$$t_n(u\,t_m(v)\,w) = t_n(u\,\overline{t_{m+1}}(v)\,w) = t_n(uvw).$$

Proof. We use induction on n. Let $n = 1$. Since the case $t = i$ is trivial, we may assume that $t = \chi$. Without loss of generality, we may assume that $u = 1$ and $v \neq 1$. It follows from Lemma 3.5 that

$$\chi_1(u\,\chi_m(v)\,w) = \chi_1\chi_m(v) = \chi_1(v) = \chi_1(uvw)$$

and the equality $\chi_1(u\,\overline{\chi_{m+1}}(v)\,w) = \chi_1(uvw)$ is obtained similarly.

Now let $n > 1$ and assume that the lemma holds for $n-1$. First we show that

$$\overline{t_{n-1}}(u\,t_m(v)\,w) = \overline{t_{n-1}}(u\,\overline{t_{m+1}}(v)\,w) = \overline{t_{n-1}}(uvw) \tag{6}$$

for all $m \geq n$ and $u, v, w \in F^1$. Indeed,

$$\begin{aligned}\overline{t_{n-1}}(u\,t_m(v)\,w) &= \overline{t_{n-1}(\overline{u\,t_m(v)\,w})} = \overline{t_{n-1}(\overline{w}\,\overline{t_m(v)}\,\overline{u})} \\ &= \overline{t_{n-1}(\overline{w}\,\overline{t_m}(\overline{v})\,\overline{u})} = \overline{t_{n-1}(\overline{w}\,\overline{v}\,\overline{u})} && \text{by induction on } n \\ &= \overline{t_{n-1}(\overline{uvw})} = \overline{t_{n-1}}(uvw),\end{aligned}$$

and the equality $t_{n-1}(u\,\overline{t_{m+1}}(v)\,w) = t_{n-1}(uvw)$ is proved similarly.

Next we show that

$$t_n(u\,t_m(v)) = t_n(u\,\overline{t_{m+1}}(v)) = t_n(uv) \tag{7}$$

for all $m \geq n$ and $u, v \in F^1$ by a secondary induction on $d = |\, c(v) \setminus c(u)\,|$.

Let $d = 0$, that is, $c(v) \subseteq c(u)$. Then

$$\begin{aligned}t_n(u\,t_m(v)) &= t_n s(u\,t_m(v))\,\sigma(u\,t_m(v))\,\overline{t_{n-1}}(u\,t_m(v)) \\ &= t_n s(u)\,\sigma(u)\,\overline{t_{n-1}}(u\,t_m(v)) && \text{since } d = 0 \\ &= t_n s(u)\,\sigma(u)\,\overline{t_{n-1}}(uv) && \text{by (6)} \\ &= t_n s(uv)\,\sigma(uv)\,\overline{t_{n-1}}(uv) && \text{since } d = 0 \\ &= t_n(uv)\end{aligned}$$

and the equality $t_n(u\,\overline{t_{m+1}}(v)) = t_n(uv)$ is proved similarly.

Now let $d > 0$ and assume that (7) holds for all values smaller than d. We can write $v = zyr$ with $uz = s(uv)$ and $y = \sigma(uv)$. By Lemma 3.3, we have $t_m(v) \in t_m(z)\, yF^1$ and so Lemma 3.2 yields

$$s(u\,t_m(v)) = u\,t_m(z), \qquad \sigma(u\,t_m(v)) = y. \tag{8}$$

Thus

$$\begin{aligned} t_n(u\,t_m(v)) &= t_n s(u\,t_m(v))\,\sigma(u\,t_m(v))\,\overline{t_{n-1}}(u\,t_m(v)) \\ &= t_n(u\,t_m(z))\,y\,\overline{t_{n-1}}(u\,t_m(v)) && \text{by (8)} \\ &= t_n(uz)\,y\,\overline{t_{n-1}}(u\,t_m(v)) && \text{by induction on } d \\ &= t_n s(uv)\,\sigma(uv)\,\overline{t_{n-1}}(uv) && \text{by (6)} \\ &= t_n(uv). \end{aligned}$$

Since by Lemma 3.1 we have

$$\overline{t_{m+1}}(v) = t_m(v)\,\varepsilon(v)\,\overline{t_{m+1}}e(v),$$

Lemma 3.2 yields that

$$s(u\,\overline{t_{m+1}}(v)) = s(u\,t_m(v)), \qquad \sigma(u\,\overline{t_{m+1}}(v)) = \sigma(u\,t_m(v))$$

and so the equality $t_n(u\,\overline{t_{m+1}}(v)) = t_n(uv)$ follows easily from the above. Thus (7) holds.

We now show that

$$t_n(u\,t_m(v)\,w) = t_n(uvw) \tag{9}$$

for all $m \geq n$ and $u, v, w \in F^1$ by induction on $e = \mid c(w) \setminus c(uv) \mid$. Let $m \geq n$, $u, v, w \in F^1$ and $r = u\,t_m(v)\,w$.

Let $e = 0$, that is, $c(w) \subseteq c(uv)$. Then

$$\begin{aligned} t_n(r) &= t_n s(r)\,\sigma(r)\,\overline{t_{n-1}}(r) \\ &= t_n s(u\,t_m(v))\,\sigma(u\,t_m(v))\,\overline{t_{n-1}}(r) && \text{since } e = 0 \\ &= s t_n(u\,t_m(v))\,\sigma t_n(u\,t_m(v))\,\overline{t_{n-1}}(r) && \text{by Lemma 3.2} \\ &= s t_n(uv)\,\sigma t_n(uv)\,\overline{t_{n-1}}(r) && \text{by (7)} \\ &= t_n s(uv)\,\sigma(uv)\,\overline{t_{n-1}}(r) && \text{by Lemma 3.2} \\ &= t_n s(uvw)\,\sigma(uvw)\,\overline{t_{n-1}}(r) && \text{since } e = 0 \\ &= t_n s(uvw)\,\sigma(uvw)\,\overline{t_{n-1}}(uvw) && \text{by (6)} \\ &= t_n(uvw). \end{aligned}$$

The equality $t_n(u\,\overline{t_{m+1}}(v)\,w) = t_n(uvw)$ is proved similarly and so the lemma holds when $e = 0$.

Let $e > 0$ and assume that (9) holds for all values smaller than e. Let $m \geq n$. Since $c(w) \nsubseteq c(uv)$, we may write $w = zyq$ with $uvz = s(uvw)$ and $y = \sigma(uvw)$. In particular, we have $\mid c(z) \setminus c(uv) \mid < \mid c(w) \setminus c(uv) \mid$. By Lemma 3.2, we have $ct_m(v) = c(v)$ and hence

$$
\begin{aligned}
t_n(r) &= t_n s(r)\,\sigma(r)\,\overline{t_{n-1}}(r) = t_n(u\,t_m(v)\,z)\,y\,\overline{t_{n-1}}(r) \\
&= t_n(uvz)\,y\,\overline{t_{n-1}}(r) && \text{by induction on } e \\
&= t_n s(uvw)\,\sigma(uvw)\,\overline{t_{n-1}}(uvw) && \text{by (6)} \\
&= t_n(uvw).
\end{aligned}
$$

This proves relation (9). The equality $t_n(u\,\overline{t_{m+1}}(v)\,w) = t_n(uvw)$ is proved similarly. □

Lemma 3.7. *Let $\varphi \in End(F)$. Then $\overline{\varphi} \in End(F)$.*

Proof. For all $u, v \in F$, we have

$$
\overline{\varphi}(uv) = \overline{\varphi(\overline{uv})} = \overline{\varphi(\overline{v}\,\overline{u})} = \overline{\varphi(\overline{v})\,\varphi(\overline{u})} = \overline{\varphi(\overline{u})}\,\overline{\varphi(\overline{v})} = \overline{\varphi}(u)\,\overline{\varphi}(v).
$$

Since $\overline{w}^* = \overline{w^*}$ for every $w \in F$, we also obtain

$$
\overline{\varphi}(w^*) = \overline{\varphi(\overline{w^*})} = \overline{\varphi(\overline{w}^*)} = \overline{(\varphi(\overline{w}))^*} = \overline{\varphi(\overline{w})}^{\,*} = (\overline{\varphi}(w))^*
$$

and so $\overline{\varphi} \in End(F)$. □

Lemma 3.8. *Let $n \geq 1$, $u, v, w, z \in F^1$ and $\varphi \in End(F)$ be such that $t_n(u) = t_n(v)$. Then $t_n(w\,\varphi(u)\,z) = t_n(w\,\varphi(v)\,z)$.*

Proof. We use induction on n. Let $n = 1$. Since the case $t = i$ is trivial, we may assume that $t = \chi$. Without loss of generality, we may assume that $w = 1$ and $u, v \neq 1$. By hypothesis, we have $\chi_1(u) = \chi_1(v) = y$ for some $y \in I$. Thus

$$
\chi_1(w\,\varphi(u)\,z) = \chi_1\varphi(u) = \chi_1\varphi(y) = \chi_1\varphi(v) = \chi_1(w\,\varphi(v)\,z)
$$

and the lemma holds for $n = 1$.

Assume now that $n > 1$ and that the lemma holds for $n - 1$. Note that $t_n(u) = t_n(v)$ implies

$$
\begin{aligned}
t_{n-1}(\overline{u}) &= t_{n-1}\overline{t_n}(\overline{u}) && \text{by Lemma 3.6} \\
&= t_{n-1}\overline{t_n(u)} = t_{n-1}\overline{t_n(v)} = t_{n-1}\overline{t_n}(\overline{v}) = t_{n-1}(\overline{v}).
\end{aligned}
$$

For all $u, v, w, z \in F^1$, we get

$$
\begin{aligned}
\overline{t_{n-1}}(w\,\varphi(u)\,z) &= \overline{t_{n-1}(\overline{w\,\varphi(u)\,z})} = \overline{t_{n-1}(\overline{z}\,\overline{\varphi(u)}\,\overline{w})} \\
&= \overline{t_{n-1}(\overline{z}\,\overline{\varphi}(\overline{u})\,\overline{w})} = \overline{t_{n-1}(\overline{z}\,\overline{\varphi}(\overline{v})\,\overline{w})} \\
&\qquad \text{by the above, Lemma 3.7 and induction on } n \\
&= \overline{t_{n-1}(\overline{z}\,\overline{\varphi(v)}\,\overline{w})} = \overline{t_{n-1}(\overline{w\,\varphi(v)\,z})} = \overline{t_{n-1}}(w\,\varphi(v)\,z).
\end{aligned}
$$

It follows that
$$\overline{t_{n-1}}(w\,\varphi(u)\,z) = \overline{t_{n-1}}(w\,\varphi(v)\,z). \tag{10}$$

We use secondary induction on $\sharp(u)$. The case $\sharp(u) = 0$ being trivial, let $\sharp(u) > 0$ and assume that $t_n(w\,\varphi(u')\,z) = t_n(w\,\varphi(v')\,z)$ whenever $t_n(u') = t_n(v')$ and $\sharp(u') < \sharp(u)$.

Now we introduce tertiary induction on $d = |\, c(z) \setminus c(w\,\varphi(u))\,|$.

Let $d = 0$, that is, $c(z) \subseteq c(w\,\varphi(u))$. Write $r = w\,\varphi(u)\,z$. If $c\varphi(u) \subseteq c(w)$, then

$$\begin{aligned}
t_n(r) &= t_n s(r)\,\sigma(r)\,\overline{t_{n-1}}(r) & \\
&= t_n s(w)\,\sigma(w)\,\overline{t_{n-1}}(r) & \text{since } c\varphi(u) \subseteq c(w)\\
&= t_n s(w)\,\sigma(w)\,\overline{t_{n-1}}(w\,\varphi(v)\,z) & \text{by (10)}\\
&= t_n(w\,\varphi(v)\,z) & \text{by symmetry.}
\end{aligned}$$

Therefore we assume that $c\varphi(u) \not\subseteq c(w)$. Then we can write $u = u'yu''$ and $\varphi(y) = qxq'$ so that
$$s(w\,\varphi(u)\,z) = w\,\varphi(u')\,q, \qquad \sigma(w\,\varphi(u)\,z) = x. \tag{11}$$
Similarly, we have a factorization $v = v'y'v''$ such that

$$c(w\,\varphi(v')) \subset c(w\,\varphi(v'y')) = c(w\,\varphi(u)\,z).$$

It follows easily from Lemma 3.3 that $i_2 t_n(a) = i_2(a)$ for every $a \in F^1$. Since $t_n(u) = t_n(v)$, we get
$$i_2(u) = i_2 t_n(u) = i_2 t_n(v) = i_2(v).$$
This yields that $y' = y$. In particular, we have

$$s(w\,\varphi(v)\,z) = w\,\varphi(v')\,q, \qquad \sigma(w\,\varphi(v)\,z) = x.$$

By Lemma 3.3, we obtain

$$t_n(u) \in t_n(u')\,yF^1, \qquad t_n(v) \in t_n(v')\,yF^1.$$

Since $t_n(u) = t_n(v)$ and $y \notin c(u') \cup c(v')$, we must have
$$t_n(u') = t_n(v'). \tag{12}$$

Therefore

$$\begin{aligned}
t_n(r) &= t_n s(r)\,\sigma(r)\,\overline{t_{n-1}}(r) & \\
&= t_n(w\,\varphi(u')\,q)\,x\,\overline{t_{n-1}}(r) & \text{by (11)}\\
&= t_n(w\,\varphi(v')\,q)\,x\,\overline{t_{n-1}}(r) & \text{by (12) and induction on } \sharp(u)\\
&= t_n(w\,\varphi(v')\,q)\,x\,\overline{t_{n-1}}(w\,\varphi(v)\,z) & \text{by (10)}\\
&= t_n(w\,\varphi(v)\,z) & \text{by symmetry,}
\end{aligned}$$

proving the case $d = 0$.

Finally, assume that $d > 0$ and $t_n(w\,\varphi(u)\,z') = t_n(w\,\varphi(v)\,z')$ whenever $|\,c(z')\backslash c(w\,\varphi(u))\,| < d$. Then we can write $z = qyq'$ with $w\,\varphi(u)\,q = s(w\,\varphi(u)\,z)$ and $y = \sigma(w\,\varphi(u)\,z)$. Since $t_n(u) = t_n(v)$, Lemma 3.2 yields that $c\varphi(u) = c\varphi(v)$ and so also $w\,\varphi(v)\,q = s(w\,\varphi(v)\,z)$ and $y = \sigma(w\,\varphi(v)\,z)$. Thus

$$\begin{aligned} t_n(r) &= t_n s(r)\,\sigma(r)\,\overline{t_{n-1}}(r) = t_n(w\,\varphi(u)\,q)\,y\,\overline{t_{n-1}}(r) \\ &= t_n(w\,\varphi(v)\,q)\,y\,\overline{t_{n-1}}(r) && \text{by induction on } d \\ &= t_n s(w\,\varphi(v)\,z)\,\sigma(w\,\varphi(v)\,z)\,\overline{t_{n-1}}(w\,\varphi(v)\,z) && \text{by (10)} \\ &= t_n(w\,\varphi(v)\,z) \end{aligned}$$

completing the inductive step for all three inductions. □

4. Operators γ_{t_n}

The purpose of this section is to construct a family of fully invariant *-band congruences on F. We start by introducing the needed notation. For $t \in \{\chi, i\}$ and $n \geq 3$, we define an operator γ_{t_n} on F by

$$\gamma_{t_n}(w) = t_{n-1}s(w)\,\sigma(w)\,\varepsilon(w)\,\overline{t_{n-1}}e(w).$$

Also let

$$\Gamma = \{\gamma_{t_n} \mid t \in \{\chi, i\}, n \geq 3\}.$$

Note that the operator γ_{t_n} is similar to the operator θ_{pq} introduced in ([7], Section 3) with the important difference that γ_{t_n} is defined by means of t_{n-1} and not by t_n.

Lemma 4.1. *For $\gamma \in \Gamma$, we have $\gamma = \overline{\gamma} = \gamma^* = \gamma^2$.*

Proof. Let $\gamma = \gamma_{t_n}$ and $w \in F$. Then

$$\begin{aligned} \overline{\gamma}(w) &= \overline{\gamma(\overline{w})} = \overline{t_{n-1}s(\overline{w})\,\sigma(\overline{w})\,\varepsilon(\overline{w})\,\overline{t_{n-1}}e(\overline{w})} \\ &= \overline{\overline{t_{n-1}}e(\overline{w})}\;\overline{\varepsilon(\overline{w})}\;\overline{\sigma(\overline{w})}\;\overline{t_{n-1}s(\overline{w})} \\ &= t_{n-1}(\overline{e(\overline{w})})\,\overline{\varepsilon}(w)\,\overline{\sigma}(w)\,\overline{t_{n-1}}(\overline{s(\overline{w})}) \\ &= t_{n-1}\overline{e}(w)\,\overline{\varepsilon}(w)\,\overline{\sigma}(w)\,\overline{t_{n-1}}\,\overline{s}(w) \\ &= t_{n-1}s(w)\,\sigma(w)\,\varepsilon(w)\,\overline{t_{n-1}}e(w) && \text{by Lemma 2.1} \\ &= \gamma(w) \end{aligned}$$

so that $\overline{\gamma} = \gamma$; the argument for the star is the same. Next

$$\begin{aligned} \gamma^2(w) &= t_{n-1}s\gamma(w)\,\sigma\gamma(w)\,\varepsilon\gamma(w)\,\overline{t_{n-1}}e\gamma(w) \\ &= t_{n-1}t_{n-1}s(w)\,\sigma(w)\,\varepsilon(w)\,\overline{t_{n-1}t_{n-1}}e(w) \\ &= t_{n-1}s(w)\,\sigma(w)\,\varepsilon(w)\,\overline{t_{n-1}}e(w) && \text{by Lemma 3.6} \\ &= \gamma(w) \end{aligned}$$

and thus also $\gamma^2 = \gamma$. $\square$

Lemma 4.2. *For* $t \in \{\chi, i\}$, $1 \leq n < m \geq 3$ *and* $u, v, w \in F^1$, *we have*

$$t_n(u\,\gamma_{t_m}(v)\,w) = t_n(uvw).$$

Proof. We use induction on n.

Let $n = 1$. Without loss of generality, we may assume that $t = \chi$, $u = 1$ and $v \neq 1$. It follows that

$$\begin{aligned} \chi_1(u\,\gamma_{t_m}(v)\,w) &= \chi_1(h_{m-1}s(v)\,\sigma(v)) \\ &= \chi_1(s(v)\,\sigma(v)) && \text{by Lemma 3.6} \\ &= \chi_1(uvw). \end{aligned}$$

Now let $n > 1$ and assume that the lemma holds for all values smaller than n. We use secondary induction on $\sharp(uvw)$. The case $\sharp(uvw) = 0$ being trivial, let $\sharp(uvw) = k > 0$ and assume that $t_n(u'\,\gamma_{t_m}(v')\,w') = t_n(u'v'w')$ whenever $\sharp(u'v'w') < k$. We have

$$\begin{aligned} \overline{t_{n-1}}(u\,\gamma_{t_m}(v)\,w) &= \overline{t_{n-1}(\overline{u\,\gamma_{t_m}(v)\,w})} = \overline{t_{n-1}(\overline{w}\,\overline{\gamma_{t_m}(v)}\,\overline{u})} \\ &= \overline{t_{n-1}(\overline{w}\,\gamma_{t_m}(\overline{v})\,\overline{u})} && \text{by Lemma 4.1} \\ &= \overline{t_{n-1}(\overline{w}\,\overline{v}\,\overline{u})} && \text{by induction on } n \\ &= \overline{t_{n-1}(\overline{uvw})} = \overline{t_{n-1}}(uvw) \end{aligned}$$

and thus

$$\overline{t_{n-1}}(u\,\gamma_{t_m}(v)\,w) = \overline{t_{n-1}}(uvw). \tag{13}$$

Now we consider two cases.

<u>Case</u>: $\sharp(uv) < k$. Let $uvz = s(uvw)$, $y = \sigma(uvw)$ and $r = u\,\gamma_{t_m}(v)\,w$. Then

$$\begin{aligned} t_n(r) &= t_n s(r)\,\sigma(r)\,\overline{t_{n-1}}(r) \\ &= t_n(u\,\gamma_{t_m}(v)\,z)\,y\,\overline{t_{n-1}}(r) && \text{since } c\gamma_{t_m}(v) = c(v) \\ &= t_n(uvz)\,y\,\overline{t_{n-1}}(r) && \text{by induction on } \sharp(uvw) \\ &= t_n s(uvw)\,\sigma(uvw)\,\overline{t_{n-1}}(uvw) && \text{by (13)} \\ &= t_n(uvw). \end{aligned}$$

<u>Case</u>: $\sharp(uv) = k$. By Lemma 3.6, we have

$$t_n(u\,t_{m-1}s(v)\,\sigma(v)) = t_n(u\,s(v)\,\sigma(v)).$$

Write $p = u\,t_{m-1}s(v)\,\sigma(v)$ and $q = u\,s(v)\,\sigma(v)$. By considering the shortest prefix with the same content, we obtain

$$t_n s(p)\,\sigma(p) = t_n s(q)\,\sigma(q). \tag{14}$$

Writing $r = u\,\gamma_{t_m}(v)\,w$, we obtain

$$\begin{aligned} t_n(r) &= t_n s(r)\,\sigma(r)\,\overline{t_{n-1}}(r) \\ &= t_n s(p)\,\sigma(p)\,\overline{t_{n-1}}(r) && \text{since } \sharp(uv) = k \\ &= t_n s(q)\,\sigma(q)\,\overline{t_{n-1}}(r) && \text{by (14)} \\ &= t_n s(uvw)\,\sigma(uvw)\,\overline{t_{n-1}}(r) && \text{since } \sharp(uv) = k \\ &= t_n s(uvw)\,\sigma(uvw)\,\overline{t_{n-1}}(uvw) && \text{by (13)} \\ &= t_n(uvw). \end{aligned}$$

□

Lemma 4.3. *For $t \in \{\chi, i\}$, $3 \le n \le m$ and $u, v, w \in F^1$, we have*

$$\gamma_{t_n}(u\,\gamma_{t_m}(v)\,w) = \gamma_{t_n}(uvw).$$

Proof. By definition of γ_{t_n}, we only need to show that

$$\eta(u\,\gamma_{t_m}(v)\,w) = \eta(uvw)$$

for $\eta \in \{t_{n-1}s, \sigma, \varepsilon, \overline{t_{n-1}}e\}$. Writing $r = u\,\gamma_{t_m}(v)\,w$, by Lemma 4.1 we have

$$\overline{r} = \overline{w}\,\overline{\gamma_{t_m}(v)}\,\overline{u} = \overline{w}\,\gamma_{t_m}(\overline{v})\,\overline{u}. \tag{15}$$

We obtain

$$\begin{aligned} t_{n-1}s(r) &= st_{n-1}(r) && \text{by Lemma 3.2} \\ &= st_{n-1}(uvw) && \text{by Lemma 4.2} \\ &= t_{n-1}s(uvw) && \text{by Lemma 3.2,} \\ \sigma(r) &= \sigma t_{n-1}(r) && \text{by Lemma 3.2} \\ &= \sigma t_{n-1}(uvw) && \text{by Lemma 4.2} \\ &= \sigma(uvw) && \text{by Lemma 3.2,} \\ \varepsilon(r) &= \sigma(\overline{r}) = \sigma(\overline{w}\,\gamma_{t_m}(\overline{v})\,\overline{u}) && \text{by (15)} \\ &= \sigma(\overline{w}\,\overline{v}\,\overline{u}) && \text{by the above} \\ &= \sigma(\overline{uvw}) = \varepsilon(uvw), \\ \overline{t_{n-1}}e(r) &= \overline{t_{n-1}(\overline{e(r)})} = \overline{t_{n-1}s(\overline{r})} \\ &= \overline{t_{n-1}s(\overline{w}\,\gamma_{t_m}(\overline{v})\,\overline{u})} && \text{by (15)} \\ &= \overline{t_{n-1}s(\overline{w}\,\overline{v}\,\overline{u})} && \text{by the above} \\ &= \overline{t_{n-1}s(\overline{uvw})} = \overline{t_{n-1}(\overline{e(uvw)})} = \overline{t_{n-1}}e(uvw). \end{aligned}$$

□

Lemma 4.4. *For $\gamma \in \Gamma$ and $u, v \in F^1$, we have $\gamma(\gamma(u)\,\gamma(v)) = \gamma(uv)$.*

Proof. Apply Lemma 4.3 twice. □

Lemma 4.5. *For $\gamma \in \Gamma$ and $w \in F^1$, we have*

$$\gamma(w^2) = \gamma(w) = \gamma(ww^*w).$$

Proof. For $\gamma = \gamma_{t_n}$ and $w \in F$, we have

$$\begin{aligned}\gamma_{t_n}(w^2) &= t_{n-1}s(w^2)\,\sigma(w^2)\,\varepsilon(w^2)\,\overline{t_{n-1}}e(w^2)\\ &= t_{n-1}s(w)\,\sigma(w)\,\varepsilon(w)\,\overline{t_{n-1}}e(w) = \gamma_{t_n}(w).\end{aligned}$$

The equality $\gamma(w) = \gamma(ww^*w)$ is proved similarly. □

Lemma 4.6. *Let $\gamma \in \Gamma$, $u, v \in F^1$ and $\varphi \in End(F)$ be such that $\gamma(u) = \gamma(v)$. Then $\gamma\varphi(u) = \gamma\varphi(v)$.*

Proof. We need to show that $\eta\varphi(u) = \eta\varphi(v)$ for $\eta \in \{t_{n-1}s, \sigma, \varepsilon, \overline{t_{n-1}}e\}$. Since $t_{n-1}\gamma_{t_n} = t_{n-1}$ by Lemma 4.2, $\gamma_{t_n}(u) = \gamma_{t_n}(v)$ yields that $t_{n-1}(u) = t_{n-1}(v)$. Thus

$$\begin{aligned}t_{n-1}s\varphi(u) &= st_{n-1}\varphi(u) && \text{by Lemma 3.2}\\ &= st_{n-1}\varphi(v) && \text{by Lemma 3.8}\\ &= t_{n-1}s\varphi(v) && \text{by Lemma 3.2,}\\ \sigma\varphi(u) &= \sigma t_{n-1}\varphi(u) && \text{by Lemma 3.2}\\ &= \sigma t_{n-1}\varphi(v) && \text{by Lemma 3.8}\\ &= \sigma\varphi(v) && \text{by Lemma 3.2.}\end{aligned}$$

By Lemma 4.1, we also have

$$\gamma_{t_n}(\overline{u}) = \overline{\gamma_{t_n}(u)} = \overline{\gamma_{t_n}(v)} = \gamma_{t_n}(\overline{v}).$$

Since $\overline{\varphi} \in End(F)$ by Lemma 3.7, it follows that

$$\begin{aligned}\varepsilon\varphi(u) &= \sigma(\overline{\varphi(u)}) = \sigma\overline{\varphi}(\overline{u}) = \sigma\overline{\varphi}(\overline{v}) && \text{by the above}\\ &= \sigma(\overline{\varphi(v)}) = \varepsilon\varphi(v),\\ \overline{t_{n-1}}e\varphi(u) &= \overline{t_{n-1}(\overline{e\varphi(u)})} = \overline{t_{n-1}s(\overline{\varphi(u)})} = \overline{t_{n-1}s\overline{\varphi}(\overline{u})}\\ &= \overline{t_{n-1}s\overline{\varphi}(\overline{v})} && \text{by the above}\\ &= \overline{t_{n-1}s(\overline{\varphi(v)})} = \overline{t_{n-1}(\overline{e\varphi(v)})} = \overline{t_{n-1}}e\varphi(v).\end{aligned}$$

□

We now collect some important properties of the operator γ.

Lemma 4.7. *Let $\gamma \in \Gamma$ and denote by $\widehat{\gamma}$ the equivalence relation on F induced by γ, i.e. $\widehat{\gamma}$ is the kernel of γ. Then $\widehat{\gamma}$ is a fully invariant *-band congruence on F.*

Proof. For any $u, v, w, z \in F$, we have

$$u \,\widehat{\gamma}\, v,\ w \,\widehat{\gamma}\, z \ \Rightarrow uw \,\widehat{\gamma}\, vz \qquad \text{by Lemma 4.4}$$

$$u \,\widehat{\gamma}\, v \ \Rightarrow u^* \,\widehat{\gamma}\, v^* \qquad \text{by Lemma 4.1}$$

$$w \,\widehat{\gamma}\, w^2 \qquad \text{by Lemma 4.5}$$

$$w \,\widehat{\gamma}\, ww^*w \qquad \text{by Lemma 4.5}$$

$$u \,\widehat{\gamma}\, v \ \Rightarrow \varphi(u) \,\widehat{\gamma}\, \varphi(v) \qquad \text{by Lemma 4.6}$$

for any *-endomorphism φ of F. □

We define a multiplication on $\gamma(F)$ by $u \star v = \gamma(uv)$. In view of Lemma 4.1, the unary *-operation on F maps $\gamma(F)$ into itself. Hence we may keep the unary operation of F restricted to $\gamma(F)$. It now follows from Lemmas 4.4 and 4.1 that γ is a *-homomorphism of F onto $\gamma(F)$ with modified operations, and thus $\gamma(F) \cong F/\widehat{\gamma}$. By Lemma 4.7, $\widehat{\gamma}$ is a fully invariant *-band congruence and hence $\gamma(F)$ is a $\mathcal{V}$-free *-band on X for some *-band variety $\mathcal{V}$. Our task in the next section is to identify the variety $\mathcal{V}$ by studying the identities in our diagram of the lattice of *-band varieties.

5. Identification of varieties

We have concluded the preceding section with the collection $\{\widehat{\gamma} \mid \gamma \in \Gamma\}$ of fully invariant *-band congruences on F. Our purpose now is to identify the variety $\mathcal{V}$ which corresponds to a given $\widehat{\gamma}$ in the usual antiisomorphism of the lattice of *-band varieties and fully invariant *-band congruences on F by means of our diagram of the former lattice. In this way, we shall include most of the diagram.

The following system of words was introduced in [4]:

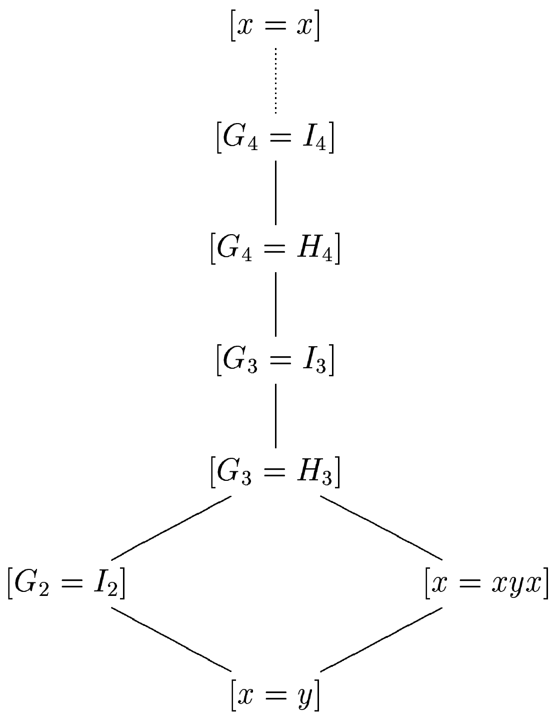
$$G_2 = x_2x_1, \qquad H_2 = x_2, \qquad I_2 = x_2x_1x_2$$

and for $n > 2$, defined inductively

$$G_n = x_n\overline{G_{n-1}}, \qquad T_n = G_nx_n\overline{T_{n-1}} \qquad\qquad (T \in \{H, I\}).$$

We assume that $t = h$ or $t = \chi$ if and only if $T = H$ and $t = i$ if and only if $T = I$.

The lattice of all *-band varieties was determined in [1]. That the bases for these varieties are as shown in the diagram was proved in [5].



The lattice of *-band varieties

Lemma 5.1. *For* $t \in \{h, i\}$ *and* $m \geq n \geq 2$, *we have*

$$t_n(G_m) = t_n(T_m), \qquad t_n(\overline{G_{m+1}}) = t_n(\overline{T_{m+1}}).$$

Proof. We use induction on n.

Let $n = 2$. The equality $t_2(G_m) = t_2(T_m)$ follows from the definitions. Since $\overline{G_{m+1}} = G_m x_{m+1}$ and $\overline{T_{m+1}} = T_m x_{m+1}\overline{G_{m+1}}$, we obtain the second equality from the first.

Now let $n > 2$ and assume that the lemma holds for all values smaller than n. Then we have

$$\begin{aligned} t_n s(T_m) &= t_n s(G_m x_m \overline{T_{m-1}}) = t_n s(G_m) && \text{since } c(G_m) = c(T_m), \\ \sigma(T_m) &= \sigma(G_m x_m \overline{T_{m-1}}) = \sigma(G_m) && \text{since } c(G_m) = c(T_m), \\ \overline{t_{n-1}}(T_m) &= \overline{t_{n-1}(\overline{T_m})} = \overline{t_{n-1}(\overline{G_m})} && \text{by induction on } n \\ &= \overline{t_{n-1}}(G_m) \end{aligned}$$

and so $t_n(G_m) = t_n(T_m)$. Also

$$\begin{aligned} t_n s(\overline{T_{m+1}}) &= t_n s(T_m x_{m+1} \overline{G_{m+1}}) = t_n(T_m) = t_n(G_m) && \text{by the above} \\ &= t_n s(G_m x_{m+1}) = t_n s(\overline{G_{m+1}}), \end{aligned}$$

$$\begin{aligned} \sigma(\overline{T_{m+1}}) &= \sigma(T_m x_{m+1} \overline{G_{m+1}}) = x_{k+1} = \sigma(G_m x_{m+1}) = \sigma(\overline{G_{m+1}}), \\ \overline{t_{n-1}}(\overline{T_{m+1}}) &= \overline{t_{n-1}(T_{m+1})} = \overline{t_{n-1}(G_{m+1})} && \text{by induction on } n \\ &= \overline{t_{n-1}}(\overline{G_{m+1}}), \end{aligned}$$

and hence $t_n(\overline{G_{m+1}}) = t_n(\overline{T_{m+1}})$. □

Lemma 5.2. *For $\{p,q\} = \{\chi, i\}$ and $n \geq 2$, we have $p_n(G_n) \neq p_n(Q_n)$.*

Proof. We use induction on n. Let $n = 2$. Since

$$\overline{\chi_1}(I_2) = x_2 \neq x_1 = \overline{\chi_1}(G_2),$$

it follows that $\chi_2(I_2) \neq \chi_2(G_2)$. Clearly, $i_2(H_2) = x_2 \neq x_2x_1 = i_2(G_2)$.

Now let $n > 2$ and assume that the lemma holds for $n-1$. We consider first the case $p = i$ and $n = 3$. If $i_3(H_3) = i_3(G_3)$, then

$$\begin{aligned} i_2(\overline{H_3}) &= i_2\overline{i_3}(\overline{H_3}) && \text{by Lemma 3.6} \\ &= i_2(\overline{i_3(H_3)}) = i_2(\overline{i_3(G_3)}) = i_2(\overline{G_3}) && \text{by symmetry.} \end{aligned}$$

However,

$$i_2(\overline{H_3}) = i_2(x_2x_3x_2x_1x_3) = x_2x_3x_1 \neq x_2x_1x_3 = i_2(x_2x_1x_3) = i_2(\overline{G_3})$$

and so $i_3(H_3) \neq i_3(G_3)$.

Assume now that $p = \chi$ or $n > 3$. If $p_n(Q_n) = p_n(G_n)$, then

$$\begin{aligned} p_{n-1}(Q_{n-1}) &= p_{n-1}s(Q_{n-1}x_n\overline{G_n}) = p_{n-1}s(\overline{Q_n}) \\ &= sp_{n-1}(\overline{Q_n}) && \text{by Lemma 3.2} \\ &= sp_{n-1}\overline{p_n}(\overline{Q_n}) && \text{by Lemma 3.6} \\ &= sp_{n-1}(\overline{p_n(Q_n)}) = sp_{n-1}(\overline{p_n(G_n)}) \\ &= p_{n-1}s(\overline{G_n}) && \text{by symmetry} \\ &= p_{n-1}s(G_{n-1}x_n) = p_{n-1}(G_{n-1}), \end{aligned}$$

contradicting the induction hypothesis. Therefore $p_n(Q_n) \neq p_n(G_n)$. □

Lemma 5.3. *Let $n \geq 3$.*
(i) $\gamma_{\chi_n}(G_{n+1}) = \gamma_{\chi_n}(H_{n+1})$, $\quad \gamma_{\chi_n}(\overline{G_n}) \neq \gamma_{\chi_n}(\overline{I_n})$.
(ii) $\gamma_{i_n}(G_n) = \gamma_{i_n}(I_n)$, $\quad \gamma_{i_n}(\overline{G_n}) \neq \gamma_{i_n}(\overline{H_n})$.

Proof. (i) We show that $\eta(H_{n+1}) = \eta(G_{n+1})$ for $\eta \in \{\chi_{n-1}s, \sigma, \varepsilon, \overline{\chi_{n-1}}e\}$. Indeed,

$$\begin{aligned} \chi_{n-1}s(H_{n+1}) &= \chi_{n-1}s(G_{n+1}x_{n+1}\overline{H_n}) = \chi_{n-1}s(G_{n+1}), \\ \sigma(H_{n+1}) &= \sigma(G_{n+1}x_{n+1}\overline{H_n}) = \sigma(G_{n+1}), \\ \varepsilon(H_{n+1}) &= \varepsilon(G_{n+1}x_{n+1}\overline{H_n}) = x_{n+1} = \varepsilon(x_{n+1}\overline{G_n}) = \varepsilon(G_{n+1}), \\ \overline{\chi_{n-1}}e(H_{n+1}) &= \overline{\chi_{n-1}}e(G_{n+1}x_{n+1}\overline{H_n}) = \overline{\chi_{n-1}}(\overline{H_n}) = \overline{\chi_{n-1}(H_n)} \\ &= \overline{\chi_{n-1}(G_n)} && \text{by Lemma 5.1} \\ &= \overline{\chi_{n-1}}(\overline{G_n}) = \overline{\chi_{n-1}}e(x_{n+1}\overline{G_n}) = \overline{\chi_{n-1}}e(G_{n+1}). \end{aligned}$$

Now suppose that $\gamma_{\chi_n}(\overline{G_n}) = \gamma_{\chi_n}(\overline{I_n})$ for some $n \geq 3$. Then we would have

$$\begin{aligned} \chi_{n-1}(I_{n-1}) &= \chi_{n-1}s(I_{n-1}x_n\overline{G_n}) = \chi_{n-1}s(\overline{I_n}) = s\gamma_{\chi_n}(\overline{I_n}) \\ &= s(\overline{\gamma_{\chi_n}(I_n)}) && \text{by Lemma 4.1} \\ &= s(\overline{\gamma_{\chi_n}(G_n)}) && \text{by hypothesis} \\ &= \chi_{n-1}s(\overline{G_n}) && \text{by symmetry} \\ &= \chi_{n-1}s(G_{n-1}x_n) = \chi_{n-1}(G_{n-1}), \end{aligned}$$

contradicting Lemma 5.2. Therefore $\gamma_{\chi_n}(\overline{G_n}) \neq \gamma_{\chi_n}(\overline{I_n})$ for every $n \geq 3$.

(ii) Similar. □

6. Main result

In order to collect all the information we need, we first introduce some notation. Let X be a nonempty set.

Let $\mathcal{RB} = [x = xyx]$. Define a mapping $\psi_{\mathcal{RB}}$ on F by

$$\psi_{\mathcal{RB}} : w \to h_2(w)\,\overline{h_2}(w).$$

On the set $\psi_{\mathcal{RB}}(F)$ define a multiplication by $xy \star wz = xz$ and a unary operation by $(xy)^* = y^*x^*$.

Let $\mathcal{S} = [G_2 = I_2]$. Assume that X is totally ordered and define a mapping $\psi_{\mathcal{S}}$ on F by

$$\psi_{\mathcal{S}} : w \to x_1x_2 \dots x_n$$

where $c(w) = \{x_1, x_2, \dots, x_n\}$ and $x_1 < x_2 < \dots < x_n$. On the set $\psi_{\mathcal{S}}(F)$ define a multiplication by

$$x_1x_2 \dots x_m \star y_1y_2 \dots y_n = z_1z_2 \dots z_k$$

where

$$\{x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n\} = \{z_1, z_2, \dots, z_k\},\ z_1 < z_2 < \dots < z_k \qquad (16)$$

and identity mapping as unary operation.

Let $\mathcal{NB} = [G_3 = H_3]$. Define a mapping $\psi_{\mathcal{NB}}$ on F by

$$\psi_{\mathcal{NB}} : w \to h_2(w)\,\psi_{\mathcal{S}}(w)\,\overline{h_2}(w).$$

On the set $\psi_{\mathcal{NB}}(F)$ define a multiplication by

$$ix_1x_2 \dots x_mj \star py_1y_2 \dots y_nq = iz_1z_2 \dots z_kq$$

in the notation (16) and a unary operation by

$$(ix_1x_2 \dots x_mj)^* = j^*x_1x_2 \dots x_mi^*.$$

For $\gamma \in \Gamma$, say $\gamma = \gamma_{t_n}$, $t \in \{h, i\}$, let $\psi_{[G_n = T_n]} = \gamma$. On the set $\gamma(F)$ define a multiplication by $u \star v = \gamma(uv)$ and consider the unary operation on F restricted to $\gamma(F)$.

Define an operator b on F inductively on $\sharp(w)$ by

$$b(w) = bs(w)\,\sigma(w)\,[bs(w^*)\,\sigma(w^*)]^* \tag{17}$$

(this is taken from [3], Section 4, where b, s and σ are denoted by b^*, s_X and σ_X, respectively). Let $\mathcal{B}$ be the variety of *-bands. Define a mapping $\psi_{\mathcal{B}}$ on F by

$$\psi_{\mathcal{B}} : w \to b(w).$$

On the set $b(F)$ define a multiplication by $u \star v = b(uv)$ and consider the unary operation on F restricted to $b(F)$.

We are finally ready for the desired result.

Theorem 6.1. *Let X be a nonempty set and $\mathcal{V}$ be a nontrivial *-band variety. Then $\psi_{\mathcal{V}}$ is a *-homomorphism of F onto $\psi_{\mathcal{V}}(F)$ which induces the least $\mathcal{V}$-congruence on F. Therefore $\psi_{\mathcal{V}}(F)$ is a $\mathcal{V}$-free *-band on X.*

Proof. The argument for the varieties $\mathcal{RB}$, $\mathcal{S}$ and $\mathcal{NB}$ is straightforward and is omitted.

Let $\gamma = \gamma_{h_n}$ with $n \geq 4$ and $\mathcal{V} = [G_n = H_n]$. The remarks at the end of the preceding section show that γ is a *-homomorphism of *-semigroups which induces the least $\mathcal{W}$-congruence on F for some *-band variety $\mathcal{W}$. By Lemma 5.3(i), we have $\gamma(G_n) = \gamma(H_n)$. By Lemma 4.6, it follows that $\gamma\varphi(G_n) = \gamma\varphi(H_n)$ for every $\varphi \in End(F)$. Thus $\gamma(F)$ satisfies the identity $G_n = H_n$ and so $\gamma(F) \in \mathcal{V}$. Therefore $\mathcal{W} \subseteq \mathcal{V}$.

Since $[G_{n-1} = I_{n-1}]$ is the variety lying immediately below $\mathcal{V}$ in the lattice of varieties of *-bands and $\gamma(G_{n-1}) \neq \gamma(I_{n-1})$ by Lemma 5.3(i), it follows that $\gamma(F) \notin [G_{n-1} = I_{n-1}]$ and so $\mathcal{W} \nsubseteq [G_{n-1} = I_{n-1}]$. Thus $\mathcal{W} = \mathcal{V}$.

The case $\gamma = \gamma_{i_n}$ is similar.

The case of $\mathcal{B}$ is essentially the content of ([3], Theorem 4.5). □

We now establish a few properties of the mappings $\psi_{\mathcal{V}}$. Let Ω be the set of nontrivial *-band varieties different from $\mathcal{S}$ and $\mathcal{NB}$. Note that Ω is a chain with least element $\mathcal{RB}$ and greatest element $\mathcal{B}$. The remaining elements are the varieties

$$\mathcal{H}_n = [G_n = H_n] \text{ for } n \geq 4, \qquad \mathcal{I}_n = [G_n = I_n] \text{ for } n \geq 3$$

with the order

$$\mathcal{H}_m \subseteq \mathcal{H}_n \Leftrightarrow \mathcal{I}_m \subseteq \mathcal{I}_n \Leftrightarrow m \leq n, \qquad \mathcal{I}_m \subset \mathcal{H}_n \Leftrightarrow m < n,$$

and $\psi_{\mathcal{H}_n} = \gamma_{h_n}$, $\psi_{\mathcal{I}_n} = \gamma_{i_n}$.

Our first result concerns properties of individual $\psi_{\mathcal{V}}$.

Proposition 6.2. *Let $\mathcal{V} \in \Omega$ and set $\psi = \psi_{\mathcal{V}}$. Then $\psi = \overline{\psi} = \psi^* = \psi^2$.*

Proof. The verification for $\mathcal{V} = \mathcal{RB}$ is straightforward and is omitted. The case of $\mathcal{V}$ different from $\mathcal{RB}$ and $\mathcal{B}$ was handled in Lemma 4.1 in terms of $\gamma \in \Gamma$. For $\mathcal{V} = \mathcal{B}$, we have to prove the corresponding statements for b. For $\overline{b} = b$, the proof of ([7], Lemma 6.1) remains valid for *-bands. An obvious variant of this proof will show that also $b^* = b$ holds. For $b^2 = b$, the proof of ([3], Lemma 2.2(iii)) remains valid in the case of *-bands as well. □

For $\mathcal{V} \in \{\mathcal{S}, \mathcal{NB}\}$, we have $\psi_{\mathcal{V}}^2 = \psi_{\mathcal{V}}$ but $\overline{\psi_{\mathcal{V}}} \neq \psi_{\mathcal{V}}$ and $\psi_{\mathcal{V}}^* \neq \psi_{\mathcal{V}}$. This is not surprising in view of the nonintrinsic nature of the definition of $\psi_{\mathcal{S}}$ and $\psi_{\mathcal{NB}}$, for it depends on the choice of the total order on X. When $\psi_{\mathcal{V}}^2 = \psi_{\mathcal{V}}$, we have that $\psi_{\mathcal{V}}(F)$ coincides with the set of fixed points of $\psi_{\mathcal{V}}$.

The form of the inductive formula (17) is quite different from that of any γ in Γ. We can modify that formula by observing that

$$\begin{aligned} b(w) &= bs(w)\,\sigma(w)\,[bs(w^*)\,\sigma(w^*)]^* \\ &= bs(w)\,\sigma(w)\,(\sigma(w^*))^*\,(bs(w^*))^* = bs(w)\,\sigma(w)\,\sigma^*(w)\,bs^*(w) \\ &= bs(w)\,\sigma(w)\,\varepsilon(w)\,be(w) \qquad\qquad \text{by Lemma 2.1} \end{aligned}$$

which is closer to the definition of γ in Γ. We can get even closer by observing that $\overline{b} = b$, for then

$$b(w) = bs(w)\,\sigma(w)\,\varepsilon(w)\,\overline{b}e(w).$$

Next we compare two *-band varieties $\mathcal{U}$ and $\mathcal{V}$ and the corresponding $\psi_{\mathcal{U}}$ and $\psi_{\mathcal{V}}$.

Proposition 6.3. *Let $\mathcal{U}, \mathcal{V}$ be *-band varieties. Then $\mathcal{U} \subseteq \mathcal{V}$ if and only if $\psi_{\mathcal{U}}\psi_{\mathcal{V}} = \psi_{\mathcal{U}}$.*

Proof. For a variety $\mathcal{V}$ of *-bands, let $\sim_{\mathcal{V}}$ denote the least $\mathcal{V}$-congruence on F. By Theorem 6.1, $\sim_{\mathcal{V}}$ is induced by $\psi_{\mathcal{V}}$. Let $\mathcal{U}, \mathcal{V}$ be *-band varieties.

Assume that $\mathcal{U} \subseteq \mathcal{V}$. Then $\sim_{\mathcal{V}} \subseteq \sim_{\mathcal{U}}$. Let $w \in F$. By Proposition 6.2, we have $\psi_{\mathcal{V}}\psi_{\mathcal{V}}(w) = \psi_{\mathcal{V}}(w)$ and so $\psi_{\mathcal{V}}(w) \sim_{\mathcal{V}} w$. Since $\sim_{\mathcal{V}} \subseteq \sim_{\mathcal{U}}$, it follows that $\psi_{\mathcal{V}}(w) \sim_{\mathcal{U}} w$ and so $\psi_{\mathcal{U}}\psi_{\mathcal{V}}(w) = \psi_{\mathcal{U}}(w)$. Hence $\psi_{\mathcal{U}}\psi_{\mathcal{V}} = \psi_{\mathcal{U}}$.

Conversely, assume that $\psi_{\mathcal{U}}\psi_{\mathcal{V}} = \psi_{\mathcal{U}}$. Let $w, z \in F$ be such that $w \sim_{\mathcal{V}} z$. Then $\psi_{\mathcal{V}}(w) = \psi_{\mathcal{V}}(z)$ and so

$$\psi_{\mathcal{U}}(w) = \psi_{\mathcal{U}}\psi_{\mathcal{V}}(w) = \psi_{\mathcal{U}}\psi_{\mathcal{V}}(z) = \psi_{\mathcal{U}}(z).$$

Hence $w \sim_{\mathcal{U}} z$ and so $\sim_{\mathcal{V}} \subseteq \sim_{\mathcal{U}}$. Thus $\mathcal{U} \subseteq \mathcal{V}$. □

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