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Varieties and Sums of Rings

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Abstract. For commutative algebras over a field, we describe all varieties closed for taking algebras which are sums of their two subalgebras in the variety.

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Properties of rings which are sums of their two subrings have been investigated by Bahturin, Beidar, Bokut, Ferrero, Fukshansky, Giambruno, Herstein, Kegel, Kepczyk, Mikhalev, Puczyłowski, Salwa, Small, and other authors (see [7] for references). In particular, several interesting results deal with PI-rings which are sums of two subrings (see [1], [2], [4]).

This note is devoted to varieties of algebras over a field F. We say that a (commutative) variety V is *closed for sums of two subalgebras* if V contains every (commutative) algebra which is a sum of two subalgebras from V.

We describe all commutative varieties closed for sums of two subalgebras. It turns out that these varieties are abundant. In contrast, it follows from the main theorem of [8] that only trivial varieties of non-commutative algebras are closed for sums of two subalgebras. A variety (of commutative algebras) is said to be *trivial* if it contains either all algebras (respectively, all commutative algebras), or only zero algebras.

A *product* of two varieties V, W is a class VW consisting of all F-algebras R with an ideal $I \in V$ such that $R/I \in W$. It is well-known that a product of varieties is a variety (see, for example, [6]). A variety is said to be *semisimple* if it is generated by a finite (possibly empty) set of finite fields. It is known that a variety is semisimple if and only if it consists of Jacobson semisimple rings (see [6]). Put $N_\ell = \mathrm{var}[x^\ell{=}0]$.

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Theorem 1. *Let F be a field of characteristic p, and let V be a nontrivial variety of commutative F-algebras. Then V is closed for sums of two subalgebras if and only if $p > 0$ and $V = N_{p^k}V_S$, where $k \geq 0$ and V_S is a semisimple variety.*

Proof. Further, by an algebra we mean a commutative F-algebra. For any algebra R, denote by $J(R)$ the Jacobson radical of R. Let V_N and V_S be the classes of all nil, respectively, semisimple algebras in V. It is known and routine to verify that V_N and V_S are varieties, too.

The 'only if' part. Suppose that V is closed for sums of two subalgebras. First we show that $V = V_NV_S$. The inclusion $V \subseteq V_NV_S$ is obvious. Indeed, every nontrivial commutative variety satisfies an identity $f(x) = 0$; whence $J(R) \in V_N$ for every $R \in V$, and $R/J(R) \in V_S$.

If $|F| = \infty$, then it is known and easy to show that V consists of nil algebras, and therefore $V = V_N = V_NV_S$, as required.

Next, we consider the case where $|F| < \infty$, and so $p > 0$.

Choose any R in V_NV_S. Let $I \in V_N$ be an ideal of R such that $R/I \in V_S$. Clearly, $I = J(R)$. In order to show that $R \in V$, it suffices to verify that every finitely generated subalgebra of R belongs to V. Hence we may assume that R is finitely generated itself. Every nontrivial variety of commutative algebras is locally finite, since it satisfies $f(x) = 0$ and $xy = yx$. Hence both I and R/I are locally finite. Therefore R is locally finite, and so R is finite dimensional. Since every finite field is perfect, the Wedderburn's classical theorem tells us that every finite dimensional algebra over F is a sum of its nil ideal I and a semisimple subalgebra $S \cong R/I$ (see [3, Theorem 72.19] or [10, §11.6]). Given that V is closed for sums of two subalgebras and $I, S \in V$, we get $R \in V$.

Thus $V = V_NV_S$. It remains to show that $p > 0$ and $V_N = N_{p^k}$ for some $k \geq 0$.

Denote by $F^*[x]$ the algebra of polynomials over F in x without constant terms. Obviously, $F^*[x]/(x^1) = 0$ belongs to V. If V contains all algebras $F^*[x]/(x^\ell)$, for all $\ell \geq 1$, then $F^*[x] \in V$, and so all algebras are in V. Therefore we may assume that there exists a maximum integer ℓ such that $F^*[x]/(x^\ell) \in V$.

If $\ell = 1$, then V does not contain $F^*[x]/(x^2)$; whence V is a semisimple variety (see [6, Theorem 5]), and $V = N_{p^0}V_S$, as required.

Further, assume that $\ell \geq 2$. Given a finitely generated algebra A satisfying $x^\ell = 0$, we claim that $A \in V$. Indeed, as we have already remarked $\dim_F(A) = n < \infty$. Since A is commutative, $A^{n\ell} = 0$. We now proceed by induction on n. The case $n = 1$ is obvious. In the inductive step $n > 1$, we first note that $A^2 \subset A$, because A is nilpotent. Choose $x_1, x_2, \ldots, x_m \in A$ such that the set $\{x_i + A^2 \mid i = 1, 2, \ldots, m\}$ is a basis of the vector space A/A^2. Set $B = \sum_{i=2}^{m} Fx_i + B^2$ and $D = \sum_{j=1}^{\ell} Fx_1^j$. Clearly, both B and D are subalgebras of A and $A = B + D$. Since $B \subset A$, $B \in V$ by the induction assumption. If $D = A$, then $A \in V$, because it is a homomorphic image of $F^*[x]/(x^\ell) \in V$. If $D \subset A$, then $D \in V$ by the induction assumption, and so $A \in V$ by the hypothesis of the theorem. Thus in both cases $A \in V$, and our claim is proved. Next, any variety is generated by its finitely generated subalgebras and so $N_\ell \subseteq V$. Clearly $N_\ell \subseteq V_N$. Given any algebra $A \in V_N$, it satisfies the identity $x^\ell = 0$ by the choice of ℓ, and so $A \in N_\ell$. We conclude that $V_N = N_\ell$.

Suppose that either $p = 0$ or ℓ is not a power of p. We set $q = 1$ if $p = 0$; otherwise $q = p^a$ where a is the largest integer such that $p^a | \ell$. Then $\binom{\ell}{q} \neq 0$ in F. Look at

$F^*[x,y]/(x^\ell, y^\ell)$. It is the sum of subalgebras $xF[x,y]/(x^\ell, y^\ell)$ and $yF[y]/(x^\ell, y^\ell)$, which are in $\mathrm{var}[x_1 \cdots x_\ell = 0]$. Therefore it belongs to V. However, if we expand $(x+y)^\ell$, then the summand $x^q y^{\ell-q}$ has the coefficient $\begin{pmatrix} \ell \\ q \end{pmatrix}$ that is nonzero. Hence $(x+y)^\ell \neq 0$ and obviously $(x+y)^{2\ell} = 0$. This contradicts the minimality of ℓ and shows that $p > 0$ and ℓ is a power of p. Thus $\ell = p^k$.

The 'if' part. Suppose that $V = N_{p^k} V_S$, where V_S is a semisimple variety.

First, we show that V_S is closed for sums of two subalgebras. Take any algebra R which is a sum of subalgebras $A, B \in V_S$. Consider a primitive homomorphic image R/I of R. We claim that it is a finite field which belongs to V_S.

Let m be the least common multiple of all n such that V_S contains the finite field $GF(p^n)$. Then $(A+I)/I, (B+I)/I \in V_S$ satisfy $x = x^{p^m}$. Since every ring of characteristic p satisfies the identity $(x+y)^p = x^p + y^p$, it follows that the identity $x = x^{p^m}$ is inherited by sums of subrings. Hence R/I satisfies $x = x^{p^m}$. Therefore R/I is a subfield of the finite field $GF(p^m)$.

The images $(A+I)/I$ and $(B+I)/I$ are subrings of R/I, and so they are finite fields. Suppose that $(A+I)/I$ and $(B+I)/I$ are not included one in another. The lattice of subfields of $GF(p^m)$ is isomorphic to the lattice of divisors of m ([9], Theorem 13.10). Therefore $(A+I)/I = GF(p^{ac})$ and $(B+I)/I = GF(p^{bc})$ for some positive integers a, b, c, where a and b are coprime, $a, b \geq 2$. Since $(A+I)/I$ and $(B+I)/I$ are subfields of $R/I = GF(p^k) \subseteq GF(p^m)$, we see that $k = abcd$ for a positive integer d.

Look at the dimensions over $GF(p^c)$. The field $(A+I)/I$ has dimension a, and $(B+I)/I$ has dimension b. Therefore $(A+I)/I + (B+I)/I$ has dimension $a+b-1$. However, R/I has dimension $\geq ab > a+b-1$. This contradiction shows that one of the fields $(A+I)/I$ and $(B+I)/I$ is contained in another, say $(A+I)/I \subseteq (B+I)/I$. Hence $R/I = (B+I)/I$ is a finite field in V_S.

Given that R is semisimple, it is a subdirect product of its primitive homomorphic images. Therefore R is a subdirect product of finite fields which are in V_S. Thus R is in V_S.

Second, we show that V is closed for sums of two subalgebras. Take any algebra $R = A+B$ with subalgebras $A, B \in V$. Given that $V = N_{p^k} V_S$, we see that $J(A), J(B) \in N_{p^k}$. Since $\mathrm{char} R = p$, it follows that all elements of the F-module $J(A) + J(B)$ satisfy $x^{p^k} = 0$. Therefore the subalgebra I of R generated by the F-module $J(A) + J(B)$ belongs to N_{p^k}. The quotient algebra R/I is a sum of two subalgebras $(A+I)/I \cong (A/J(A))/(I/J(A)) \in V_S$ and $(B+I)/I \in V_S$. As we have proved, V_S is closed for sums of two subalgebras. Therefore $R/I \in V_S$. Thus $R \in V$. This completes our proof. □

The variety $V = \mathrm{var}[x^2 = x^3]$ of algebras over $F = GF(2)$ contains both F and the two-element algebra $F^0 = \{0, a\}$ with zero multiplication. However, it does not contain the ideal extension $(F^0)^1 = F + F^0$, because $(1+a)^2 = 1$ and $(1+a)^3 = 1+a$. Thus not all commutative varieties are products of their nil and semisimple subvarieties, and not all commutative varieties are closed for sums of two subalgebras.

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