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# Construction of Set Theoretic Complete Intersections via Semigroup Gluing

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**Abstract.** In this note we present a very simple, but powerful, technique for finding monomial varieties which are set theoretic complete intersections. The technique is based on the concept of gluing semigroups that was defined by J. C. Rosales in [6] and used by K. Fischer, W. Morris and J. Shapiro in [2] to characterize complete intersection affine semigroup rings.
There are several techniques in the literature proving that certain varieties are set theoretic complete intersections but all of them preserve the dimension of the variety and are mainly results about curves, see [5, 7] and the references in there. The technique presented here does not preserve necessarily the dimension of the variety and it can combine the known results to produce set theoretic complete intersection varieties of any dimension, see Examples 4 and 5.

Let $T = \{\alpha_1, \ldots, \alpha_n\}$ be a set of nonzero elements in $N^m$. Let $S(T) = \{l_1\alpha_1 + \ldots + l_n\alpha_n : l_1, \ldots, l_n \in N\}$ denote the affine semigroup generated by $T$, $G(T) = \{k_1\alpha_1 + \ldots + k_n\alpha_n : k_1, \ldots, k_n \in Z\}$ denote the group generated by $T$ and $r$ denote the $rank_Z G(T)$. Let $\phi$ be the homomorphism of semigroup algebras from $K[X_1, \ldots, X_n]$ to the polynomial ring $K[t_1, \ldots, t_m]$ induced by $T$, that is: $\phi(X_i) = t^{\alpha_i}$. The kernel of $\phi$ is the ideal of $T$ and is denoted by $I(T)$, while the set of zeroes $V(I(T))$ in $K^n$ is called the *monomial variety* of $T$. $K[S(T)] = \phi(K[X_1, \ldots, X_n])$ is the affine semigroup ring of $S(T)$.

We will write $rad(I)$ for the radical of $I$. The arithmetical rank of $I(T)$, written $ara(I(T))$, is the smallest integer $s$ for which there exist elements $f_1, f_2, \cdots, f_s$ in $I(T)$, such that $rad(I(T)) = rad(f_1, f_2, \cdots, f_s)$. The ideal $I(S)$ is called *set theoretic complete intersection* if $ara(I(T)) = n - r$.

Let $T_1$ and $T_2$ be two nonempty subsets of $T$ such that $T = T_1 \cup T_2$ and $T_1 \cap T_2 = \emptyset$. $T$ is said to be a *gluing* of $T_1$ and $T_2$ if there exists a nonzero $a \in S(T_1) \cap S(T_2)$ such that $G(a) = G(T_1) \cap G(T_2)$. In terms of the ideals this is equivalent to $I(T) = I(T_1) + I(T_2) + < F_a >$, see [6], where $F_a$ is a binomial in the form $M_1 - M_2$, $M_1$ involves variables corresponding to $T_1$, $M_2$ involves variables corresponding to $T_2$, and $\phi(M_1) = t^a = \phi(M_2)$.

**Lemma 1.** *Suppose that $T$ is the gluing of $T_1$ and $T_2$. Then*

$$ara(I(T)) \leq ara(I(T_1)) + ara(I(T_2)) + 1.$$

*Proof.* Suppose that $ara(I(T_1)) = s_1$ and $ara(I(T_2)) = s_2$, therefore there exist $f_1, \ldots, f_{s_1}$, $g_1, \ldots, g_{s_2}$ such that $I(T_1) = rad(f_1, \ldots, f_{s_1})$ and $I(T_2) = rad(g_1, \ldots, g_{s_2})$. The result follows from the observation that $I(T) = rad(f_1, \ldots, f_{s_1}, g_1, \ldots, g_{s_2}, F_a)$, since $I(T) = I(T_1) + I(T_2) + < F_a >$.

**Theorem 2.** *If both $I(T_1), I(T_2)$ are set theoretic complete intersections then $I(T)$ is set theoretic complete intersection.*

*Proof.* Let $n_1$ be the number of elements in $T_1$ and $n_2$ the number of elements in $T_2$, we have $n = n_1 + n_2$. Let $r_1$ be the rank of $G(T_1)$ and $r_2$ the rank of $G(T_2)$. Since $G(a) = G(T_1) \cap G(T_2)$ and $rank_Z G(a) = 1$, we have $r + 1 = r_1 + r_2$. Then we have $n - r = (n_1 - r_1) + (n_2 - r_2) + 1$.

Now suppose that both $I(T_1), I(T_2)$ are set theoretic complete intersections. Therefore $ara(I(T_1)) = n_1 - r_1$ and $ara(I(T_2)) = n_2 - r_2$. Then, from the Lemma 1 and the previous remarks we have that $n - r \leq ara(I(T)) \leq ara(I(T_1)) + ara(I(T_2)) + 1 = (n_1 - r_1) + (n_2 - r_2) + 1 = n - r$, that means $I(T)$ is set theoretic complete intersection.

Based on this theorem one can develop a technique producing set theoretic complete intersection monomial varieties. One can make any combination of the known results to produce new examples and iterate this procedure many times. Note also that we get explicitly the defining equations of the new variety, provided that we know the defining equations of the initial two.

**Example 3.** Affine monomial curves are set theoretic complete intersections in positive characteristic, while in characteristic zero the general problem is still open, even for monomial curves in $A_K^4$.

In [4] D. Patil proves that any affine monomial curve $(t^{n_1}, t^{n_2}, \ldots, t^{n_s})$, for which $s - 1$ numbers among $n_1, \ldots, n_s$ form an arithmetic sequence, is set theoretic complete intersection in $A_K^s$, while in [3] W. Gastinger proves that an almost complete intersection monomial curve $(t^{m_1}, t^{m_2}, t^{m_3}, t^{m_4})$ is set theoretic complete intersection in $A_K^4$. Note that only for special values of the exponents $m_1, m_2, m_3, m_4$ the curve is an almost complete intersection. Choose any positive integers $m, n$ prime to each other such that $n \in S(n_1, \ldots, n_s)$ and $m \in S(m_1, \ldots, m_4)$, where $s - 1$ numbers among $n_1, \ldots, n_s$ form an arithmetic sequence and $(t^{m_1}, t^{m_2}, t^{m_3}, t^{m_4})$ is an almost complete intersection. From Theorem 2, the monomial curve $(t^{mn_1}, \ldots, t^{mn_s}, t^{nm_1}, \ldots, t^{nm_4})$ in $A_K^{s+4}$ is set theoretic complete intersection.

**Example 4.** In [1] S. Eliahou proves that the curve $(t^4, t^6, t^7, t^9)$ is set theoretic complete intersection in $A^4_C$. In [7] it was proved that the Eliahou's curve is the set theoretic complete intersection in $A^4_K$ of $X_2^3 - X_1X_3^2$, $X_3^5 - 3X_2^2X_3^2X_4 + 3X_1X_2X_3X_4^2 - X_1^2X_4^3$ and $X_1^9 - 5X_1^6X_2^2 + 10X_1^4X_2X_3^2 - 10X_1^2X_3^4 + 5X_2^3X_4^2 - X_4^4$. Also the curve $(u^{13}, u^5v^8, u^2v^{11}, v^{13})$ in $P^3_K$ is arithmetically Cohen-Macaulay, therefore it is the set theoretic complete intersection of $Y_2^3 - Y_1Y_3Y_4$ and $Y_3^{13} - 3Y_2^2Y_3^8Y_4^3 + 3Y_1Y_2Y_3^4Y_4^7 - Y_1^2Y_4^{11}$. From Theorem 2 we have that the semigroup $S((13,0),(5,8),(2,11),(0,13),(4,4),(6,6),(7,7),(9,9))$ is set theoretic complete intersection, since

$$(13,13) \in S((13,0),(5,8),(2,11),(0,13)) \cap S((4,4),(6,6),(7,7),(9,9)) \text{ and}$$
$$G((13,13)) = G((13,0),(5,8),(2,11),(0,13)) \cap G((4,4),(6,6),(7,7),(9,9)).$$

Therefore the monomial surface

$$V = (u^{13}, u^5v^8, u^2v^{11}, v^{13}, u^4v^4, u^6v^6, u^7v^7, u^9v^9)$$

in $A^8_K$ is set theoretic complete intersection and

$$\begin{aligned} I(V) = rad(&X_2^3 - X_1X_3^2, \\ &X_3^5 - 3X_2^2X_3^2X_4 + 3X_1X_2X_3X_4^2 - X_1^2X_4^3, \\ &X_1^9 - 5X_1^6X_2^2 + 10X_1^4X_2X_3^2 - 10X_1^2X_3^4 + 5X_2^3X_4^2 - X_4^4, \\ &X_1X_4 - Y_1Y_4, Y_2^3 - Y_1Y_3Y_4, \\ &Y_3^{13} - 3Y_2^2Y_3^8Y_4^3 + 3Y_1Y_2Y_3^4Y_4^7 - Y_1^2Y_4^{11}). \end{aligned}$$

**Example 5.** One can use the same result two or more times, for example in [7] it was proved that all monomial curves in $P^4_K$ (surfaces in $A^5_K$) of the form

$$(u^{k+7l}, u^{7l}v^k, u^{6l}v^{k+l}, u^{4l}v^{k+3l}, v^{k+7l})$$

are set theoretic complete intersections. From Theorem 2 we have that the monomial varieties of the form

$$(u^{k_1+7l_1}, u^{7l_1}v^{k_1}, u^{6l_1}v^{k_1+l_1}, u^{4l_1}v^{k_1+3l_1}, v^{k_1+7l_1}, v^{k_2+7l_2}, v^{7l_2}t^{k_2}, v^{6l_2}t^{k_2+l_2}, v^{4l_2}t^{k_2+3l_2}, t^{k_2+7l_2})$$

are set theoretic complete intersections.

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