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Hochschild Cohomology Rings of Algebras $k[X]/(f)$

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Abstract. Let k be a commutative ring with unity and let $f \in k[X]$ a monic polynomial. We determine the ring structure of the Hochschild cohomology for the k-algebra $k[X]/(f)$. This generalizes results of [4] on the Hochschild cohomology rings of modular group algebras of cyclic groups over fields.

1. Introduction

The multiplicative structure of the Hochschild cohomology of an algebra is not known in general and is difficult to calculate, even in examples. In [4] we determined the structure of the Hochschild cohomology for commutative modular group algebras over a field. With a different and more direct method this was recently generalized by Cibils and Solotar to commutative group algebras over arbitrary commutative rings k, showing that there exists an algebra isomorphism $HH^*(kG) \cong kG \otimes_k H^*(G,k)$ for any finite abelian group G, [2].

In this paper we deal with the Hochschild cohomology of algebras of the form $k[X]/(f)$ where k is a commutative ring and $f \in k[X]$ is a monic polynomial. This includes the case of group algebras of cyclic groups and the results of [4] are obtained as special cases. But apart from group algebras we also obtain results on the Hochschild cohomology rings for a much larger class of algebras.

Our approach relies on the construction of projective resolutions of algebras $k[X]/(f)$ as bimodules over itself given in [3]. The authors in [3] were mainly interested in cyclic homology and therefore studied only the additive structure of Hochschild cohomology. So the main topic of this article is to introduce the multiplicative structure in this context.

For the convenience of the reader we review in this section the basic definitions on Hochschild cohomology. Let k be a commutative ring with unity and let A be an associative k-algebra. In the sequel any commutative ring is assumed to have a unit element. By $A^{en} = A \otimes_k A^{op}$ we denote the enveloping algebra of A. Recall that every A-bimodule is a (left-)A^{en}-module, and vice versa. For all $i \geq 0$ set $A^{\otimes i} = A \otimes_k \ldots \otimes_k A$, the i-fold tensor product, and $C^i = C^i(A) = Hom_k(A^{\otimes i}, A)$. Define a differential operator $d^i : C^i \to C^{i+1}$ by setting

$$(d^i f)(a_1 \otimes \ldots \otimes a_{i+1}) = a_1 f(a_2 \otimes \ldots \otimes a_{i+1}) +$$
$$+ \sum_{j=1}^{i} (-1)^j f(a_1 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_{i+1}) + (-1)^{i+1} f(a_1 \otimes \ldots \otimes a_i) a_{i+1}$$

for all $a_1, \ldots, a_{i+1} \in A$ and $f \in C^i$. Then $(C^i, d^i)_{i \geq 0}$ is easily seen to be a complex of A^{en}-modules. The additive group $HH^i(A) = ker(d^i)/im(d^{i-1})$ is called the *i-th Hochschild cohomology group of A*.

There is an equivalent definition of these cohomology groups via certain *Ext*-functors. For all $i \geq -1$ set $\tilde{C}_i = \tilde{C}_i(A) = A^{\otimes i+2}$ with the obvious A^{en}-action and define a differential by setting

$$\tilde{d}_i(a_0 \otimes \ldots \otimes a_{i+1}) = \sum_{j=0}^{i} (-1)^j a_0 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_{i+1}$$

for all $a_0, \ldots, a_{i+1} \in A$. Then $(\tilde{C}_i, \tilde{d}_i)_{i \geq 0}$ is a projective resolution of A as an A^{en}-module ([1], 2.11). It is usually called the standard resolution. Applying the contravariant functor $Hom_{A^{en}}(-, A)$ to this resolution yields $Hom_{A^{en}}(\tilde{C}_*(A), A) \cong C^*(A)$. Thus by passing to cohomology one gets $HH^i(A) \cong Ext^i_{A^{en}}(A, A)$ for all $i \geq 0$.

In addition to the additive structure one can define a multiplication for the cochains. The *cup product* of two cochains $f \in C^i(A)$ and $g \in C^j(A)$ is by definition the cochain $f \cup g \in HH^{i+j}(A)$ given by

$$(f \cup g)(a_1 \otimes \ldots \otimes a_{i+j}) = f(a_1 \otimes \ldots \otimes a_i) \cdot g(a_{i+1} \otimes \ldots \otimes a_{i+j})$$

for all $a_1, \ldots, a_{i+j} \in A$. By a straightforward calculation one verifies $d^{i+j}(f \cup g) = d^i f \cup g + (-1)^i f \cup d^j g$ for all $f \in C^i(A),\ g \in C^j(A)$. Thus the cup product of two cocycles is again a cocycle, which implies that the cohomology class of $f \cup g$ only depends on the cohomology classes of f and g. So the cup product induces a well-defined product on Hochschild cohomology $\cup : HH^i(A) \times HH^j(A) \to HH^{i+j}(A)$. This product turns the graded k-vector space $HH^*(A) = \bigoplus_{i \geq 0} HH^i(A)$ into a graded k-algebra.

2. A periodic resolution

Let $f = X^n + f_{n-1}X^{n-1} + \ldots + f_1 X + f_0 \in k[X]$ be a monic polynomial. We consider the corresponding k-algebra $A_f = k[X]/(f)$ where (f) denotes the ideal of $k[X]$ generated by f. Without further notice we often set in the sequel $A = A_f$ for abbreviation.

In [3], the authors construct periodic resolutions for the algebras A_f as bimodules over itself and they give homotopy equivalences between these resolutions and the standard resolution defined in Section 1.

We first fix some notation. By $P, Q, P_i, \dots$ we denote polynomials in $k[X]$, but also their cosets in A if there are no ambiguities possible. The following maps will appear in the sequel:

$T : A \to A^{en}\ ,\ \ T(P) = 1 \otimes P - P \otimes 1\ ,$
$\epsilon_0 : A^{\otimes s+2} \to A^{\otimes s+3}\ ,\ \ \epsilon_0(a_0 \otimes \dots \otimes a_{s+1}) = 1 \otimes a_0 \otimes \dots \otimes a_{s+1}\quad (for\ s \geq 0)\ ,$
$\mu : A^{en} \to A\ ,\ \ \mu(P \otimes Q) = PQ\ .$

As f is monic, for any $P \in k[X]$ there exist polynomials $\bar{P}, \tilde{P} \in k[X]$ with $P = \bar{P}f + \tilde{P}$ and $deg\ \tilde{P} < deg\ f$.

Proposition 2.1. ([3], 1.3) *Let $A = A_f = k[X]/(f)$ where $f = \sum_{i=0}^{n} f_i X^i$ is a monic polynomial. Consider the sequence*

$$\bar{C}_* = \bar{C}_*(A) :\ \ \dots \xrightarrow{d_3} A^{\otimes 2} \xrightarrow{d_2} A^{\otimes 2} \xrightarrow{d_1} A^{\otimes 2} \xrightarrow{\mu} A \longrightarrow 0$$

where $d_{2r+1}(P \otimes Q) = (P \otimes Q)T(X)\ ,\ \ d_{2r}(P \otimes Q) = (P \otimes Q)(\sum_{i=1}^{n} f_i \sum_{j=0}^{i-1} X^j \otimes X^{i-j-1})$ for all $r \geq 0$. Then $\bar{C}_*(A)$ is a projective resolution of A as an A^{en}-module.

Thus the Hochschild cohomology of A_f is the cohomology of the complex $Hom_{A^{en}}(\bar{C}_*(A_f), A_f)$.

For $f \in k[X]$ denote by $f' \in k[X]$ the (formal) derivative of f; for $g \in k[X]$ let $Ann(g) = Ann_A(g) = \{a \in A \mid ga = 0\}$ be the annihilator of g.

Proposition 2.2. *Let $A = k[X]/(f)$ as above.*

1. *Using the identification $Hom_{A^{en}}(A^{\otimes 2}, A) \cong A$, the complex $Hom_{A^{en}}(\bar{C}_*(A), A)$ takes the form*
$$A \xrightarrow{0} A \xrightarrow{f'} A \xrightarrow{0} A \xrightarrow{f'} A \xrightarrow{0} A \xrightarrow{f'} A \longrightarrow \dots$$
where $A \xrightarrow{f'} A$ is the map $P \mapsto Pf'$ given by multiplication with the derivative f'.
2. *The Hochschild cohomology groups of A are*
$$HH^i(A) = \begin{cases} A & \text{if } i = 0 \\ Ann_A(f') & \text{if } i \text{ odd} \\ A/(f') & \text{if } i \text{ even} \end{cases}$$

Proof. We identify $Hom_{A^{en}}(A^{\otimes 2}, A) \cong A$ by $g \mapsto g(1 \otimes 1)$. Then for all $i \geq 0$ and $g \in Hom_{A^{en}}(A^{\otimes 2}, A)$ one has

$$d^*_{2i+1}(g)(1 \otimes 1) = g \circ d_{2i+1}(1 \otimes 1) = g(1 \otimes X - X \otimes 1) = g(1 \otimes 1) \cdot X - X \cdot g(1 \otimes 1) = 0$$

as g is an A^{en}-homomorphism and A is commutative. Similarly,

$$\begin{aligned} d^*_{2i}(g)(1 \otimes 1) &= g(\sum_{i>0} f_i \sum_{j=0}^{i-1} X^j \otimes X^{i-j-1}) \\ &= \sum_{i>0} f_i \sum_{j=0}^{i-1} g(X^j \otimes X^{i-j-1}) \\ &= g(1 \otimes 1) \sum_{i>0} f_i \sum_{j=0}^{i-1} X^{i-1} = g(1 \otimes 1) \cdot f' \end{aligned}$$

for all $i \geq 0$. This proves the first part; the second assertion is an easy consequence. □

We now turn our attention to the multiplicative structure of $HH^*(A_f)$. In order to be able to compute the cup product (which is given in terms of the standard resolution) we consider homotopy equivalences between the standard resolution and the resolution of Proposition 2.1. These were given in [3] in terms of Hochschild homology; we sketch the construction and transfer their results to cohomology.

Define A^{en}-homomorphisms $g_* : \tilde{C}_* \to \bar{C}_*$ inductively as follows

$g_0 = Id : A^{\otimes 2} \to A^{\otimes 2}$ the identity map;
$g_1 : A^{\otimes 3} \to A^{\otimes 2}$, $g_1(1 \otimes P \otimes 1) = -\frac{T(P)}{T(X)}$;
$g_2 : A^{\otimes 4} \to A^{\otimes 2}$, $g_2(1 \otimes P_1 \otimes P_2 \otimes 1) = -1 \otimes \overline{P_1 P_2}$;
$g_s : A^{\otimes s+2} \to A^{\otimes 2}$, $g_s(1 \otimes P_1 \otimes \ldots \otimes P_s \otimes 1) = g_{s-2}(1 \otimes P_1 \otimes \ldots \otimes P_{s-2} \otimes 1) \cdot g_2(1 \otimes P_{s-1} \otimes P_s \otimes 1)$ for $s > 2$.

The next result describes the induced homomorphisms $g_i^* = Hom_{A^{en}}(g_i, A)$. For homology this was shown in [3]; a proof of the cohomological version needed here can be found in [4].

Lemma 2.3. *Identifying $Hom_{A^{en}}(A^{\otimes 2}, A) \cong A$ and $Hom_{A^{en}}(A^{\otimes s+2}, A) \cong C^s(A)$ for $s > 0$ the following formulas hold for all $r \geq 0$ and all $P, P_1, \ldots, P_{2r+1} \in A$:*
$g_{2r}^*(P)(P_1 \otimes \ldots \otimes P_{2r}) = P(-1)^r \prod_{i=1}^r \overline{P_{2i-1} P_{2i}}$,
$g_{2r+1}^*(P)(P_1 \otimes \ldots \otimes P_{2r+1}) = P(-1)^{r+1} P_1' \prod_{i=1}^r \overline{P_{2i} P_{2i+1}}$.

Conversely, define A^{en}-homomorphisms $h_* : \bar{C}_* \to \tilde{C}_*$ by setting for all $r \geq 0$

$$h_{2r}(1 \otimes 1) = (-1)^r \sum_{i_1, \ldots, i_r = 1}^{n} (f_{i_1} \ldots f_{i_r}) \sum_{k_1, \ldots, k_r = 1}^{i_1 - 1, \ldots, i_r - 1} 1 \otimes X^{k_1} \otimes X \otimes \ldots \otimes X^{k_r} \otimes X \otimes X^c;$$

$$h_{2r+1}(1 \otimes 1) = (-1)^{r+1} \sum_{i_1, \ldots, i_r = 1}^{n} (f_{i_1} \ldots f_{i_r}) \sum_{k_- 1, \ldots, k_r = 1}^{i_1 - 1, \ldots, i_r - 1} 1 \otimes X \otimes X^{k_1} \otimes \ldots \otimes X^{k_r} \otimes X \otimes X^c,$$

where we have set $c = \sum_{j=1}^r i_j - \sum_{j=1}^r k_j - r$ for abbreviation. Again we are mainly interested in the induced maps $h_i^* = Hom_{A^{en}}(h_i, A)$.

Lemma 2.4. *Setting $c = \sum_{j=1}^r i_j - \sum_{j=1}^r k_j - r$ for abbreviation the following formulas hold for all $r \geq 0$:*

$$h_{2r}^*(\alpha)(1 \otimes 1) = (-1)^r \sum_{i_1, \ldots, i_r = 1}^{n} (f_{i_1} \ldots f_{i_r}) \sum_{k_1, \ldots, k_r = 1}^{i_1 - 1, \ldots, i_r - 1} X^c \cdot \alpha(1 \otimes X^{k_1} \otimes X \otimes \ldots \otimes X^{k_r} \otimes X \otimes 1)$$

for $\alpha \in Hom_{A^{en}}(A^{\otimes 2r+2}, A)$, and

$$h_{2r+1}^*(\alpha)(1 \otimes 1) = (-1)^{r+1} \sum_{i_1, \ldots, i_r = 1}^{n} (f_{i_1} \ldots f_{i_r}) \sum_{k_1, \ldots k_r = 1}^{i_1 - 1, \ldots, i_r - 1} X^c \cdot \alpha(1 \otimes X \otimes X^{k_1} \otimes \ldots \otimes X^{k_r} \otimes X \otimes 1)$$

for $\alpha \in Hom_{A^{en}}(A^{\otimes 2r+3}, A)$.

Proof. By the above formula for h_{2r} one has for all $r \geq 0$ and $\alpha \in Hom_{A^{en}}(A^{\otimes 2r+2}, A)$:

$$h^*_{2r}(\alpha)(1 \otimes 1) = \alpha((-1)^r \sum_{i_1,\ldots,i_r=1}^{n} (f_{i_1} \ldots f_{i_r}) \sum_{k_1,\ldots,k_r=1}^{i_1-1,\ldots,i_r-1} 1 \otimes X^{k_1} \otimes X \otimes \ldots \otimes X^{k_r} \otimes X \otimes X^c)$$

$$= (-1)^r \sum_{i_1-1,\ldots,i_r-1}^{n} (f_{i_1} \ldots f_{i_r}) \sum_{k_1,\ldots,k_r=1}^{i_1-1,\ldots,i_r-1} X^c \cdot \alpha(1 \otimes X^{k_1} \otimes X \otimes \ldots \otimes X^{k_r} \otimes X \otimes 1)$$

because α is an A^{en}-homomorphism and A is commutative.

The second assertion is shown similarly. □

3. The even cohomology ring

In this section we determine the structure of the subring $HH^{ev}(A) = \bigoplus_{i \geq 0} HH^{2i}(A)$ of elements of $HH^*(A_f)$ of even degree. Therefore we have to develop formulas for the cup product of even elements.

The A^{en}-homomorphisms g_* and h_* of complexes are homotopy equivalences [3], so they induce isomorphisms in cohomology. By Proposition 2.2 we know $HH^0(A) \cong A$ and $HH^{2i}(A) \cong A/(f')$ for all $i > 0$, i.e. the elements of even degree in $HH^*(A)$ are represented by polynomials in $k[X]$. The strategy for computing the cup product of two elements is as follows. Map the elements with g_* to the level of the standard resolution, compute the cup product according to the definition of Section 1 and then map the result back with h_*.

Lemma 3.1. *Let $A = A_f = k[X]/(f)$ as above. Then the cup product on the even Hochschild cohomology ring of A is induced by multiplication in A.*

Proof. Assume $Q_1 \in HH^{2r}(A)$, $Q_2 \in HH^{2s}(A)$ for some $r, s \geq 0$. For abbreviation, $Q_1 \cup Q_2$ denotes the element $g^*(Q_1) \cup g^*(Q_2) \in C^*(A)$. Then by definition of the cup product and by Lemma 2.3

$$\begin{aligned} (Q_1 \cup Q_2)(P_1 \otimes \ldots \otimes P_{2(r+s)}) &= g^*_{2r}(Q_1)(P_1 \otimes \ldots \otimes P_{2r}) \cdot g^*_{2s}(Q_2)(P_{2r+1} \otimes \ldots \otimes P_{2(r+s)}) \\ &= (-1)^r Q_1 \prod_{i=1}^{r} \overline{P_{2i-1}P_{2i}} \cdot (-1)^s Q_2 \prod_{i=r+1}^{r+s} \overline{P_{2i-1}P_{2i}} \\ &= (-1)^{r+s} Q_1 Q_2 \prod_{i=1}^{r+s} \overline{P_{2i-1}P_{2i}} \end{aligned}$$

for all $P_1, \ldots, P_{2(r+s)} \in A$. Setting $b = \sum_{j=1}^{r+s} i_j - \sum_{j=1}^{r+s} k_j - r - s$ it follows by Lemma 2.4

$h^*_{2(r+s)}(Q_1 \cup Q_2)(1 \otimes 1) =$

$$\begin{aligned} &= (-1)^{r+s} \sum_{i_1,\ldots,i_{r+s}=1}^{n} (f_{i_1} \ldots f_{i_{r+s}}) \sum_{k_1,\ldots,k_{r+s}=1}^{i_1-1,\ldots,i_{r+s}-1} X^b (Q_1 \cup Q_2)(X^{k_1} \otimes X \otimes \ldots \otimes X^{k_{r+s}} \otimes X) \\ &= (-1)^{r+s} \sum_{i_1,\ldots,i_{r+s}=1}^{n} (f_{i_1} \ldots f_{i_{r+s}}) \sum_{k_1,\ldots,k_{r+s}=1}^{i_1-1,\ldots,i_{r+s}-1} X^b (-1)^{r+s} Q_1 Q_2 \prod_{i=1}^{r+s} \overline{X^{k_i+1}} \\ &= Q_1 Q_2 \ , \end{aligned}$$

because $\overline{X^{k_i+1}} = \delta_{k_i,n-1}$. □

Now we are in the position to give a presentation for the even Hochschild cohomology ring of $k[X]/(f)$ by generators and relations.

Theorem 3.2. *Let k be a commutative ring and let $f \in k[X]$ be a monic polynomial with corresponding k-algebra $A_f = k[X]/(f)$. Then*

$$HH^{ev}(A_f) \cong k[x,z]/(f(x), f'(x)z)$$

where deg $x = 0$ and deg $z = 2$.

Proof. As $HH^0(A_f) \cong A_f$ and cup product in degree zero is multiplication in A_f, the coset $x = X + (f)$ generates $HH^0(A_f)$; the only relation is $f(x) = 0$.

Take $z \in HH^2(A_f) \cong A_f/(f')$ to be the coset of the constant polynomial $1 \in k[X]$. Then $HH^2(A_f)$ is generated by z and $HH^0(A_f)$, again because the cup product is induced by multiplication. As $HH^2(A_f) \cong A_f/(f')$ one gets the relation $f'(x)z = 0$.

In higher degrees $HH^{2i}(A_f)$ is generated by the i-fold cup product $z^i = z \cup \ldots \cup z$ and $HH^0(A_f)$. The corresponding relation $f'(x)z^i = 0$ is a consequence of the above relation in degree two. □

Putting some restrictions on k we already obtain the complete Hochschild cohomology ring of kC_n from Theorem 3.2. Note that the following result applies in particular to integral group algebras.

Corollary 3.3. *Let k be an integral domain and let C_n be a cyclic group of order n. If the characteristic of k does not divide n then*

$$HH^*(kC_n) \cong k[x,z]/(x^n - 1, nx^{n-1}z)$$

where deg $x = 0$ and deg $z = 2$.

Proof. Under the given assumptions the group cohomology of C_n with trivial coefficient module k vanishes in all odd degrees. This implies $HH^{2i+1}(kC_n) = 0$ for all $i \geq 0$ ([1], 2.11.2). Thus $HH^*(kC_n) = HH^{ev}(kC_n)$ and the assertion follows from Theorem 3.2. □

4. The cup product in odd degrees

We have already seen that for arbitrary f the cup product in $HH^{ev}(A_f)$ is given by multiplication in A_f. The situation in odd degrees is more difficult as we will see in this section. The aim is to prove the following result. Note that by Proposition 2.2 the elements of the Hochschild cohomology groups are represented by polynomials.

Lemma 4.1. *Let $f = \sum_{i=0}^n f_i X^i \in k[X]$ be monic and set $A = A_f = k[X]/(f)$.*
If i or j is even then the cup product $\cup : HH^i(A) \times HH^j(A) \to HH^{i+j}(A)$ is induced by multiplication in A.
If i and j are odd then the cup product is given by the formula

$$Q_1 \cup Q_2 = -Q_1 Q_2 \sum_{i_1=1}^{n} (\sum_{k_1=1}^{i_1-1} k_1) f_{i_1} X^{i_1-2}.$$

Proof. If i and j are even this was proved in Lemma 3.1. So let $Q_1 \in HH^{2r+1}(A)$, $HH^{2s}(A)$ for some $r, s \geq 0$. Then by Lemma 2.3

$$\begin{aligned}
&(Q_1 \cup Q_2)(P_1 \otimes \ldots \otimes P_{2(r+s)+1}) = \\
&\quad = g^*_{2r+1}(Q_1)(P_1 \otimes \ldots \otimes P_{2r+1}) \cdot g^*_{2s}(Q_2)(P_{2r+2} \otimes \ldots \otimes P_{2(r+s)+1}) \\
&\quad = (-1)^{r+1} Q_1 P'_1 \prod_{i=1}^{r} \overline{P_{2i}P_{2i+1}} \cdot (-1)^s Q_2 \prod_{i=r+1}^{r+s} \overline{P_{2i}P_{2i+1}} \\
&\quad = (-1)^{r+s+1} Q_1 Q_2 P'_1 \prod_{i=1}^{r+s} \overline{P_{2i}P_{2i+1}}.
\end{aligned}$$

Applying the induced map h^* and setting $b = \sum_{j=1}^{r+s} i_j - \sum_{j=1}^{r+s} k_j - r - s$ it follows by Lemma 2.4

$$\begin{aligned}
&h^*_{2(r+s)+1}(Q_1 \cup Q_2)(1 \otimes 1) = \\
&= (-1)^{r+s+1} \sum_{i_1,\ldots,i_{r+s}=1}^{n} (f_{i_1} \cdots f_{i_{r+s}}) \sum_{k_1,\ldots,k_{r+s}}^{i_1-1,\ldots,i_{r+s}-1} X^b (Q_1 \cup Q_2)(X \otimes X^{k_1} \otimes X \otimes \ldots \otimes X^{k_{r+s}} \otimes X) \\
&= (-1)^{r+s+1} \sum_{i_1,\ldots,i_{r+s}=1}^{n} (f_{i_1} \cdots f_{i_{r+s}}) \sum_{k_1,\ldots,k_{r+s}=1}^{i_1-1,\ldots,i_{r+s}-1} X^b \cdot (-1)^{r+s+1} Q_1 Q_2 X' \prod_{i=1}^{r+s} \overline{X^{k_i+1}} \\
&= Q_1 Q_2
\end{aligned}$$

as $\overline{X^{k_i+1}} = \delta_{k_i,n-1}$.

Assume now $Q_1 \in HH^{2r+1}(A)$, $Q_2 \in HH^{2s+1}(A)$ for some $r, s \geq 0$. Then by Lemma 2.3

$$\begin{aligned}
&(Q_1 \cup Q_2)(P_1 \otimes \ldots \otimes P_{2(r+s+1)}) = \\
&\quad = g^*_{2r+1}(Q_1)(P_1 \otimes \ldots \otimes P_{2r+1}) \cdot g^*_{2s+1}(Q_2)(P_{2r+2} \otimes \ldots \otimes P_{2(r+s+1)}) \\
&\quad = (-1)^{r+1} Q_1 P'_1 \prod_{i=1}^{r} \overline{P_{2i}P_{2i+1}} \cdot (-1)^{s+1} Q_2 P'_{2r+2} \prod_{i=r+1}^{r+s} \overline{P_{2i+1}P_{2i+2}}.
\end{aligned}$$

Setting $a = \sum_{j=1}^{r+s+1} i_j - \sum_{j=1}^{r+s+1} k_j - r - s - 1$ it follows by Lemma 2.4

$$\begin{aligned}
&h^*_{2(r+s+1)}(Q_1 \cup Q_2)(1 \otimes 1) = \\
&= (-1)^{r+s+1} \sum_{i_1,\ldots,i_{r+s+1}=1}^{n} (f_{i_1} \cdots f_{i_{r+s+1}}) \sum_{k_1,\ldots,k_{r+s+1}=1}^{i_1-1,\ldots,i_{r+s+1}-1} X^a (Q_1 \cup Q_2)(X^{k_1} \otimes X \otimes \ldots \otimes X^{k_{r+s+1}} \otimes X) \\
&= (-1)^{r+s+1} \sum_{i_1,\ldots,i_{r+s+1}=1}^{n} (f_{i_1} \cdots f_{i_{r+s+1}}) \sum_{k_1,\ldots,k_{r+s+1}=1}^{i_1-1,\ldots,i_{r+s+1}-1} X^a (-1)^{r+s+2} Q_1 Q_2 (X^{k_1})' X' \prod_{i=2}^{r+s+1} \overline{X^{k_i+1}} \\
&= -Q_1 Q_2 \sum_{i_1=1}^{n} f_{i_1} \sum_{k_1=1}^{i_1-1} X^{i_1-k_1-1} k_1 X^{k_1-1} \\
&= -Q_1 Q_2 \sum_{i_1=1}^{n} (\sum_{k_1=1}^{i_1-1}) f_{i_1} X^{i_1-2}
\end{aligned}$$

which completes the proof of the lemma. □

5. Polynomials with zero derivative

A consequence of Theorem 3.2 is that if $f' = 0$ then the even Hochschild cohomology ring of A_f is a polynomial ring over $HH^0(A_f)$. In this section we determine the structure of the whole cohomology ring of A_f for certain polynomials with derivative $f' = 0$ using the explicit formulas for the cup product given in Lemma 4.1. Note that non-zero polynomials with zero derivative can only occur in positive characteristic.

Lemma 5.1. *Let k be a commutative ring of characteristic $p > 0$. Assume that $f \in k[X]$ is a monic polynomial with $f' = 0$, i.e., f has the form $f = \sum_{j=0}^{m} f_{jp}X^{jp} \in k[X]$. For $A_f = k[X]/(f)$ we then have $HH^i(A_f) = A_f$ for all $i \geq 0$. The cup product $\cup : HH^i(A_f) \times HH^j(A_f) \to HH^{i+j}(A_f)$ is induced by multiplication in A_f if i or j is even; if i and j are odd then $\cup$ is given by*

$$Q_1 \cup Q_2 = \begin{cases} Q_1Q_2(\sum_{j\, odd} f_{2j}X^{2j-2}) & \text{if } p = 2 \\ 0 & \text{if } p \neq 2 \end{cases}$$

Proof. As $f' = 0$ we get $HH^i(A_f) = A_f$ for all $i \geq 0$ by Proposition 2.2. If i or j is even then $\cup : HH^i(A_f) \times HH^j(A_f) \to HH^{i+j}(A_f)$ is multiplication in A_f by Lemma 4.1. So it remains to consider the case where i and j are odd. If $Q_1 \in HH^i(A_f)$, $Q_2 \in HH^j(A_f)$ then

$$\begin{aligned} Q_1 \cup Q_2 &= -Q_1Q_2 \sum_{i_1=1}^{mp} (\sum_{k_1=1}^{i_1-1} k_1) f_{i_1} X^{i_1-2} \\ &= -Q_1Q_2 \sum_{j=1}^{m} (\sum_{k_1=1}^{jp-1} k_1) f_{jp} X^{jp-2} \end{aligned}$$

If $p \neq 2$ then $\sum_{k_1=1}^{jp-1} k_1 \equiv 0 \ (mod\ p)$ for all $j \geq 1$. Therefore $Q_1 \cup Q_2 = 0$ in this case.

We finally consider the case $p = 2$. Then

$$\sum_{k_1=1}^{2j-1} k_1 \equiv \begin{cases} 0 \ (mod\ 2) & \text{if } j \text{ even} \\ 1 \ (mod\ 2) & \text{if } j \text{ odd} \end{cases}$$

Consequently, $Q_1 \cup Q_2 = Q_1Q_2(\sum_{j\, odd} f_{2j}X^{2j-1})$. □

We are now able to describe the cohomology rings $HH^*(A_f)$.

Theorem 5.2. *Let k be a commutative ring with characteristic $p > 0$. Assume $f \in k[X]$ is monic and $f' = 0$, i.e., $f = \sum_{j=0}^{m} f_{jp}X^{jp} \in k[X]$. Then the Hochschild cohomology ring of $A_f = k[X]/(f)$ has the following structure:*

If $p \neq 2$ then $HH^*(A_f) \cong A_f[y, z]/(y^2)$ where $deg\ y = 1$ and $deg\ z = 2$.

If $p = 2$ then $HH^*(A_f) \cong k[x, y, z]/(f(x),\ y^2 - (\sum_{j\, odd} f_{2j}x^{2j-2})z)$ where $deg\ x = 0$, $deg\ y = 1$ and $deg\ z = 2$.

Proof. Take $y \in HH^1(A_f)$ to be the coset of the constant polynomial $1 \in k[X]$. In both cases the cup product $\cup : HH^1(A_f) \times HH^0(A_f) \to HH^1(A_f)$ is multiplication in A_f and therefore $HH^1(A_f) = \{y \cup Q \mid Q \in HH^0(A_f)\}$ is generated by y and $HH^0(A_f)$.

First we consider the case $p \neq 2$ or $p = 2$ and $\sum_{j\ odd} f_{2j}X^{2j-2} = 0$. Then $y \cup y = 0$ by Lemma 5.1. Therefore one needs another generator z in degree two. Take $z \in HH^2(A_f)$ as the coset of $1 \in k[X]$. For all $i \geq 0$ the i-fold cup product $z^i = z \cup \ldots \cup z$ corresponds to 1. Thus $HH^{2i}(A_f) = \{z^i \cup Q \mid Q \in HH^0(A_f)\}$ and $HH^{2i+1}(A_f) = \{z^i \cup P \mid P \in HH^1(A_f)\}$ are generated by y and z over $HH^0(A_f)$. Obviously the only relation is $y^2 = 0$.

Assume now $p = 2$ and $\sum_{j\ odd} f_{2j}X^{2j-2} \neq 0$. Then $y \cup y \in HH^2(A_f)$ is the coset of the polynomial $\sum_{j\ odd} f_{2j}X^{2j-2} \in k[X]$. We shall need a generator z in degree two corresponding to the constant polynomial $1 \in k[X]$. As in the previous case, $HH^*(A_f)$ is generated by y and z over $HH^0(A_f)$. Obviously, the only defining relation is $y^2 - (\sum_{j\ odd} f_{2j}x^{2j-2})z = 0$. □

6. The general case (in characteristic $\neq 2$)

Let $f = g_1^{a_1} \cdots g_r^{a_r}$ be the decomposition of f in irreducible factors in $k[X]$. By the Chinese Remainder Theorem $A_f \cong A_{g_1^{a_1}} \times \ldots \times A_{g_r^{a_r}}$. It is well known that Hochschild cohomology behaves well with respect to direct products, i.e.

$$HH^*(A_f) \cong HH^*(A_{g_1^{a_1}}) \times \ldots \times HH^*(A_{g_r^{a_r}}).$$

So the general case reduces to the study of $HH^*(A_f)$ where $f = g^a$ for an irreducible $g \in k[X]$. This will be done in this section. But for technical reasons we have to exclude the case of characteristic 2.

So assume *char* $k \neq 2$. Let $g \in k[X]$ be irreducible and monic and $f = g^a$ for some $a \in \mathbb{N}$. If $a = 1$ then $HH^*(A_f) \cong A_f$ concentrated in degree zero. So we may assume $a \geq 2$. One has $f' = ag^{a-1}g'$; if $f' = 0$ then we already know the structure of $HH^*(A_f)$ by Theorem 5.2. Thus we may assume $f' \neq 0$. In particular, $a \not\equiv 0 \ (mod\ p)$, $g' \neq 0$ and g is separable. Then $HH^{2i+1}(A_f) = Ann(f') = (g)/(g^a) \leq A_f$ for all $i \geq 0$ and as before $HH^{2i}(A_f) = A_f/(f')$ for $i > 0$.

After these introductory remarks we begin with building up a presentation for $HH^*(A_f)$. Let $x \in HH^0(A_f)$ be the coset of the polynomial $X \in k[X]$. Obviously, x generates $HH^0(A_f)$ as a k-algebra; the only relation is $f(x) = 0$. Take $t \in HH^1(A_f)$ corresponding to $g + (g^a) \in Ann(f')$. Then $HH^1(A_f) = (g)/(g^a)$ is generated by t and x; an obvious relation is $tg^{a-1}(x) = 0$. To deal with powers of t we need the following auxiliary result.

Lemma 6.1. *Let k be a commutative ring with characteristic $\neq 2$ in which 2 is invertible. Assume $g \in k[X]$ is irreducible, $f = g^a = \sum_{i=0}^n f_iX^i$ for some $a \in \mathbb{N}$ and set $S = \sum_{j=2}^n f_j \sum_{k=1}^{j-1} kX^{j-2} \in k[X]$. Then g^{a-2} divides S.*

Proof. As the characteristic of k is not 2 one gets

$$S = \sum_{j=2}^{n} f_j \sum_{k=1}^{j-1} kX^{j-2} = \sum_{j=2}^{n} f_j \frac{j(j-1)}{2} X^{j-2} = \frac{1}{2}f''.$$

Every root of g is a root of f with multiplicity a and thus a root of $S = \frac{1}{2}f''$ with multiplicity $a - 2$. □

The expression S appears in the formula for the cup product of elements of odd degree (cf. Lemma 4.1). The element $t^2 = t \cup t \in HH^2(A_f)$ is represented by $-g^2 S$. By the preceding lemma $f = g^a$ divides $-g^2 S$ and therefore $t^2 = 0$ in $HH^2(A_f) = A_f/(f')$. So one needs at least one additional generator to generate the cohomology ring. Take $z \in HH^2(A_f)$ to be the element represented by the constant polynomial $1 \in k[X]$. Then $HH^2(A_f)$ is generated by z and x because the cup product in even degrees is induced by multiplication. One gets the relation $zf'(x) = 0$. The set $\{x, t, z\}$ generates $HH^*(A_f)$ as a graded k-algebra. In fact, the i-fold cup product $z^i \in HH^{2i}(A_f)$ is represented by $1 \in k[X]$; therefore $HH^{2i}(A_f)$ is generated by z^i and x and $HH^{2i+1}(A_f)$ is generated by $z^i \cup t$ and x. It is not hard to see that all the relations occuring in higher degrees are consequences of the relations given above. Thus we have proved the following result.

Theorem 6.2. *Let k be a commutative ring with characteristic $\neq 2$ in which 2 is invertible. Let $g \in k[X]$ be irreducible, separable and monic and let $f = g^a$ for some $a \geq 2$. If $f' \neq 0$ then the Hochschild cohomology ring of $A_f = k[X]/(f)$ has the following structure*

$$HH^*(A_f) \cong k[x,t,z]/(f(x),\ tg^{a-1}(x),\ t^2,\ zf'(x))$$

where deg $x = 0$, deg $t = 1$ and deg $z = 2$. □

Remark 6.3. 1. If one restricts the above result to the even subring, i.e. set $t = 0$, then $HH^*(A_f) \cong k[x,z]/(f(x),\ zf'(x))$ as in Theorem 3.2.

2. Note that the case $a = 1$ which is not covered by Theorem 6.2 was already dealt with in Section 2. If $a = 1$ then $HH^*(A_f) \cong k[X]/(f)$ concentrated in degree zero.

It remains open to deal with the case of characteristic 2. The problem seems to be that one does not have a result like Lemma 6.1 at hand. This makes it difficult to handle the powers of the generator t. Although it may be possible to compute specific examples we were not able to find a general presentation for $HH^*(A_f)$.

7. Truncated polynomial algebras

In this final section we apply our results to an important class of examples. For $n \geq 2$ let $A_n = k[X]/(X^n)$ be the truncated polynomial algebra. Note that this class contains modular group algebras of cyclic p-groups, whose Hochschild cohomology rings were determined in [4].

The structure of the even Hochschild cohomology rings of A_n are given by Theorem 3.2:

$$HH^{ev}(A_n) \cong k[x,z]/(x^n,\ nx^{n-1}z)$$

where $deg\, x = 0$ and $deg\, z = 2$. The structure of the odd part depends on the characteristic of the ground ring k.

Case 1: $p := char\, k \mid n$. The structure of $HH^*(A_n)$ is given by Theorem 5.2:

$$HH^*(A_n) \cong \begin{cases} k[x,y,z]/(x^n,\ y^2) & \text{if } p \neq 2 \text{ or } p = 2 \text{ and } t \equiv 0\,(mod\ 4) \\ k[x,y,z]/(x^n,\ y^2 - x^{n-2}z) & \text{if } p = 2 \text{ and } t \equiv 2\,(mod\ 4) \end{cases}$$

where $deg\, x = 0$, $deg\, y = 1$ and $deg\, z = 2$.

Case 2: $p := char\, k \nmid n$. In this case we have the following result:

$$HH^*(A_n) \cong k[x,t,z]/(x^n,\ nx^{n-1}z,\ tx^{n-1},\ t^2).$$

For $p \neq 2$ this follows from Theorem 6.2. But the presentation is also true in characteristic 2; in fact, for the special polynomial X^n the formula in 4.1 for multiplication of odd degree elements becomes in characteristic 2

$$Q_1 \cup Q_2 = \begin{cases} 0 & \text{if } n \equiv 0,1\,(mod\ 4) \\ Q_1Q_2X^{n-2} & \text{if } n \equiv 2,3\,(mod\ 4) \end{cases}$$

It follows that the square of the generator t chosen in Section 6 is 0. The other relations are proved verbatim as for $p \neq 2$. We summarize the results on the structure of the Hochschild cohomology rings of truncated polynomial algebras.

Theorem 7.1. *Let k be a commutative ring with $char\, k = p \geq 0$. For $n \geq 2$ consider the truncated polynomial algebra $A_n = k[X]/(X^n)$. Then the Hochschild cohomology ring of A_n has the following structure*

$$HH^*(A_n) \cong \begin{cases} k[x,y,z]/(x^n, y^2) & \text{if } p \mid n, \text{ and} \\ & \text{if } p \neq 2 \text{ or } p = 2 \text{ and } t \equiv 0\,(mod\ 4) \\ k[x,y,z]/(x^n, y^2 - x^{n-2}z) & \text{if } p \mid n, \text{ and } p = 2 \text{ and } t \equiv 2\,(mod\ 4) \\ k[x,y,z]/(x^n, nx^{n-1}z, yx^{n-1}, y^2) & \text{if } p \nmid n \end{cases}$$

where $deg\, x = 0$, $deg\, y = 1$ and $deg\, z = 2$.

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