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Generators of Order Two for S_n and its Two Double Covers

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1. Introduction

This paper considers the minimum number of involutions, $i(G)$, required to generate both of the double covers G of the symmetric group. In particular, explicit generators, of order two, for each of the groups are also introduced. These generators may, for example, be useful for implementation in Magma-Cayley or GAP.

As a starting point we observe that if $i(G) = 1$ then G is cyclic and if $i(G) = 2$ then G is dihedral. Hence for any group of order greater than two that is not isomorphic to a dihedral group we have $i(G) \geq 3$.

It is also well-known that the symmetric group S_n, $n \geq 3$, may be generated by the two cycles $(1, 2)$ and $(1, 2, 3, \dots, n)$. But as we may write $(1, 2, 3 \dots, n)$ as the product of, multiplying from the left,

$$(1, n-1)(2, n-2)\dots(r, n-r) \quad \text{and} \quad (1, n)(2, n-1)\dots(t, n+1-t)$$

where r is the integer part of $(n-1)/2$ and t the integer part of $n/2$, it is clear that S_n, $n \geq 4$, may be generated by three involutions. Moreover, for $n \geq 4$, $i(S_n) = 3$.

2. $i(\tilde{S}_n)$

This double cover of S_n, which lifts a transposition of S_n to an element of order 4, will be denoted by $\tilde{S}_n$. So that $\tilde{S}_n$ is the group with generators $z, r_1, r_2, \dots, r_{n-1}$ and relations

$$z^2 = 1,$$

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$$\begin{aligned} zr_i &= r_i z, \quad r_i^2 = z \quad \text{for} \quad 1 \le i \le n-1, \\ (r_j r_{j+1})^3 &= z \quad \text{for} \quad 1 \le j \le n-2, \\ r_k r_h &= z r_h r_k \quad \text{for} \quad \mid h-k \mid > 1 \quad \text{and} \quad 1 \le h, k \le n-1. \end{aligned}$$

For computations in $\tilde{S}_n$ we will use a method first described by Conway and others at Cambridge (Atlas [2]). This method is outlined in a paper by David B. Wales [5]. In this the elements of $\tilde{S}_n$ are products of the form $\pm[\sigma_i]$, where the σ_i are disjoint cycles in S_n and $\pm[\sigma_i]$ are the corresponding lifts in $\tilde{S}_n$.

Definition 2.1. *For distinct elements $a_1, \ldots, a_m$ we define $[a_1 a_2 \ldots a_m] = a_1 a_2 \ldots a_m a_1$. We call $\pm[a_1 a_2 \ldots a_k]$ signed cycles in $\tilde{S}_n$. Each is a lift of the cycle $(a_1 a_2 \ldots a_k)$ in S_n.*

In fact each a_i corresponds to an element of a subgroup of a Clifford algebra which is isomorphic to $\tilde{S}_n$. But the following rules are sufficient to enable the calculation of products of disjoint signed cycles in $\tilde{S}_n$ (these appear as 2.3 and 2.4 in [5]).

$$\begin{aligned} [a_i] &= -1, \\ [a_1 a_2 \ldots a_m] &= (-1)^{m+1}[a_2 a_3 \ldots a_m a_1], \\ [a_1 a_2 \ldots a_{m-1}] a_m &= (-1)^m a_m [a_1 a_2 \ldots a_{m-1}]. \end{aligned}$$

In particular, these are used in [1] to prove the following proposition.

Proposition 2.2. *Any product of k disjoint signed transpositions in $\tilde{S}_n$ has order two if the integer part of $(k+1)/2$ is a multiple of two, and order four otherwise.*

Hence in $\tilde{S}_n$ an involution is of the form $\pm\pi$ where π is a signed cycle consisting of k disjoint transpositions and k is congruent to 0 or 3 modulo 4. Also as the only other element of order two in $\tilde{S}_n$ is -1, we have immediately that $\tilde{S}_4$ and $\tilde{S}_5$ may not be generated by involutions.

Also, as we will be making repeated use of this fact, it is convenient at this point to note that factoring out $\tilde{S}_n$ by the subgroup $Z = \langle 1, -1 \rangle$ recovers S_n. That is $\tilde{S}_n / Z \cong S_n$.

Proposition 2.3. *For $n \ge 3$, $\tilde{S}_n$ may be generated by $a = \pm[1,2]$ and $b = \pm[1,2,\ldots,n]$.*

Proof. As S_n is generated by $(1,2)$ and $(1,2,3,\ldots,n)$ it follows that a and b will generate at least one, up to parity of sign, of every type of signed cycle. Hence we only need show that we may also generate -1. But $a^2 = -1$. □

Proposition 2.4. $i(\tilde{S}_6) = 5$ *and* $i(\tilde{S}_7) = 3$.

Proof. That $i(\tilde{S}_6) = 5$ is proved in Section 3. However, in order to give specific generators, we note that $(1,2)(3,4)(5,6)$, $(1,3)(2,4)(5,6)$, $(1,4)(2,3)(5,6)$, $(1,5)(2,6)(3,4)$ and $(1,6)(2,3)(4,5)$ generate S_6. Thus, as in Proposition 2.3, the corresponding signed cycles will generate $\tilde{S}_6$.

For $\tilde{S}_7$ we only need note that

$$(1,2)(3,4)(5,6), \quad (1,4)(3,5)(2,7) \quad \text{and} \quad (2,3)(3,7)(1,6)$$

generate S_7. Hence the corresponding signed cycles generate $\tilde{S}_7$. □

Proposition 2.5. *For* $8 \le n \le 12$*, the following involutions generate* $\tilde{S}_n$*. When* $n = 8$

$$a = [1,2][3,8][4,7][5,6]\,,\ b = [1,3][4,8][5,7] \quad and \quad c = [3,8][4,7][5,6].$$

For $9 \le n \le 12$ $\quad a = [1,2][3,9][4,8][5,7],\ b = [1,3][4,9][5,8][6,7]$

$$and \quad c = \begin{cases} [3,9][4,8][5,7] & when \quad n = 9, \\ [3,10][4,8][5,7] & when \quad n = 10, \\ [3,10][4,11][5,7] & when \quad n = 11, \\ [3,10][4,11][5,12] & when \quad n = 12. \end{cases}$$

Proof. For $n = 8$ and 9 we have $ac = \pm[1,2]$ and $ab = \pm[1,2,\ldots,n]$ from which the result follows.

When $n = 10,\ 11$ or 12, $(ac)^3 = \pm[1,2]$ and $ab = \pm[1,2,\ldots,9]$ so for each n we may generate the subgroup $\tilde{S}_9$, in particular the signed cycles $d = [1,3]$, $e = [1,4]$ and $f = [1,5]$.

It only remains to show that $[1,2,\ldots,n]$ can also be generated for each n. But when $n = 10$ we have $abcdc = \pm[1,2,\ldots,10]$, when $n = 11$, $abcdec = \pm[1,2,\ldots,11]$ and when $n = 12$, $abcdefc = \pm[1,2,\ldots,12]$. □

Proposition 2.6. $\quad i(\tilde{S}_{13}) = 4.$

Proof. Note that in $\tilde{S}_{13}$ the only involutions are -1 and signed cycles of type 2^3 and 2^4. Also we require at least 12 signed transpositions in our generators to ensure that all the numbers from 1 to 13 have some link. However we cannot use three signed cycles of type 2^4, the minimum needed, as they are all even and thus cannot generate any odd signed cycles. Thus $i(\tilde{S}_{13}) > 3$. That $i(\tilde{S}_{13}) = 4$ follows by noting that $(1,12)(2,11)(3,10)(4,9)$, $(5,8)(6,7)(1,13)(2,12)$, $(3,11)(4,10)(5,9)(6,8)$ and $(1,2)(4,11)(5,6)$ will generate S_{13}. Thus the corresponding signed cycles generate $\tilde{S}_{13}$. □

Proposition 2.7. *For* $14 \le n \le 16$*, the following involutions generate* $\tilde{S}_n$*.*

$$\begin{array}{lrl}
n = 14 & a = & [1,2][3,14][4,13][5,12][6,11][7,10][8,9], \\
 & b = & [1,3][4,14][5,13][2,9], \\
 & c = & [6,12][7,11][8,10][2,9]. \\
n = 15 & a = & [2,15][3,14][4,13][5,12][6,11][7,10][8,9], \\
 & b = & [1,2][3,15][4,14][5,13][6,12][7,11][8,10], \\
 & c = & [3,14][4,13][6,11][7,10]. \\
n = 16 & a = & [2,16][3,15][4,14][5,13][6,12][7,11][8,10], \\
 & b = & [1,2][3,16][4,15][5,14][6,13][7,12][8,11][9,10], \\
 & c = & [3,16][4,15][5,14][6,13][7,12][8,11][9,10].
\end{array}$$

Proof. For $n = 16$, $bc = \pm[1,2]$ and $ab = \pm[1,2,\ldots,16]$ from which the result follows. Now for $n = 14$, $abc = \pm[1,2,\ldots,14]$ and for $n = 15$, $ab = \pm[1,2,\ldots,15]$. Thus in both cases we need only show that we may also generate $\pm[1,2]$. But when $n = 15$, $(ab)^4((abc)^2bab)^{13}(ab)^{-4} = \pm[2,9]$ and $(b. \pm [2,9])^3 = \pm[1,2]$. While for $n = 14$, $d = ((ba)^2(ca)^2bc)^{15} = \pm[1,5][7,13]$ and $f = (abcedb)^9 = \pm[5,6]$, from which we obtain $(abc)^4 f (abc)^{10} = \pm[1,2]$. □

It is convenient at this point to introduce some notation.

Definition 2.8. *We will denote the following product of $\delta+1$ signed transpositions*

$$[\alpha,\beta][\alpha+1,\beta-1][\alpha+2,\beta-2]\ldots[\alpha+\delta,\beta-\delta] \quad \textit{by} \quad T(\alpha,\alpha+\delta,\alpha+\beta).$$

As an example of this notation, in the previous proposition, we could express the generators for $\tilde{S}_{16}$ as

$$a=T(2,8,18), \quad b=[1,2]T(3,9,19) \quad \text{and} \quad c=T(3,9,19).$$

Proposition 2.9. *For $n \geq 17$, the following involutions, which are dependent on the value of n modulo 8, generate $\tilde{S}_n$.*

$$\begin{array}{lll}
n\equiv 1 & a= & [1,2]T(3,(n+1)/2,n+3),\\
 & b= & [1,3]T(4,(n+3)/2,n+4),\\
 & c= & T(3,(n+1)/2,n+3).\\
n\equiv 2 & a= & [1,2]T(3,n/2,n+2),\\
 & b= & [1,3]T(4,(n+2)/2,n+3),\\
 & c= & [3,n]T(4,n/2,n+2).\\
n\equiv 3 & a= & [1,2]T(3,(n-1)/2,n+1),\\
 & b= & [1,3]T(4,(n+1)/2,n+2),\\
 & c= & [3,n][4,n-1]T(5,(n-1)/2,n+1).\\
n\equiv 4 & a= & [1,2]T(3,(n-2)/2,n),\\
 & b= & [1,3]T(4,n/2,n+1),\\
 & c= & T(3,5,n+3)T(6,(n-2)/2,n).\\
n\equiv 5 & a= & [1,2]T(3,(n-3)/2,n-1),\\
 & b= & [1,3]T(4,(n-1)/2,n),\\
 & c= & T(3,6,n+3)T(7,(n-3)/2,n-1).\\
n\equiv 6 & a= & [1,2]T(3,(n-4)/2,n-2),\\
 & b= & [1,3]T(4,(n-2)/2,n-1),\\
 & c= & T(3,7,n+3)T(8,(n-4)/2,n-2).\\
n\equiv 7 & a= & [1,2]T(3,(n-5)/2,n-3),\\
 & b= & [1,3]T(4,(n-3)/2,n-2),\\
 & c= & T(3,8,n+3)T(9,(n-5)/2,n-3).\\
n\equiv 0 & a= & [1,2]T(3,(n-6/2,n-4),\\
 & b= & [1,3]T(4,(n-4)/2,n-3),\\
 & c= & T(3,9,n+3)T(10,(n-6)/2,n-4).
\end{array}$$

Note that when $n=24$ we have $(n-6)/2<10$. So we define $T(10,(n-6)/2,n-4)$ to be 1.

Proof. For $n \not\equiv 1$ we have $(ac)^3=\pm[1,2]$ and when $n\equiv 1$, $ac=\pm[1,2]$. Hence we only need show that $\pm[1,2,\ldots,n]$ is also generated in each case. When $n\equiv 1$, we have directly that

$ab = \pm[1, 2, \dots, n]$. For the remaining values of n we note that when

$$\begin{array}{lll} n \equiv 2 & ab = \pm[1,2,\dots,n-1] & \text{so } \tilde{S}_{n-1} \text{ is generated,} \\ n \equiv 3 & ab = \pm[1,2,\dots,n-2] & \text{so } \tilde{S}_{n-2} \text{ is generated,} \\ n \equiv 4 & ab = \pm[1,2,\dots,n-3] & \text{so } \tilde{S}_{n-3} \text{ is generated,} \\ n \equiv 5 & ab = \pm[1,2,\dots,n-4] & \text{so } \tilde{S}_{n-4} \text{ is generated,} \\ n \equiv 6 & ab = \pm[1,2,\dots,n-5] & \text{so } \tilde{S}_{n-5} \text{ is generated,} \\ n \equiv 7 & ab = \pm[1,2,\dots,n-6] & \text{so } \tilde{S}_{n-6} \text{ is generated.} \end{array}$$

In particular, as $n \geq 17$, we may generate for each n the signed cycles $d = [1,9]$, $e = [1,8]$, $f = [1,7]$, $g = [1,6]$, $h = [1,5]$, $j = [1,4]$, $k = [1,3]$. Thus we may obtain $\pm[1,2,\dots,n]$ from $abckc$ when $n \equiv 2$, $abckjc$ when $n \equiv 3$, $abchjkc$ when $n \equiv 4$, $abcghjkc$ when $n \equiv 5$, $abcfghjkc$ when $n \equiv 6$, $abcefghjkc$ when $n \equiv 7$ and $abcdefghjkc$ when $n \equiv 0$. □

The previous propositions give directly the following theorem.

Theorem 2.10. *For $n \geq 7$ and $n \neq 13$ $\quad i(\tilde{S}_n) = 3$.*

3. $i(\hat{S}_n)$

We denote by $\hat{S}_n$ the double cover of S_n that lifts a transposition of S_n to an element of order 2. Hence $\hat{S}_n$ is the group with generators $z, r_1, r_2, \dots, r_{n-1}$ and relations

$$\begin{aligned} z^2 &= 1, \\ zr_i &= r_i z, \quad r_i^2 = 1 \quad \text{for} \quad 1 \leq i \leq n-1, \\ (r_j r_{j+1})^3 &= 1 \quad \text{for} \quad 1 \leq j \leq n-2, \\ r_k r_h &= z r_h r_k \quad \text{for} \quad \mid h-k \mid > 1 \quad \text{and} \quad 1 \leq h,k \leq n-1. \end{aligned}$$

For computations in $\hat{S}_n$ we multiply each of the generators of the Clifford Algebra $C(\Omega)$, where $\Omega = \{1, 2, \dots, n\} \bigcup \{\delta\}$ (see [5]), by the complex number i to obtain an algebra over $\mathbf{C}$ generated by $f_1, f_2, \dots f_n, f_\delta$ where $f_j^2 = 1$ and $f_j f_k = -f_k f_j$ for $j \neq k$. The subgroup of this complex algebra generated by

$$(f_1 - f_2)/\sqrt{2},\ (f_2 - f_3)/\sqrt{2}, \dots, (f_n - f_\delta)/\sqrt{2}$$

is isomorphic to $\hat{S}_n$.

By identifying D_{a_j} with $(f_j - f_\delta)/\sqrt{2}$, where a_j are distinct in $\Omega \setminus \delta$, the following two relations are readily verified.

$$D_{a_j} D_{a_j} = 1 \quad \text{and} \quad D_{a_1} D_{a_2} \dots D_{a_m} D_{a_1} = (-1)^{m+1} D_{a_2} D_{a_3} \dots D_{a_m} D_{a_1} D_{a_2}.$$

Now, misusing notation, as in [5], we may write these as

$$a_j a_j = 1 \quad \text{and} \quad a_1 a_2 \dots a_m a_1 = (-1)^{m+1} a_2 a_3 \dots a_m a_1 a_2.$$

We are now in a position to define a signed cycle in $\hat{S}_n$.

Definition 3.1. *For distinct elements $a_1, \ldots, a_m$ we define $\langle a_1 a_2 \ldots a_m \rangle = a_1 a_2 \ldots a_m a_1$. We call $\pm \langle a_1 a_2 \ldots a_k \rangle$ signed cycles in $\hat{S}_n$. Each is a lift of the cycle $(a_1 a_2 \ldots a_k)$ in S_n.*

Using this definition with the above relations we obtain the following rules for multiplying signed cycles in $\hat{S}_n$.

$$\begin{aligned} \langle a_j \rangle &= 1, \\ \langle a_1 a_2 \ldots a_m \rangle &= (-1)^{m+1} \langle a_2 a_3 \ldots a_m a_1 \rangle, \\ \langle a_1 a_2 \ldots a_{m-1} \rangle a_m &= (-1)^m a_m \langle a_1 a_2 \ldots a_{m-1} \rangle. \end{aligned}$$

As an example of the multiplication of signed cycles we recover our presentation for $\hat{S}_n$ as follows: Let $r_1 = \langle 1, 2 \rangle$, $r_2 = \langle 2, 3 \rangle, \ldots, r_{n-1} = \langle n-1, n \rangle$ and $z = -1$. Clearly $z^2 = 1$ and

$$\begin{aligned} zr_j &= -\langle j, j+1 \rangle = r_j z, \quad \text{for} \quad 1 \le j \le n-1, \\ r_j^2 &= \langle j, j+1 \rangle \langle j, j+1 \rangle = j, j+1, j, j, j+1, j = 1, \\ (r_j r_{j+1})^3 &= (\langle j, j+1 \rangle \langle j+1, j+2 \rangle)^3 = (-\langle j+1, j \rangle \langle j+1, j+2 \rangle)^3 \\ &= (-j+1, j, j+2, j+1)^3 = -j+1 \langle j, j+2 \rangle \langle j+2, j \rangle j+1 \\ &= j+1 \langle j+2, j \rangle \langle j+2, j \rangle j+1 \\ &= j+1, j+1 = 1 \quad \text{for} \qquad 1 \le j \le n-2 \\ \text{and} & \\ r_j r_k &= \langle j, j+1 \rangle \langle k, k+1 \rangle = -\langle k, k+1 \rangle \langle j, j+1 \rangle \\ &= z r_k r_j \quad \text{for} \quad \mid j-k \mid > 1 \quad \text{and} \quad 1 \le j, k \le n-1. \end{aligned}$$

Note that, for $n \ge 4$, $\hat{S}_n \not\cong \tilde{S}_n$ if $n \ne 6$, see [3]. Also note that we will again be taking advantage of the fact that by factoring out $\hat{S}_n$ by the subgroup $Z = \langle 1, -1 \rangle$ we recover S_n. That is $\hat{S}_n / Z \cong S_n$.

Proposition 3.2. *Any product of k disjoint signed transpositions in $\hat{S}_n$ has order two if the integer part of $k/2$ is a multiple of two, and order four otherwise.*

Proof. Let S denote a product of k disjoint signed transpositions so that we have, for $k \ge 2$,

$$S = \pm \langle a_1, b_1 \rangle \langle a_2, b_2 \rangle \ldots \langle a_k, b_k \rangle,$$

where a_i and b_i are distinct integers in the signed transpositions of $\hat{S}_n$. Then a straight forward induction proof gives, for $k > 1$,

$$S^2 = (-1)^{k-1}(-1)^{k-2} \ldots (-1)^2(-1).$$

While for $k = 1$, $\langle a_1, b_1 \rangle \langle a_1, b_1 \rangle = 1$. □

So in $\hat{S}_n$ an involution is of the form $\pm\pi$ where π is a signed cycle consisting of k disjoint transpositions and k is congruent to 0 or 1 modulo 4.

Proposition 3.3. *For $n \ge 4$, $\hat{S}_n$ may be generated by $a = \pm \langle 1, 2 \rangle$ and $b = \pm \langle 1, 2, \ldots, n \rangle$.*

Proof. As in Proposition 2.3 we need only show that -1 is generated. Now in $\hat{S}_n$ we may generate either

$$g = +\langle 1,2\rangle\langle 3,4\rangle \qquad \text{or} \quad h = -\langle 1,2\rangle\langle 3,4\rangle\,,$$

but in either case $g^2 = h^2 = -1$. □

The investigation into the value of $i(\hat{S}_n)$ closely follows that of the previous section, except that here involutions are products of k signed transpositions for $k \equiv 0$ or 1 modulo 4. So in particular $\pm\langle 1,2\rangle$ is an involution in this double cover.

Hence in $\hat{S}_5$, $\hat{S}_6$ and $\hat{S}_7$ the only elements of order two are -1 and signed transpositions of the form $\pm\langle a,b\rangle$. Thus we have immediately that $i(\hat{S}_5) = 4$, $i(\hat{S}_6) = 5$ (which, as $\hat{S}_6 \cong \tilde{S}_6$, implies $i(\tilde{S}_6) = 5$) and $i(\hat{S}_7) = 6$, generators being $\langle 1,2\rangle, \dots, \langle 1,n\rangle$. Also, as the only involutions in $\hat{S}_8$ are -1 and signed cycles of type 2^1 and 2^4, it is readily verified (via GAP [4]) that $i(\hat{S}_8) > 3$. But as

$$\langle 1,2\rangle\,, \quad \langle 1,6\rangle\,, \quad \langle 1,2\rangle\langle 3,8\rangle\langle 4,7\rangle\langle 5,6\rangle \quad \text{and} \quad \langle 1,6\rangle\langle 2,3\rangle\langle 4,8\rangle\langle 5,7\rangle$$

generate $\hat{S}_8$ we have $i(\hat{S}_8) = 4$.

Proposition 3.4. *For $n \geq 9$, when $n \equiv 1$, 2 or 3 modulo 8, $\hat{S}_n$ may be generated by three involutions.*

Proof. We only need apply the decomposition referred to in the introduction to see that we may express $\pm\langle 1,2,\dots,n\rangle$ as the product ab where

$$a = \langle 1,n-1\rangle\langle 2,n-2\rangle\dots\langle r,n-r\rangle\,, \quad b = \langle 1,n\rangle\langle 2,n-1\rangle\dots\langle t,n+1-t\rangle\,.$$

Hence if we take a, b along with $\langle 1,2\rangle$ the result follows. □

Proposition 3.5. *For $12 \leq n \leq 16$, the following involutions generate $\hat{S}_n$.*
When $n = 16$

$$\begin{aligned} a &= \langle 1,15\rangle\langle 2,14\rangle\langle 3,13\rangle\langle 6,10\rangle\langle 7,9\rangle \\ b &= \langle 4,12\rangle\langle 5,11\rangle\langle 7,10\rangle\langle 2,15\rangle\langle 3,14\rangle \\ c &= \langle 4,13\rangle\langle 5,12\rangle\langle 6,11\rangle\langle 1,16\rangle\langle 8,9\rangle\,. \end{aligned}$$

For $12 \leq n \leq 15$

$$a = \langle 1,2\rangle\langle 3,11\rangle\langle 4,10\rangle\langle 5,9\rangle\langle 6,8\rangle\,,\ b = \langle 1,3\rangle\langle 4,11\rangle\langle 5,10\rangle\langle 6,9\rangle\langle 7,8\rangle$$

$$\textit{and} \quad c = \begin{cases} \langle 3,12\rangle\langle 4,10\rangle\langle 5,9\rangle\langle 6,8\rangle & \textit{when} \quad n = 12, \\ \langle 3,12\rangle\langle 4,13\rangle\langle 5,9\rangle\langle 6,8\rangle & \textit{when} \quad n = 13, \\ \langle 3,12\rangle\langle 4,13\rangle\langle 5,14\rangle\langle 6,8\rangle & \textit{when} \quad n = 14, \\ \langle 3,12\rangle\langle 4,13\rangle\langle 5,14\rangle\langle 6,15\rangle & \textit{when} \quad n = 15. \end{cases}$$

Proof. For $n = 16$ we have $abc = \pm\langle 1,2,\ldots,16\rangle$ so we need only show that $\langle 1,2\rangle$ is also generated. But

$$(abc)^6((c(ab)^4cb)^3(b(abcb)^3b)^3)^3(abc)^{-6} = \pm\langle 1,4\rangle \quad \text{and}$$
$$((abc)^2b(abc)^3)^2 \pm\langle 1,4\rangle\,((abc)^2b(abc)^3)^{-2} = \pm\langle 1,2\rangle\,.$$

When $n = 12$, 13, 14 or 15, $(ac)^3 = \pm\langle 1,2\rangle$ and $ab = \pm\langle 1,2,\ldots,11\rangle$ so for each n we may generate the subgroup $\hat{S}_{11}$, in particular the signed cycles $d = \langle 1,3\rangle$, $e = \langle 1,4\rangle$, $f = \langle 1,5\rangle$ and $g = \langle 1,6\rangle$.

Hence we need only show that $\langle 1,2,\ldots,n\rangle$ can also be generated for each n. But when $n = 12$ we have $abcdc = \pm\langle 1,2,\ldots,12\rangle$. When $n = 13$, $abcdec = \pm\langle 1,2,\ldots,13\rangle$. When $n = 14$, $abcdefc = \pm\langle 1,2,\ldots,14\rangle$ and when $n = 15$, $abcdefgc = \pm\langle 1,2,\ldots,15\rangle$. □

We again make use of the notation $T(\alpha,\alpha+\delta,\alpha+\beta)$, as in the previous section, but here this represents the $\delta+1$ signed transpositions

$$\langle\alpha,\beta\rangle\,\langle\alpha+1,\beta-1\rangle\,\langle\alpha+2,\beta-2\rangle\ldots\langle\alpha+\delta,\beta-\delta\rangle\,.$$

Proposition 3.6. *For $n \geq 17$, the following involutions, which are dependent on the value of n modulo 8, generate $\hat{S}_n$.*

$$\begin{array}{lll}
n\equiv 0 & a = & \langle 1,2\rangle\,T(3,(n-4)/2,n-2),\\
 & b = & \langle 1,3\rangle\,T(4,(n-2)/2,n-1),\\
 & c = & T(3,7,n+3)T(8,(n-4)/2,n-2).\\
n\equiv 4 & a = & \langle 1,2\rangle\,T(3,n/2,n+2),\\
 & b = & \langle 1,3\rangle\,T(4,(n+2)/2,n+3),\\
 & c = & \langle 3,n\rangle\,T(4,n/2,n+2).\\
n\equiv 5 & a = & \langle 1,2\rangle\,T(3,(n-1)/2,n+1),\\
 & b = & \langle 1,3\rangle\,T(4,(n+1)/2,n+2),\\
 & c = & \langle 3,n\rangle\,\langle 4,n-1\rangle\,T(5,(n-1)/2,n+1).\\
n\equiv 6 & a = & \langle 1,2\rangle\,T(3,(n-2)/2,n),\\
 & b = & \langle 1,3\rangle\,T(4,n/2,n+1),\\
 & c = & T(3,5,n+3)T(6,(n-2)/2,n).\\
n\equiv 7 & a = & \langle 1,2\rangle\,T(3,(n-3)/2,n-1),\\
 & b = & \langle 1,3\rangle\,T(4,(n-1)/2,n),\\
 & c = & T(3,6,n+3)T(7,(n-3)/2,n-1).
\end{array}$$

Proof. For each n here we have $(ac)^3 = \pm\langle 1,2\rangle$, so we only need show that $\pm\langle 1,2,\ldots,n\rangle$ is also generated in each case. Now for

$$\begin{array}{lll}
n\equiv 4 & ab = \pm\langle 1,2,\ldots,n-1\rangle & \text{so } \hat{S}_{n-1} \text{ is generated,}\\
n\equiv 5 & ab = \pm\langle 1,2,\ldots,n-2\rangle & \text{so } \hat{S}_{n-2} \text{ is generated,}\\
n\equiv 6 & ab = \pm\langle 1,2,\ldots,n-3\rangle & \text{so } \hat{S}_{n-3} \text{ is generated,}\\
n\equiv 7 & ab = \pm\langle 1,2,\ldots,n-4\rangle & \text{so } \hat{S}_{n-4} \text{ is generated,}\\
n\equiv 0 & ab = \pm\langle 1,2,\ldots,n-5\rangle & \text{so } \hat{S}_{n-5} \text{ is generated.}
\end{array}$$

In particular, as $n \geq 17$, we may generate for each n the signed cycles $d = \langle 1, 7 \rangle$, $e = \langle 1, 6 \rangle$, $f = \langle 1, 5 \rangle$, $g = \langle 1, 4 \rangle$ and $h = \langle 1, 3 \rangle$. Thus we may obtain $\pm \langle 1, 2, \ldots, n \rangle$ from $abchc$ when $n \equiv 4$, $abcghc$ when $n \equiv 5$, $abcfghc$ when $n \equiv 6$, $abcefghc$ when $n \equiv 7$ and $abcdefghc$ when $n \equiv 0$. □

These last three propositions, along with the fact that $(1, 2)$, $(1, 3)$ and $(1, 4)$ generate S_4, give directly the following theorem.

Theorem 3.7. *For $n = 4$ and $n \geq 9$,* $\quad i(\hat{S}_n) = 3$.

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