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ON A SINGULARLY PERTURBED, COUPLED ELLIPTIC-ELLIPTIC PROBLEM

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To Professor Silviu Sburlan, at his 60' anniversary

Abstract

The behavior of the solution of the below problem (E_ϵ), (BC_ϵ), (TC_ϵ) is studied when the small parameter ϵ tends to 0.

1. Introduction.

We consider the following coupled boundary value problem of elliptic-elliptic type, denoted by P_ϵ:

$$\begin{cases} -\epsilon u''(x) + \alpha(x)u'(x) + \beta(x)u(x) = f(x), \; x \in (a, b), \\ -\big(\mu(x)v'(x)\big)' + \alpha(x)v'(x) + \beta(x)v(x) = g(x), \; x \in (b, c), \end{cases} \tag{E_ϵ}$$

with homogeneous Dirichlet boundary conditions

$$u(a) = v(c) = 0 \tag{BC_ϵ}$$

and transmission conditions at $x = b$

$$u(b) = v(b), \quad \epsilon u'(b) = (\mu v')(b). \tag{TC_ϵ}$$

The transmission conditions at $x = b$ express the continuity of the solution and of the flux.

The following assumptions will be required in the following:

(A_1) $a, \ b, \ c \in \mathbb{R}, \ a < b < c, \ \epsilon > 0$ is a small parameter;

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(A_2) $\alpha \in H^1(a,c),\ \beta \in L^\infty(a,c),\ \mu \in H^1(a,c)$;

(A_3) $\alpha(x) \leq \alpha_0 < 0$ in $[a,c]$, $\mu(x) \geq \mu_0 > 0$ in $[b,c]$, $\beta - \frac{\alpha'}{2} \geq 0$ a.e. in (a,c);

$(A_4) f \in L^2(a,b), g \in L^2(b,c)$.

The aim of this paper is to investigate the problem P_ϵ for ϵ going to zero from the view point of singular perturbation theory. This is a singularly perturbed problem with respect to the norm of uniform convergence and the boundary layer is the point $x = a$. To have an idea about this matter, let us consider the particular case when α, β, μ are constant functions. If the solution (u, v) of (P_ϵ) converges in $C[a,b] \times C[b,c]$ to (U, V), then it can easily be seen that (U, V) satisfies

$$\begin{cases} \alpha U' + \beta U = f, \quad \text{in} \quad (a,b), \\ -\mu V'' + \alpha V' + \beta V = g, \quad \text{in} \quad (b,c), \\ U(a) = 0, U(b) = V(b), \\ V'(b) = 0, V(c) = 0. \end{cases}$$

The condition $U(a) = 0$ is not satisfied in general (this exceeds the number of conditions allowed). This fact is not acceptable from a physical point of view. Actually we shall see that indeed, in the point $x = a$, the solution (u, v) has a singular behaviour as ϵ goes to zero. The corresponding unperturbed (reduced) problem, denoted by P_0, is the following [6]:

$$\begin{cases} \alpha(x)U'(x) + \beta(x)U(x) = f(x), \quad x \in (a,b) \\ -\ \big(\mu(x)V'(x)\big)' + \alpha(x)V'(x) + \beta(x)V(x) = g(x), \quad x \in (b,c), \end{cases} \tag{E_0}$$

$$V(c) = 0, \tag{BC_0}$$

$$\begin{cases} U(b) = V(b), \\ V'(b) = 0. \end{cases} \tag{TC_0}$$

This problem will be reobtained below by using the Vishik-Lusternik method and it is the same as that derived in [6] by using a different way.

In [1], [2] we studied a similar transmission of type elliptic-elliptic but we considered that $\alpha \geq \alpha_0 > 0$. Here we are assuming $\alpha < 0$, hence the asymptotic behavior is different from the case $\alpha > 0$: actually we cannot have a convergence of $u(a)$ to $U(a)$, in general. A comment is needed about the conditions $(TC_0), (BC_0)$. Note that the problem P_0 has no conditions at all for $x = a$ and needs two conditions at b, in this case, when $\alpha < 0$.

In Section 2 we shall derive a formal zero-th order asymptotic expansion for the solution (u, v) of the problem (P_ϵ). The interface point $x = a$ is a boundary layer and the expansion of u contains a corresponding corrector (boundary layer function).

In Section 3 we shall investigate the existence and uniqueness of the solutions to the problems (P_ϵ) and P_0.

Finally, Section 4 is devoted to obtaining some estimates for the remainder terms of the expansion established in Section 2 with respect to the uniform convergence topology.

2. A formal asymptotic expansion for the solution of P_ϵ

The classical perturbation theory (see [7] for details) can be adapted to our specific singular perturbation problem. Following this theory, we are going to derive formally an expansion of the solution $(u,\ v)$ of (P_ϵ) of the form:

$$\begin{cases} u(x) = U(x) + \theta_1(\zeta) + \rho_{1\epsilon}(x), & x \in [a, b], \\ v(x) = V(x) + \theta_2(\zeta) + \rho_{2\epsilon}(x), & x \in [b, c], \end{cases} \tag{2.1}$$

where $\zeta{:=}\epsilon^{-1}(x - a)$ is the fast variable; $(U,\ V)$ is the zero-th order term of the regular series; $\theta_1,\ \theta_2$ are boundary layer functions (correctors); $\rho_{1\epsilon},\ \rho_{2\epsilon}$ denote the remainder terms of zero-th order.

We substitute formally in (E_ϵ) $(u,\ v)$ given by (2.1) and then we identify the coefficients of ϵ^k $(k = -1, 0)$, separately those depending on x from those depending on ζ. So, we get

$$\begin{cases} \alpha(x)U'(x) + \beta(x)U(x) = f(x), \quad a < x < b, \\ -(\mu(x)V'(x))' + \alpha(x)V'(x) + \beta(x)V(x) = g(x), \quad b < x < c. \end{cases} \tag{E_0}$$

For $\theta_1{=}\theta_1(\zeta)$ we derive the equation

$$\theta_1''(\zeta) - \alpha(a)\theta_1'(\zeta) = 0. \tag{2.2}$$

Eq. (2.2) and the fact that θ_1 is a boundary layer function (in particular, $\theta_1(\zeta) \longrightarrow 0,\ \ as\ \ \zeta \longrightarrow \infty$) implies that $\theta_1(\zeta) = ke^{\alpha(a)\zeta}$, where k is a constant which will be determined from (BC_ϵ). Also, we can deduce that $\theta_2 = 0$.

For the remainder terms we have the equations

$$\begin{cases} -\varepsilon\ \rho_{1\epsilon}''(x) + \alpha(x)\rho_{1\epsilon}'(x) + \beta(x)\rho_{1\epsilon}(x) = \\ = \epsilon U''(x) + \epsilon^{-1}\big(\alpha(x) - \alpha(a)\big)\theta_1'\big(\zeta(x)\big) - \beta(x)\theta_1\big(\zeta(x)\big), \quad a < x < b, \\ -\big((\mu\rho_{2\epsilon}')(x)\big)' + \alpha(x)\rho_{2\epsilon}'(x) + \beta(x)\rho_{2\epsilon}(x) = 0, \quad b < x < c, \end{cases} \tag{ER}$$

where $\zeta(x) = (x-a)/\epsilon$.

From (BC_ϵ), we can derive

$$k = -U(a), \text{ hence } \theta_1(\xi) = -U(a)e^{\alpha(a)\zeta} \tag{2.3}$$

$$V(c) = 0, \tag{BC_0}$$

$$\begin{cases} \rho_{1\epsilon}(a) = 0, \\ \rho_{2\epsilon}(c) = 0. \end{cases} \tag{BCR}$$

By replacing (2.1) into $(TC)_\epsilon$, we can see that U and V satisfy the transmission conditions

$$\begin{cases} U(b) = V(b), \\ V'(b) = 0, \end{cases} \tag{TC_0}$$

and for the remainder terms we have

$$\begin{cases} \rho_{1\epsilon}(b) = \rho_{2\epsilon}(b) + U(a)e^{\alpha(a)\theta_1(\zeta(b))}, \\ \epsilon\rho_{1\epsilon}'(b) = -\epsilon U'(b) + U(a)\alpha(a)e^{\alpha(a)\theta_1(\zeta(b))} + \big(\mu\rho_{2\epsilon}'\big)(b). \end{cases} \tag{TCR}$$

Summarizing, the reduced problem, P_0, is $(E_0)-(BC_0)-(TC_0)$, while the problem satisfied by the remainder terms is $(ER)-(BCR)-(TCR)$. As we shall see later on, the last problem is satisfied by $(\rho_{1\epsilon},\ \rho_{2\epsilon})$ in a generalized sense.

3. Existence and regularity for the problems (P_ϵ) and P_0.

For the problem (P_ϵ) we have the following result, whose proof is essentially known (see [1]):

Proposition 3.1. *Assume that (A_1)–(A_4) are satisfied. Then, the problem P_ϵ admits a unique solution $(u,v) \in H^2(a,b)\times H^2(b,c)$ satisfying (E_ϵ) a.e. in (a,b) and in (b,c), respectively, as well as (BC_ϵ) and (TC_ϵ).*

In the following, we are going to investigate the reduced problem (P_0). In fact, we can split it in two separate problems, with the unknowns U and V, respectively. Clearly, V is a solution of $(E_0)_2$, with the boundary conditions

$$V'(b) = 0, V(c) = 0. \tag{3.1}$$

By the Lax-Milgram lemma, there exists a unique solution $V\in H^2(b,c)$ of this problem. Obviously, Eq. $(E_0)_1$, with $U(b) = V(b)$, has a unique solution $U \in H^1(a,b)$. Therefore, we have the following result

Proposition 3.2. *Assume that $(A_1)-(A_4)$ are satisfied. Then, the problem P_0 has a unique solution $U\in H^1(a,b)$, $V\in H^2(a,c)$, which satisfies (E_0) a.e. in (a,b) and (b,c), respectively, as well as (BC_0) and (TC_0).*

If we denote

$$\tilde{\rho}_{1\epsilon}(x):=\rho_{1\epsilon}(x)+A_\epsilon x+B_\epsilon,\ A_\epsilon:=(b-a)\chi_\epsilon,\ B_\epsilon:=-aA_\epsilon, \tag{3.2}$$

where $\chi_\epsilon=-U(a)e^{\alpha(a)\theta_1(\zeta(b)}$ then we have $\tilde{\rho}_{1\epsilon}(a)=0$, $\tilde{\rho}_{1\epsilon}(b)=\rho_{2\epsilon}(b)$ and taking into account $(P_\epsilon),(P_0)$, we can see that

$$\rho_\epsilon:=\begin{cases}\tilde{\rho}_{1\epsilon} & \text{in } [a,b]\\ \rho_{2\epsilon} & \text{in } (b,c]\end{cases},$$

satisfies $\rho_\epsilon\in H_0^1(a,c)$ and

$$\epsilon\int_a^b\tilde{\rho}_{1\epsilon}'\varphi'dx+\int_b^c\mu{\rho_{2\epsilon}}'\varphi'dx+\int_a^b\alpha\tilde{\rho}_{1\epsilon}'\varphi dx+\int_b^c\alpha{\rho_{2\epsilon}}'\varphi dx$$

$$=-\epsilon\int_a^b U'\varphi'dx-\int_a^c h_\epsilon\varphi dx+\epsilon A_\epsilon\varphi(b)+\alpha(a)\varphi(b)\chi_\epsilon,\quad\forall\ \varphi\in H_0^1(a,c), \tag{3.3}$$

where

$$h_\epsilon(x):=\begin{cases}\beta(x)\theta_1\big((\zeta(x)\big)+\alpha(a)\big(\alpha(x)-\alpha(a)\big)\theta_1\big(\zeta(x)\big)+\\ +\Big[A_\epsilon\alpha(x)+\beta(x)(A_\epsilon x+B_\epsilon)\Big],\quad\text{in } (a,b),\\ 0,\quad\text{in } (b,c).\end{cases}$$

Indeed, by (E_ϵ) and (E_0), we obtain

$$\int_a^b\epsilon u'\varphi'dx+\int_a^b\alpha u'\varphi dx+\int_a^b\beta u\varphi dx+\int_b^c(\mu v')\varphi'dx+$$

$$+\int_b^c\alpha v'\varphi dx+\int_b^c\beta v\varphi dx=\int_a^b f\varphi dx+\int_b^c g\varphi dx,\ \forall\ \varphi\in H_0^1(a,c),$$

$$\int_a^b\alpha U'\varphi dx+\int_a^b\beta U\varphi dx=\int_a^b f\varphi dx,$$

$$\int_b^c(\mu V')\varphi'dx+\int_b^c\alpha V'\varphi dx+\int_b^c\beta V\varphi dx-\ \mu V'\varphi\big|_b^c=\int_b^c\varphi g dx,\ \forall\ \varphi\in H_0^1(a,c).$$

Now, subtracting the last two equalities from the first one, we obtain that

$$\int_a^b\epsilon\Big(U'+\frac{d}{dx}\theta_1(\zeta(x))+\varrho_{1\epsilon}'\Big)\varphi'dx+\int_a^b\alpha\Big(\frac{d}{dx}\theta_1(\zeta(x))+\rho_{1\epsilon}'\Big)\varphi dx+$$

$$+\int_a^b \beta\Big(\theta_1(\zeta(x))+\rho_{1\epsilon}\Big)\varphi dx+$$

$$+\int_b^c (\mu\rho_{2\epsilon}')\varphi' dx+\int_b^c \alpha\rho_{2\epsilon}'\varphi dx+\int_b^c \beta\rho_{2\epsilon}\varphi dx-(\mu V')(b)\varphi(b)=0.$$

From

$$\epsilon\int_a^b \varphi'\frac{d}{dx}\theta_1(\zeta(x))dx = \ \epsilon\varphi(x)\frac{d}{dx}\theta_1(\zeta(x))\big|_a^b-\epsilon\int_a^b \varphi\frac{d^2}{dx^2}\theta_1(\zeta)dx=$$

$$=\varphi(b)\chi_\epsilon\alpha(a)-\epsilon^{-1}\alpha(a)^2\int_a^b \varphi\theta_1(\zeta(x))dx,$$

we can see that,

$$\int_a^b \epsilon\big(\ \tilde\rho_{1\epsilon}{}'-A_\epsilon\varphi'\big)dx+\int_a^b \alpha\big(\ \tilde\rho_{1\epsilon}{}'-A_\epsilon-\epsilon^{-1}\alpha(a)\theta_1(\zeta)\big)\varphi dx+$$

$$+\int_a^b \beta\big(\theta_1(\xi)+\tilde\rho_{1\epsilon}-A_\epsilon x-B_\epsilon)\big)\varphi dx+$$

$$+\int_b^c \mu\rho_{2\epsilon}'\varphi' dx+\int_b^c \alpha\rho_{2\epsilon}'\varphi dx+\int_b^c \beta\rho_{2\epsilon}\varphi dx-$$

$$-\epsilon^{-1}\alpha(a)^2\int_a^b \varphi\theta_1(\zeta)dx=-\epsilon\int_a^b U'\varphi' dx+\alpha(a)\varphi(b)\chi_\epsilon.$$

In conclusion, we obtain (3.3).

An elementary computation shows that, if $\rho_{1\epsilon}$, $\rho_{2\epsilon}$ are smooth functions, then they satisfy problem $(ER)-(BCR)-(TCR)$ in a classical sense.

4. Estimates for the remainder terms

The main result of this section is

Theorem 4.1. *Assume that* $(A_1)-(A_4)$ *are satisfied and* α *is Lipschitzian in* $[a,b]$ *(i.e.,* $\exists\ L{>}0$*, such that* $|\alpha(x)-\alpha(y)\ |{\le}L|x-y|$*,* $\forall\ x,y{\in}[a,b]$*). Then, for every* $\epsilon{>}0$*, the problem* (P_ϵ) *has a unique solution* $(u,v)\in H^2(a,b){\times}H^2(b,c)$ *of the form* (2.1)*, where* $col(U,V){\in}H^1(a,b){\times}H^2(b,c)$ *is the solution of* (P_0)*,* θ_1 *is given by* (2.3)*,* $\theta_2=0$ *and* $(\rho_{1\epsilon},\rho_{2\epsilon})\in H^1(a,b){\times}H^2(b,c)$*.*

In addition, we have the estimates $\|\rho_{1\epsilon}\|_{C[a,b]}{=}O(\sqrt{\epsilon})$*,* $\|\rho_{2\epsilon}\|_{C[b,c]}{=}O(\sqrt{\epsilon})$*.*

Proof. By Propositions 3.1 and 3.2, $(\rho_{1\epsilon},\rho_{2\epsilon}){\in}H^1(a,b){\times}H^2(b,c)$. For the sake of simplicity, we assume that $\beta{-}\alpha'/2{\ge}\gamma_0{>}0$ a.e. in (a,c). If we choose in (3.3) $\varphi{=}\rho_\epsilon{\in}H_0^1(a,c)$,, we can see that

$$\epsilon\int_a^b\ (\tilde{\rho}'_{1\epsilon})^2dx+\int_b^c\mu\ (\rho'_{2\epsilon})^2dx+\int_a^b(\beta-\alpha'/2)\ \ \tilde{\rho}_{1\epsilon}{}^2dx+$$

$$+\int_b^c(\beta-\alpha'/2)\rho_{2\epsilon}^2dx=-\epsilon\int_a^b U'\ \ \tilde{\rho}_{1\epsilon}{}'dx-\int_a^b h_\epsilon\ \tilde{\rho}_{1\epsilon}dx+\gamma_\epsilon, \tag{4.1}$$

where $\gamma_\epsilon=\alpha(a)\varphi(b)\ \tilde{\rho}_{1\epsilon}(b)+\epsilon A_\epsilon\ \tilde{\rho}_{1\epsilon}(b)$. In the case $\beta{-}\alpha'/2{\ge}0$ a.e. in (a,c), we choose in (3.3)

$$\varphi(x):=\begin{cases} e^{-x}\ \tilde{r}_{1\epsilon}(x) & \text{in } [a,b] \\ e^{-b}r_{2\epsilon}(x) & \text{in } (b,c] \end{cases}$$

and we can use a slight modification of our reasoning below. Denote by $\|\cdot\|_1$, $\|\cdot\|_2$ the norms of $L^2(a,b)$, $L^2(b,c)$, respectively. As $\beta-\ \alpha'/2{\ge}\gamma_0{>}0$ a.e. in (a,c) and $\mu{\ge}\mu_0\ {>}0$ in $[b,c]$, it follows from (4.1) that

$$\epsilon\|\ \tilde{\rho}_{1\epsilon}{}'\|_1^2+\mu_0\|\rho'_{2\epsilon}\|_2^2+\gamma_0\big(\|\tilde{\rho}_{1\epsilon}\|_1^2+\|\rho_{2\epsilon}\|_2^2\big)\le$$

$$\le(1/2)\Big[\epsilon\|U'\|_1^2+\epsilon\|\tilde{\rho}'_{1\epsilon}\|_1^2+\gamma_0^{-1}\|h_\epsilon\|_1^2+\gamma_0\|\tilde{\rho}_{1\epsilon}\|_1^2\Big]+|\ \gamma_\epsilon|. \tag{4.2}$$

In the following, we can show that $\|h_\epsilon\|_1{=}O(\sqrt{\epsilon})$ and $\gamma(\epsilon){=}\ O(\epsilon^k)$, $\forall k{\ge}0$. From equations (E_ϵ), we obtain

$$\epsilon\|u'\|_1^2+\mu_0\|v'\|_2^2+\gamma_0\big(\|u\|_1^2+\|v\|_2^2\big)\le$$

$$\le(1/2)\Big[\gamma_0^{-1}\big(\|f\|_1^2+\|g\|_2^2\big)+\gamma_0\big(\|u\|_1^2+\|v\|_2^2\big)\Big]. \tag{4.3}$$

This implies that $\|u\|_1{=}O(1)$, $\|v\|_2 = O(1)$ and $\|v'\|_2{=}=O(1)$, $\epsilon\ \|u'\|_2^2{=}O(1)$. Since $v(c){=}0$ and $H^1(b,c){\subset}C[b,c]$, with a compact injection, we get $\|v\|_{C[b,c]}{=}O(1)$. For γ_ϵ, we have

$$\lim_{\epsilon\to 0} \epsilon^{-k}\gamma_\epsilon = 0, \text{ for every } k \geq 0,$$

and therefore $\gamma_\epsilon = O(\epsilon^k)$. Now, as $\beta{\in}L^\infty(a,c)$, $\mid A_\epsilon \mid, \mid B_\epsilon \mid = O(\epsilon^k), k \geq 1$, α is Lipschitzian in $[a,b]$, one gets by an easy computation that $\|h_\epsilon\|_1^2 = O(\epsilon)$.

Now, from (4.2), it follows

$$\|\tilde{\rho}'_{1\epsilon}\|_1 = O(1), \quad \|\tilde{\rho}_{1\epsilon}\|_1 = O(\sqrt{\epsilon}), \quad \|\rho'_{2\epsilon}\|_2 = O(\sqrt{\epsilon}), \quad \|\rho_{2\epsilon}\|_2 = O(\sqrt{\epsilon}),$$

therefore

$$\|\rho_{1\epsilon}\|_1 \leq \ \|\tilde{\rho}_{1\epsilon}\|_1 + \ \|(A_\epsilon x + B_\epsilon)\omega(\epsilon)\|_1 = O(\sqrt{\epsilon}). \tag{4.4}$$

From

$$\rho_{2\epsilon}^2(x) = (-1/2)\int_x^c {\rho_{2\epsilon}}'(s)\rho_{2\epsilon}(s)ds \leq 2\ \|\rho_{2\epsilon}\|_2\ \ \|\rho_{2\epsilon}\|_2,$$

one gets

$$\|\rho_{2\epsilon}\|_{C[b,c]} = O(\sqrt{\epsilon}). \tag{4.5}$$

In that which follows, we shall prove that $\|\rho_{1\epsilon}\|_{C[b,c]}{=}O(\sqrt{\epsilon})$. To do that, we integrate $(E_\epsilon)_1$ on $[y,b]$, $y{\in}[a,b]$:

$$\epsilon\big(u'(y) - u'(b)\big) + \int_y^b \alpha(s)u'(s)ds + \int_y^b \beta(s)u(s)ds = \int_y^b f(s)ds. \tag{4.6}$$

By replacing

$$u_(x) = U(x) + \theta_1\big(\zeta(x)\big) + \rho_{1\epsilon}(x),$$

in (4.6), one obtains

$$\epsilon\ \rho'_{1\epsilon}(y) + \int_y^b \alpha(s)\rho'_{1\epsilon}(s)ds + \int_y^b \beta(s)\rho_{1\epsilon}(s)ds =$$

$$= \epsilon\Big[U'(y) + \frac{d}{dy}\theta_1\big(\zeta(y)\big)\Big] -$$

$$-\int_y^b \Big[\alpha(s)\Big(\frac{d}{ds}\theta_1\big(\zeta(s)\big)\Big) + \beta(s)\Big(\theta_1\big(\zeta(s)\big)\Big)\Big]ds + \epsilon u'(b)\text{a.e. in } (a,x). \tag{4.7}$$

Now we multiply the above inequality by $\rho'_{1\epsilon}(y)$ and then integrate on $[a,x]$.

Finally, we obtain that

$$\big(\alpha(x)/2\big)\ \rho_{1\epsilon}^2(x)+\tilde{b}(x)\rho_{1\epsilon}(x)+\tilde{c}(x)=0, \tag{4.8}$$

where

$$\tilde{b}(x):=-\int_x^b\Big(\alpha'(y)-\beta(y)\Big)(\rho_{1\epsilon}(y)dy-\epsilon u'(b)-\alpha(b)\theta_1(\zeta(b)+\theta_1(\zeta(y)),$$

$$\tilde{c}(x):=-\int_a^x\big(\alpha'/2-\beta\big)(y)\tilde{\rho}_{1\epsilon}^2(y)dy-$$

$$-\int_a^x\ \rho_{1\epsilon}'(y)\Big[\epsilon U'(y)-\big(\alpha(y)-\alpha(a)\big)\theta_1\big(\zeta(y)\big)\Big]dy-$$

$$-\int_a^x\Big[\Big(\beta(y)-\alpha'(y)\Big)\theta_1\big(\zeta(y)\big)\rho_{1\epsilon}(y)dy-\epsilon\int_a^x\ \big(\rho_{1\epsilon}'\big)^2(y)dy.$$

By (4.4), one obtains that $\tilde{c}(x){=}O(\epsilon)$. Integrating (4.6) on $[a,b]$, one obtains $\epsilon u'(b){=}O(\sqrt{\epsilon})$, hence $\tilde{b}(x){=}O(\sqrt{\epsilon})$.

Finally, from

$$(1/2)\alpha(x)\rho_{1\epsilon}^2(x)+O(\sqrt{\epsilon})\rho_{1\epsilon}(x)+O(\epsilon)=0 \text{ a.e. in } (a,b),$$

$$\alpha(x)/2\le\alpha_0/2,<0\ \text{ in } [a,b],$$

we can deduce that $\mid\rho_{1\epsilon}(x)\mid\ \le C\ \sqrt{\epsilon}$, $x\in[a,b]$, so we have $\|r_{1\epsilon}\|_{C[a,b]}=O(\sqrt{\epsilon})$.

References

[1] L. Barbu and Gh. Morosanu, On a class of singularly perturbed, coupled boundary value problems, *Math. Sci. Res. Hot-Line* **4**(6) (2000), 25-37.

[2] L. Barbu and Gh. Moroşanu, Asymptotic Analysis of Some Boundary Value Problems with Singular Perturbations,*Editura Academiei, Bucharest*, 2000 (in Romanian).

[3] C.A. Coclici, Gh. Moroşanu and W.L. Wendland, On the viscous-viscous and the viscous-inviscid interactions in Computational Fluid Dynamics, *Comput. Visualization Sci.* **2** (1999), 95-105.

[4] C.A. Coclici, Gh. Moroşanu and W.L. Wendland, One-dimensional singularly perturbed coupled boundary value problems, *Math. Sci. Res. Hot-Line* **3**(10) (1999), 1-21.

[5] C.A. Coclici, Gh. Moroşanu and W.L. Wendland, The coupling of hyperbolic and elliptic boundary value problems with variable coefficients, *Math. Meth. Appl. Sci.* **23** (2000), 401-440.

[6] F. Gastaldi and A. Quarteroni, On the coupling of hyperbolic and parabolic systems: analytical and numerical approach, *Appl. Numer. Math.* **6** (1990), 3-31.

[7] A.B. Vasilieva, V.F. Butuzov and I.V. Kalashev, *The Boundary Function Method for Singular Perturbation Problems*, SIAM, Philadelphia, 1995.

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