An. Şt. Univ. Ovidius Constanţa Vol. 15(1), 2007, 47–54

ABOUT RELATIONSHIP BETWEEN GENERALIZED STRUCTURABLE ALGEBRAS AND LIE RELATED TRIPLES

Camelia Ciobanu

Abstract

In this paper, we present some results about relationship between generalized structurable algebras and Lie related triples.

Introduction

The generalized structurable algebras were investigated in some papers ([3], [4], [5], [6]) and they contain the classes of Clifford algebras, Lie algebras, alternative algebras, Poisson algebras.

One of our aims is to give examples of generalized structurable algebras because these algebras are valuable in characterizing physically relevant phenomena.

Firstly we define the generalized structurable algebra and their standard embedding structurable algebras, with examples.

In Section 2, we consider the Lie related triple done by N. Jacobson [2].

Finally, we investigate the relationship between generalized structurable algebras and Lie related triples.

We are interested in algebras and triple systems with finite or infinite dimension over a commutative associative ring of scalars, k, without 2-torsion or 3-torsion, unless it is otherwise specified.

Key Words: Lie algebra, Jordan algebra, Clifford algebra, inner derivation
2000 Mathematical Subject Classification: 16W10,16W25,17C40
Received: January, 2007

1 Preliminaries

In this section we present some examples of generalized structurable algebras.

A generalized structurable algebra over a ring of scalars k is a non-associative algebra A equipped with a bilinear derivation $D(x,y)\,(\neq 0)$ for which the following conditions are satisfied:

$$D\left(x,y\right)=-D\left(y,x\right) \tag{1.1}$$

$$D\left(xy,z\right)+D\left(yz,x\right)+D\left(zx,y\right)=0, \text{ for all } x,y,z\in A. \tag{1.2}$$

Examples.

i) Let $(\mathcal{L},[,])$ be a Lie algebra over k. Then $\mathcal{L}$ is a generalized structurable algebra equipped with a derivation:

$$D\left(x,y\right):=ad\left[x,y\right].$$

ii) Let C be a Clifford algebra over k. Then C is a generalized structurable algebra equipped with a derivation:

$$D\left(x,y\right):=\left[L(x),L(y)\right]+\left[L(x),R(y)\right]+\left[R(x),R(y)\right]. \tag{1.3}$$

Since the Clifford algebra is an associative algebra, it follows that the associative algebras are contained in a class of generalized structurable algebras equipped with the above derivation (1.3).

iii) Let $(J,\circ)$ be a commutative Jordan algebra over k. Then J is a generalized structurable algebra equipped with a derivation:

$$D\left(x,y\right):=\left[L(x),L(y)\right],$$

where $\left[L(x),L(y)\right]z=x\circ\left(y\circ z\right)-y\circ\left(x\circ z\right), L(x)z=x\circ z.$

iv) Now, we consider $(U,^{-})$ being a unital algebra with an involution $^{-}$ over a field k of characteristic different from $2,3$.

Then $(U,^{-})$ is said to be structurable if, for all $x,y,z\in U$,

$$\left[L(x,y),L(z,w)\right]=L\left(<xyz>,w\right)-L\left(z,<yzw>\right),$$

where $L\left(x,y\right)z=<xyz>:=\left(x\overline{y}\right)z+\left(z\overline{y}\right)x-\left(z\overline{x}\right)y\,(\text{See [1]})$.

Then U is a generalized structurable algebra equipped with the derivation

$$D\left(x,y\right):=\frac{1}{3}\left(\left[\left[x,y\right]+\left[\overline{x},\overline{y}\right],z\right]\right)+\left[z,y,x\right]-\left[z,\overline{x},\overline{y}\right], \tag{1.4}$$

where

$$\left[x,y\right]:=xy-yx,\left[x,y,z\right]:=\left(xy\right)z-x\left(yz\right).$$

We note that the identities (1.1) and (1.2) hold for a structurable algebra U.

Thus any structurable algebra is a generalized structurable algebra. The derivation induced by (1.4) is said to be *an inner derivation* of the structurable algebra. The set of all such derivations is denoted by $InnDerU$.

Theorem 1.1.([3], [4]) *Let A be a generalized structurable algebra over k equipped with a derivation $D(x,y)$ satisfying*

$$\Sigma_{(x,y,z)}\left(D\left(x,y\right)z+[[x,y],z]\right)=0$$

and

$$[D\left(z,w\right),D(x,y)]=D\left(D(z,w)x,y\right)+D\left(x,D\left(z,w\right)y\right)\ \text{for}\ x,y,z,w\in A.$$

Then the vector space

$$L(A):=InnDerA\oplus A$$

is a Lie algebra with respect to the new bracket operation $[,]^*$

$$[D+x,E+y]^*:=[D,E]+D\left(x,y\right)+Dy-Ex+[x,y] \tag{1.5}$$

for $D,E\in InnDerA,\ x,y\in A$.

The Lie algebra obtained in this theorem is said to be the standard embedding Lie algebra associated with the generalized structurable algebra.

We may present as a generalization of Theorem 1.1 the following theorem.

Theorem 1.2.(Extended property for generalized structurable algebra).
Let A be a generalized structurable algebra over k equipped with the derivation $D\left(x,y\right)$ such that the derivation satisfies the relation $[D,D\left(x,y\right)]=D\left(Dx,y\right)+D\left(x,Dy\right)$, for all $D,D\left(x,y\right)\in InnDerA$. Then the vector space $V:=InnDerA\oplus A$ is a generalized structurable algebra equipped with a new product and new derivation defined by

$$[X,Y]^*:=D_1x_2-D_2x_1+[x_1,x_2]+[D_1,D_2]+D\left(x_1,x_2\right)$$

$$D^*\left(X,Y\right)Z:=$$

$$[[D_1,D_2],D_3]+[D\left(x_1,x_2\right),D_3]+D\left(x_1,x_2\right)x_3+[D_1,D_2]\,x_3$$

for $X=D_1+x_1,\ Y=D_2+x_2,\ Z=D_3+x_3,\ D_i\in InnDerA,\ x_i\in A$.

The generalized structurable algebra obtained from Theorem 1.2 is said to be *the standard embedding generalized structurable algebra associated with the generalized structurable algebra.*

Remark. *We note that Theorem 1.2 can be generalized to super or graded concepts.* (See [3],[4],[5]).

2 Lie Related Triples

In this section, we recall some concepts related to triality in octonion algebras as developed in [2] and [5].

Let $(A,^-)$ be a non associative algebra with involution over a ring of scalars k. Denote by $gl(A)$ the Lie algebra of endomorphisms of A.
If $\mathcal{F} \in gl(A)$, we define $\overline{\mathcal{F}}$ by $\overline{\mathcal{F}}(x) = (\mathcal{F}(\overline{x}))^-$.

We say that

$$F = (F_1, F_2, F_3) \in gl(A)^{(3)} := gl(A) \oplus gl(A) \oplus gl(A)$$

is a *partial Lie related triple product,* if

$$\overline{F}_1(ab) = F_2(a)b + aF_3(b), \tag{2.1}$$

for all $a, b \in A$ and we denote the set of all partial Lie related triples by J_0.

Example. Let C be a Cayley algebra over a ring k of characteristic $\neq 2, 3$, with norm $n(x)$ and let $\sigma(8, n)$ be the orthogonal Lie algebra of all F in $End(C)$ which are skew relative to $n(x)$. Then for every F_1 in $\sigma(8, n)$ there are unique F_2, F_3 in $\sigma(8, n)$ satisfying $F_1(xy) = F_2(x)y + x(F_3(y))$ for all $x, y \in C$. We also say that $F = (F_1, F_2, F_3)$ is *a Lie related triple,* if

$$(F_i, F_j, F_k) \in J_0 \tag{2.2}$$

for all (i, j, k) which are cyclic permutations of $(1, 2, 3)$.

The set of Lie related triples is denoted by J. It is easy to show that both J_0 and J are Lie subalgebras of $gl(A)^{(3)}$.

A particular triple is $F = (F_1, F_2, F_3)$, where

$$\begin{aligned} F_i &= L_{\overline{b}}L_a - L_{\overline{a}}L_b \\ F_j &= R_{\overline{b}}R_a - R_{\overline{a}}R_b \\ F_k &= R_{\left(\overline{a}b - \overline{b}a\right)} + L_bL_{\overline{a}} - L_aL_{\overline{b}}, \end{aligned} \tag{2.3}$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$ and $a, b \in A$, with $L_a(b) = ab = R_b(a)$.

We denote the k-space of all such triples by J_I and say $F \in J_I$ is an inner triple of the algebra $(A,^-)$.

3 Relationship between generalized structurable algebras and Lie related triples

In this section, we shall investigate the construction of a generalized structurable algebra.

Let A be a nonassociative algebra with involution over a ring of scalars k and $A[ij],\ i \leq i \neq j \leq 3$. be a copies of A with

$$a[ij] = -\gamma_i \gamma_j^{-1} a[ji],\ \gamma_i, \gamma_j \in K^*.$$

We form an algebra

$$B := A\,[12] \oplus A\,[23] \oplus A\,[31]\,,$$

defined by

$$[a\,[ij]\,, b\,[jk]] = -\,[b\,[jk]\,, a\,[ij]] = ab\,[ik]\,, \tag{3.1}$$

for distinct i, j, k and all other products are identically zero.

Then we put

$$D\,(a\,[ij]\,, b\,[ij]) := \gamma_i \gamma_j^{-1} F, \tag{3.2}$$

where $F\,(a\,[ij]) = F_k(a)\,[ij]$, and $F = (F_1, F_2, F_3)$ in as (2.2).

In fact, for $D\,(a\,[ij]\,, b\,[ij])$, we have

$$D\,(a\,[ij]\,, b\,[ij])\,d = \gamma_i \gamma_j^{-1}\,(F_k(d), F_i(d), F_j(d))\,.$$

Therefore we have the following.

Theorem 3.1. *Let A, B be as above. If the identities (2.3) and (3.3) hold, where*

$$\begin{aligned} &\overline{c}\,((\overline{a}b)\,d) - (ab)(cd) + \left(d(\overline{b}c)\right)\overline{a} - \\ &-(da)(bc) + d((ca)b) - d(\overline{b}(\overline{c}a)) + \\ &+b((ca)d) - (\overline{c}a)(\overline{b}d) = 0, \end{aligned} \tag{3.3}$$

then $(B, [,])$ is a generalized structurable algebra equipped with the derivation defined by the identity (3.2) .

Hereafter we say that $D\,(a[ij], b[ij])$ is an inner derivation of B.

Next we shall consider a vector space $L(B) = InnDerB \oplus B$ by linearity extending the product on $InnDerB$, by defining as follows:

$$\begin{aligned} &[a[ij], b[jk]]^* = -\,[b[jk], a[ij]]^* = \\ &= ab\,[ik] + D\,(a[ij], b[jk])\,, \end{aligned}$$

$$[F, a[ij]]^* = -[a[ij]F]^* = F_k(a)[ij],$$
$$[a[ij], b[ij]]^* = \gamma_i\gamma_j^{-1}F, \tag{3.4}$$

where F is as in (3.2), therefore $D(a[ij], b[jk]) = 0$ for distinct i, j, k. Then the following result can be proved.

Theorem 3.2. *Let B be as in Theorem 3.1. If the identities (2.3) and (3.3) hold on B, then the vector space $L(B) = InnDerB \oplus B$ is a generalized structurable algebra equipped with a new product and a new derivation defined by*

$$[X, Y]^* := D_1x_2 - D_2x_1 + [x_1, x_2] + [D_1, D_2] + D(x_1, x_2),$$
$$D^*(X, Y)Z := [[D_1, D_2], D_3] + [D(x_1, x_2), D_3] +$$
$$+D(x_1, x_2)x_3 + [D_1, D_2]x_3,$$

for $X = D_1 + x_1,\ Y = D_2 + x_2,\ Z = D_3 + x_3,\ D_i \in InnDerB,\ x_i \in B$.

Furthermore $L(B)$ becomes a Lie algebra, i.e., $L(B)$ is the standard embedding Lie algebra associated with B.

Proof. From the definition of a Lie related triple, it follows that

$$D^*(X, Y[Z, W]^* = [D^*(X, Y)Z, W]^* + [Z, D^*(X, Y)W]^*.$$

From the identity (3.3), it follows that

$$D^*\left([X, Y]^*, Z\right) + D^*\left([Y, Z]^*, X\right) + D^*\left([Z, X]^* Y\right) = 0.$$

Therefore the vector space

$$L(B) = InnDerB \oplus B$$

is a generalized structurable algebra equipped with the derivation $D^*(X, Y)$. Furthermore, we can show that $L(B)$ is a Lie algebra. In fact, due to the identity (2.3), we obtain

$$[D(X, Y), D(U, V)] = D(D(X, Y)U, V) + D(U, D(X, Y)).$$

On the other hand, for the case of $X = a[ij],\ Y = b[ij],\ Z = c[ij]$ with distinct i, j, k, we have

$$D(X, Y)Z = D(a[ij], b[ij])c[kj] = \gamma_i\gamma_jF_i(c[kj]) =$$
$$= -\gamma_i\gamma_j^{-1}F_i\left(\gamma_k\gamma_j^{-1}\overline{c}[jk]\right)\right) = -\gamma_i\gamma_j^{-1}\gamma_k\gamma_j^{-1}F_i(\overline{c})[jk].$$

Similarly, we have

$$[[Y, Z], X] = [b[ij], c[kj], a[ij]] =$$

$$= -\gamma_k \gamma_j^{-1} \gamma_i \gamma_j^{-1} \overline{a}(b\overline{c})[jk],$$

$$[[Z, X], Y] = \gamma_i \gamma_j^{-1} \gamma_k \gamma_j^{-1} \overline{b}(a\overline{c})[jk],$$

and from, the definition, it follows that

$$D(Y, Z)X = [[X, Y], Z] = D(Z, X)Y = 0.$$

Hence from the identity (2.3), we obtain

$$\sum_{(X,Y,Z)} D(X, Y)Z + [[X, Y], Z] = 0.$$

Similarly, it holds for the other cases. This completes the proof.

Remark. We note that the identity for Lie related triples is equivalent to the identity for the derivation $D(X, Y)$.

References

[1] B.N. Allison and J.R. Faulkner, *Nonassociative algebras for Steinberg unitary Lie algebra*, J. Algebra, **161** (1993), 1-19.

[2] N. Jacobson, *Triality and Lie algebras of type D_4*, Rend. Circ. Mat. Palermo, **13** (1968), 1-25.

[3] N. Kamiya, *On a generalization of structurable algebras*, Algebras, Groups and Geometries, **9** (1932), 35-47.

[4] N. Kamiya, *On Freudenthal-Kantor triple systems and generalized structurable algebra*, in: "Math. and Its Applications", **303** (1994), 198-203.

[5] N. Kamiya, *On generalized structurable algebras and Lie related triples*, Advances in Appl. Algebras, **5**, (1995),No.2, 127-140.

[6] R.D. Shafer, *Introduction to nonassociative algebras*, Academic Press., 1966.

Department of Mathematics and Informatics,
"Mircea cel Batran" Naval Academy,
1, Fulgerului Street
8700 Constanta
Romania E-mail: gica@al.math.unibuc.ro