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LIMITING BEHAVIOR AND ANALYTICITY OF TWO SPECIAL TYPES OF INFEASIBLE WEIGHTED CENTRAL PATHS IN SEMIDEFINITE PROGRAMMING

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Abstract. The central path is the most important concept in the theory of interior point methods. It is an analytic curve in the interior of the feasible set which tends to an optimal point at the boundary. The analyticity properties of the paths are connected with the analysis of the superlinear convergence of the interior point algorithms for semidefinite programming. In this paper we study the analyticity of two special types of weighted central paths in semidefinite programming, under the condition of the existence of the strictly complementary solution.

1. Introduction

Denote S^n the vector space of all $n \times n$ symmetric matrices. In this paper we consider the following primal-dual pair SDP problems in the standard form

$$\begin{array}{llcll} & \text{minimize} & \mathbf{X} \bullet \mathbf{C} & & \\ (1) & \text{subject to} & \mathbf{A}^i \bullet \mathbf{X} & = & b_i, \quad \text{for all } i = 1, \dots, m, \\ & & \mathbf{X} & \succeq & 0, \end{array}$$

and

$$\begin{array}{llcll} & \text{maximize} & b^T y & & \\ (2) & \text{subject to} & \sum_{i=1}^{m} \mathbf{A}^i y_i + \mathbf{S} & = & \mathbf{C}, \\ & & \mathbf{S} & \succeq & 0, \end{array}$$

where the data consists of $\mathbf{C} \in S^n, b \in R^m$ and $\mathbf{A}^i \in S^n$ for all $i = 1, \dots, m$. The primal variable is $\mathbf{X} \in S^n$ and the dual variable consists of $(\mathbf{S}, y) \in S^n \times R^m$. We will denote S^n_+ and S^n_{++} the sets of positive semidefinite and positive definite matrices, respectively. We will write $X \succeq 0$ or $X \succ 0$, if $X \in S^n_+$, or $X \in S^n_{++}$ respectively.

Given fixed $W \in S^n_{++}$, $\Delta b \in R^m$ and $\Delta C \in S^n$, our aim is to study two types of weighted central path, which are implicitly defined by the $\mu > 0$ following

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parameterized system of nonlinear equations

$$\mathbf{A}^i \bullet \mathbf{X} = b_i + \mu \triangle b_i, \qquad i = 1, \dots, m, \mathbf{X} \succ 0, \tag{3}$$

$$\sum_{i=1}^{m} \mathbf{A}^i y_i + \mathbf{S} = \mathbf{C} + \mu \triangle \mathbf{C}, \qquad \mathbf{S} \succ 0, \tag{4}$$

$$\Phi_j(\mathbf{X}, \mathbf{S}) = \sqrt{\mu}\mathbf{W}. \tag{5}$$

Here $\Phi_j(\mathbf{X}, \mathbf{S})$, $j \in \{1, 2\}$, is a symmetrization map $\Phi_j : S^n_{++} \times S^n_{++} \to S^n$ which symmetrizes the product $\mathbf{XS}$, defined by:

$$\Phi_1(\mathbf{X}, \mathbf{S}) := (\mathbf{X}^{\frac{1}{2}}\mathbf{S}^{\frac{1}{2}} + \mathbf{S}^{\frac{1}{2}}\mathbf{X}^{\frac{1}{2}})/2, \tag{6}$$

$$\Phi_2(\mathbf{X}, \mathbf{S}) := ({\mathbf{U}_\mathbf{S}}^T \mathbf{L}_\mathbf{X} + {\mathbf{L}_\mathbf{X}}^T \mathbf{U}_\mathbf{S})/2, \tag{7}$$

where $\mathbf{X}^{\frac{1}{2}}$ and $\mathbf{L}_\mathbf{X}$ denote the square root and the lower Cholesky factor of the positive definite matrix $\mathbf{X}$, respectively, and $\mathbf{S}^{\frac{1}{2}}$ and $\mathbf{U}_\mathbf{S}$ denote the square root and the upper Cholesky factor of the positive definite matrix $\mathbf{S}$, respectively. The existence of these paths was established in [**13**] (see also [**17, 18**]). It was shown that these paths are well defined for weights $\mathbf{W} \in \mathcal{M}_{\frac{1}{3\sqrt{2}}}$, where

$$\mathcal{M}_{\frac{1}{3\sqrt{2}}} = \left\{ \mathbf{M} \in S^n_{++}; \exists \nu : \|\mathbf{M} - \nu \mathbf{I}\|_F < \frac{1}{3\sqrt{2}} \right\}$$

($\|.\|_F$ is the Frobenius norm defined for $\mathbf{A} \in R^{n \times n}$ as $\|\mathbf{A}\|_F = \sqrt{\mathrm{tr}(\mathbf{A}^T\mathbf{A})}$) and for a suitable choice of parameters $(\triangle b, \triangle \mathbf{C})$. It can be shown that if the condition number $\kappa(\mathbf{W}) < \frac{3\sqrt{2n}+1}{3\sqrt{2n}-1}$, then $\mathbf{W} \in \mathcal{M}_{\frac{1}{3\sqrt{2}}}$ (see [**18**, Lemma 3.3.1 and Proposition A.2.7(a)]). Therefore, under the mentioned conditions, the system (3)–(5) has a unique solution $p_j(\mu) = (\mathbf{X}(\mu), y(\mu), \mathbf{S}(\mu))$ for every $\mu > 0$.

If $(\mathbf{W}, \triangle b, \triangle \mathbf{C}) = (\mathbf{I}, 0, 0)$, then both the paths defined in (3)–(5) are identical to the central path associated with the problems (1), (2) (see [**18**, Lemma 3.4.3]). Properties of the central path, including the limiting behavior and the analyticity, were studied in the works [**4, 6, 7, 8, 9, 16**]. In linear programming, the notion of the central path can be easily extended to the notion of the weighted central path – by defining the weighted logarithmic barrier functions. This approach was possible only for a special type of the weighted path in SDP, associated with so-called Cholesky type symmetrization and positive diagonal weight, see [**1**]. A general approach was presented by authors of [**13**], where various types of weighted central paths were defined implicitly as a solution of the system consisting of (3), (4) and an equation of the form $\Phi(\mathbf{X}, \mathbf{S}) = \phi(\mu)\mathbf{W}$. Besides the paths studied in this paper, also paths associated with symmetrizations $\Phi_{AHO}(\mathbf{X}, \mathbf{S}) := (\mathbf{XS} + \mathbf{SX})/2$, $\Phi_{SR}(\mathbf{X}, \mathbf{S}) := \mathbf{X}^{\frac{1}{2}}\mathbf{S}\mathbf{X}^{\frac{1}{2}}$, $\Phi_{CH}(\mathbf{X}, \mathbf{S}) := {\mathbf{L}_\mathbf{X}}^T \mathbf{S} \mathbf{L}_\mathbf{X}$ were discussed. The existence of these paths was studied in the works [**13, 14, 17**]. The results concerning the limiting behavior and analyticity were obtained under the assumption of the existence of the strictly complementary solution. (An optimal solution $(\mathbf{X}, y, \mathbf{S})$ of the problems (1), (2) is called strictly complementary, if $\mathbf{X} + \mathbf{S} \succ 0$.) The analyticity of the weighted paths at the boundary point was studied by several authors. In the papers [**12, 15**] it was shown that the paths associated with the symmetrization Φ_{AHO} is an analytic function of μ at $\mu = 0$. The authors of [**11**] proved that the weighted path associated with the square-root-type symmetrization Φ_{SR} is analytic at $\mu = 0$ as a function of $\sqrt{\mu}$. Finally, in the work [**2**] it was shown that the weighted path associated with the Cholesky-type symmetrization Φ_{CH} and positive diagonal weight is an analytic function of μ at $\mu = 0$. In the paper [**10**] (see also [**18**]) the weighted path associated with Cholesky-type symmetrization and a suitable symmetric positive definite weight was studied and it was proved that this path is analytic at $\mu = 0$ as a function of $\sqrt{\mu}$. Moreover, it was shown that the weighted paths (associated with both − the square-root-type and Cholesky-type symmetrization) are analytic functions of μ (at the boundary point) if and only if the weight matrix is block diagonal. The aim of this paper is to complete the above results and to show that the weighted central paths associated with symmetrizations (6) and (7) are analytic at $\mu = 0$ as a function of $\sqrt{\mu}$.

1.1. Notation

Denote R_{++} the set of all positive real numbers, i.e. $R_{++} = (0, \infty)$. The vector space of all symmetric $n \times n$ matrices is denoted by S^n. We will write $\mathbf{A} \succeq 0$, or $\mathbf{A} \succ 0$ if $\mathbf{A}$ is positive semidefinite or positive definite, respectively. The cone of all positive semidefinite (definite) matrices is denoted by S^n_+ (S^n_{++}). Similarly, we will denote L^n and U^n the vector spaces of all lower and upper triangular matrices. The cones of all matrices from L^n with nonnegative (positive) diagonal entries are denoted L^n_+ (L^n_{++}) and the cones of all matrices from U^n with nonnegative (positive) diagonal entries are denoted U^n_+ (U^n_{++}). For given matrices $\mathbf{A}, \mathbf{B} \in R^{p\times q}$, the standard inner product is defined by $\mathbf{A} \bullet \mathbf{B} = \mathrm{tr}(\mathbf{A}^T\mathbf{B})$, where tr(.) denotes the trace of a matrix. The Frobenius norm of $\mathbf{B} \in R^{p\times q}$ is defined as $\|\mathbf{B}\|_F = \sqrt{\mathbf{B} \bullet \mathbf{B}}$. The spectral norm on $R^{n\times n}$ is defined as $\|\mathbf{B}\|_2 = \max_i\{\lambda_i(\mathbf{B}\mathbf{B}^T)\}$.

For a matrix function $\mathbf{A} : R_{++} \to R^{p\times q}$ we will use the standard $\mathcal{O}$-notation, that is, if $f : R_{++} \to R_{++}$ is a real function, we will write $\mathbf{A}(\mu) = \mathcal{O}(f(\mu))$ if it holds $\|\mathbf{A}(\mu)\|_F \le \gamma f(\mu)$ for some a positive constant γ and a small $\mu > 0$. Moreover, for matrix function $\mathbf{A} : R_{++} \to S^n$ we will write $\mathbf{A}(\mu) = \Theta(f(\mu))$ if there exists a constant $\alpha > 0$ such that $\frac{\mathbf{A}(\mu)}{f(\mu)} - \frac{1}{\alpha}\mathbf{I} \succeq 0$ and $\alpha\mathbf{I} - \frac{\mathbf{A}(\mu)}{f(\mu)} \succeq 0$. Similarly, for matrix function $\mathbf{A} : R_{++} \to L^n$ we will write $\mathbf{A}(\mu) = \Theta(f(\mu))$ if there exists a constant $\alpha > 0$ such that $\frac{\mathbf{A}(\mu)}{f(\mu)} - \frac{1}{\alpha}\mathbf{I} \in L^n_+$ and $\alpha\mathbf{I} - \frac{\mathbf{A}(\mu)}{f(\mu)} \in L^n_+$ and for matrix function $\mathbf{A} : R_{++} \to U^n$ we will write $\mathbf{A}(\mu) = \Theta(f(\mu))$ if there exists a constant $\alpha > 0$ such that $\frac{\mathbf{A}(\mu)}{f(\mu)} - \frac{1}{\alpha}\mathbf{I} \in U^n_+$ and $\alpha\mathbf{I} - \frac{\mathbf{A}(\mu)}{f(\mu)} \in U^n_+$.

2. Preliminaries

2.1. Assumptions

In this paper we will consider the following assumptions:

Assumption (A1): The matrices $\mathbf{A}_1, \dots, \mathbf{A}_m$ are linearly independent.

Assumption (A2): The parameters $\triangle b, \triangle \mathbf{C}$ are such that there exists $\mathbf{W}_0 \in \mathcal{M}_{\frac{1}{3\sqrt{2}}}$ and $\mu_0 > 0$ such that the system (3)–(5) is solvable for $\mathbf{W} = \mathbf{W}_0$ and $\mu = \mu_0$.

Assumption (A3): There exists a strictly complementary solution for (1), (2), that is a triple $(\mathbf{X}^*, y^*, \mathbf{S}^*)$ which is feasible and satisfies $\mathbf{X}^*\mathbf{S}^* = 0$ and $\mathbf{X}^* + \mathbf{S}^* \succ 0$.

Both of the assumptions (A1) and (A2), together with the assumption of the existence of the (not necessarily complementary) solution of the problems (1), (2) imply the welldefinedness of the central path which is stated in the following theorem.

Theorem 2.1. *Assume (A1), (A2) and that there exists a solution of the primal-dual pair* (1), (2). *Then, for any* $\mu \in (0, \mu_0\rangle$ *and any* $\mathbf{W} \in \mathcal{M}_{\frac{1}{3\sqrt{2}}}$, *there exists a unique solution* $(\mathbf{X}(\mu), y(\mu), \mathbf{S}(\mu))$ *of the system* (3)–(5). *Moreover, the path* $\mu \to (\mathbf{X}(\mu), y(\mu), \mathbf{S}(\mu))$ *is an analytic function for* $\mu > 0$.

For the proof see e.g. [**13, 17, 18**].

The Assumption (A1) ensures the one-to-one correspondence between the dual variables y and $\mathbf{S}$.

The Assumption (A2) is not restrictive – there always exist $\triangle b, \triangle \mathbf{C}$ such that this assumption is satisfied. We can choose $\mathbf{W}_0 \in \mathcal{M}_{\frac{1}{3\sqrt{2}}}$ and $\mu_0 > 0$ and pick up $(\mathbf{X}^0, y^0, \mathbf{S}^0) \in S^n_{++} \times R^m \times S^n_{++}$ such that

$$\Phi_j(\mathbf{X}^0, \mathbf{S}^0) = \sqrt{\mu_0}\mathbf{W}_0,$$

where $j \in \{1, 2\}$. If we set

$$\triangle b_i = \frac{\mathbf{A}_i \bullet \mathbf{X}^0 - b_i}{\sqrt{\mu_0}} \qquad \text{for all } i = 1, \dots, m$$
$$\triangle \mathbf{C} = \frac{\sum_{i=1}^m \mathbf{A}^i y_i^0 + \mathbf{S}^0 - \mathbf{C}}{\sqrt{\mu_0}},$$

then the triple $(\mathbf{X}^0, y^0, \mathbf{S}^0)$ is a solution of the system (3)–(5).

The Assumption (A3) is restrictive, though it is necessary for our analysis of the limiting behavior of the paths. It is also commonly used in the analysis of the superlinear convergence of the interior-point algorithms. Moreover, the results of [**3**] indicate that without this assumption the analytical properties of the central paths would be very difficult to describe. In linear programming, as a special case of semidefinite programming, the existence of an optimal solution implies the existence of a strictly complementary solution, but in general in SDP this is not necessarily true.

For readers convenience, we now provide an example of an (nonlinear) SDP problem satisfying the assumptions (A1)–(A3).

Example 2.1. Let $m = n = 3$, $\mathbf{X} = (x_{ij}), \mathbf{S} = (s_{ij}) \in S^3, y \in R^3$ be the unknown variables and let the data be given as follows:

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \mathbf{A}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathbf{A}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ b = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}.$$

Clearly, the data matrices $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ are linearly independent.
The primal SDP problem is equivalent to

$$\begin{array}{rl} \text{minimize} & -x_{11} \\ \text{subject to} & x_{31} = x_{32} = x_{33} = 0 \\ & x_{11} + x_{22} = 2 \\ & x_{11}x_{22} - x_{12}^2 \geq 0. \end{array}$$

It can be easily seen that the optimal solution of the problem is $\mathbf{X}^* = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

The dual SDP problem is equivalent to

$$\begin{array}{rl} \text{maximize} & 2y_1 \\ \text{subject to} & y_1 + s_{11} = -1 \\ & y_1 + s_{22} = y_2 + s_{33} = y_3 + s_{13} = 0 \\ & s_{12} = s_{23} = 0 \\ & s_{ii} \geq 0, \qquad i = 1, 2, 3 \\ & s_{11}s_{33} - s_{13}^2 \geq 0 \end{array}$$

The optimal solution set of the problem is

$$\mathcal{D}^* = \left\{ (y^*, \mathbf{S}^*) \ \middle| \ y^* = (-1, -a, 0), \quad \mathbf{S}^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}; \ a \geq 0 \right\}.$$

For any $a > 0$ the tripple $(\mathbf{X}^*, y^*, \mathbf{S}^*)$ is a strictly complementary optimal solution of the primal-dial pair of SDP problems.

Let $\mathbf{W}_0 = \mathbf{I}, \mu_0 = 1$. Then $\mathbf{X}^0 = \mathbf{S}^0 = \mathbf{I}$ satisfy the equality $\Phi_j(\mathbf{X}^0, \mathbf{S}^0) = \mathbf{I}$ for $j = 1, 2$. Let $y^0 = (0, 0, 0)$. The parameters

$$\triangle b = (0, 1, 0), \quad \triangle \mathbf{C} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

satisfy the Assumption (A2).

Let $(\mathbf{X}^*, y^*, \mathbf{S}^*)$ be a strictly complementary optimal solution. Since $\mathbf{X}^*\mathbf{S}^* = 0$, the matrices $\mathbf{X}^*, \mathbf{S}^*$ commute and therefore there exists an orthogonal matrix $\mathbf{Q}$ such that the matrices $\mathbf{Q}\mathbf{X}^*\mathbf{Q}^T$, $\mathbf{Q}\mathbf{S}^*\mathbf{Q}^T$ are diagonal. Therefore, without loss of generality (applying an orthogonal transformation on the data, if necessary), we may assume that

$$\mathbf{X}^* = \begin{pmatrix} \Lambda_B^* & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{S}^* = \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_N^* \end{pmatrix},$$

where $\Lambda_B^* = \mathrm{diag}(\lambda_1^*, \ldots, \lambda_{|B|}^*) \succ 0$, $\Lambda_N^* = \mathrm{diag}(\lambda_{|B|+1}^*, \ldots, \lambda_n^*) \succ 0$.

Let $(\hat{\mathbf{X}}, \hat{y}, \hat{\mathbf{S}})$ be another (not necessarily strictly complementary) optimal solution of the primal-dual pair (1), (2). From the complementarity property it follows that any optimal solution pair $(\hat{\mathbf{X}}, \hat{\mathbf{S}})$ is in the form

$$\hat{\mathbf{X}} = \begin{pmatrix} \hat{\mathbf{X}}_B & 0 \\ 0 & 0 \end{pmatrix}, \qquad \hat{\mathbf{S}} = \begin{pmatrix} 0 & 0 \\ 0 & \hat{\mathbf{S}}_N \end{pmatrix},$$

where $\hat{\mathbf{X}}_B \succeq 0$, $\hat{\mathbf{S}}_N \succeq 0$.

In what follows, we will assume that any square symmetric matrix $\mathbf{M} \in S^n$ has the partition

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_B & \mathbf{M}_V \\ \mathbf{M}_V^T & \mathbf{M}_N \end{pmatrix} \tag{8}$$

and we will denote $|B| \times |B|$ the dimension of the square block $\mathbf{M}_B$ and $|N| \times |N|$ the dimension of the square block $\mathbf{M}_N$.

2.2. Asymptotic behavior

In the following we give results concerning the asymptotic behavior of the blocks $\mathbf{X}_B(\mu)$, $\mathbf{X}_V(\mu)$, $\mathbf{X}_N(\mu)$, $\mathbf{S}_B(\mu)$, $\mathbf{S}_V(\mu)$, $\mathbf{S}_N(\mu)$ of the matrix functions $\mathbf{X}(\mu), \mathbf{S}(\mu)$, and also the asymptotic behavior of the blocks of the functions $[\mathbf{X}(\mu)]^{\frac{1}{2}}$, $[\mathbf{X}(\mu)]^{\frac{1}{2}}$, $\mathbf{L}_{\mathbf{X}(\mu)}$ and $\mathbf{U}_{\mathbf{S}(\mu)}$ for $\mu \to 0$. All the properties hold for both paths studied in this paper.

The results stated in this section can be proved using the standard techniques (see e.g. [**12, 11, 15**]), therefore they are omitted. For details see [**18**].

Proposition 2.1. *For $\mu \in (0, \mu_0\rangle$ sufficiently small it holds*

$$\mathbf{X}(\mu) = \mathcal{O}(1), \quad y(\mu) = \mathcal{O}(1), \quad \mathbf{S}(\mu) = \mathcal{O}(1).$$

Proposition 2.2. *The weighted paths posses the following asymptotic behavior:*

$$\mathbf{X}(\mu) = \begin{pmatrix} \Theta(1) & \mathcal{O}(\sqrt{\mu}) \\ \mathcal{O}(\sqrt{\mu}) & \Theta(\mu) \end{pmatrix}, \quad \mathbf{S}(\mu) = \begin{pmatrix} \Theta(\mu) & \mathcal{O}(\sqrt{\mu}) \\ \mathcal{O}(\sqrt{\mu}) & \Theta(1) \end{pmatrix}. \tag{9}$$

Denote

$$\mathbf{Y}(\mu) := [\mathbf{X}(\mu)]^{\frac{1}{2}}, \qquad \mathbf{Z}(\mu) := [\mathbf{S}(\mu)]^{\frac{1}{2}} \tag{10}$$

the square roots of the matrices $\mathbf{X}(\mu)$ and $\mathbf{S}(\mu)$, which exist and are uniquely defined. Obviously

$$\begin{aligned}
\mathbf{X}_B(\mu) &= \mathbf{Y}_B^2(\mu) + \mathbf{Y}_V(\mu)\mathbf{Y}_V^T(\mu),\\
\mathbf{S}_B(\mu) &= \mathbf{Z}_B^2(\mu) + \mathbf{Z}_V(\mu)\mathbf{Z}_V^T(\mu),\\
\mathbf{X}_V(\mu) &= \mathbf{Y}_B(\mu)\mathbf{Y}_V(\mu) + \mathbf{Y}_V(\mu)\mathbf{Y}_N(\mu),\\
\mathbf{S}_V(\mu) &= \mathbf{Z}_B(\mu)\mathbf{Z}_V(\mu)] + \mathbf{Z}_V(\mu)\mathbf{Z}_N(\mu),\\
\mathbf{X}_N(\mu) &= \mathbf{Y}_N^2(\mu) + \mathbf{Y}_V^T(\mu)\mathbf{Y}_V(\mu),\\
\mathbf{S}_N(\mu) &= \mathbf{Z}_N^2(\mu) + \mathbf{Y}_V^T(\mu)\mathbf{Y}_V(\mu).
\end{aligned} \tag{11}$$

The asymptotic behavior of the square roots is stated in the following proposition.

Proposition 2.3. *It holds*

$$\mathbf{Y}(\mu) = \begin{pmatrix} \Theta(1) & \mathcal{O}(\sqrt{\mu}) \\ \mathcal{O}(\sqrt{\mu}) & \Theta(\sqrt{\mu}) \end{pmatrix}, \quad \mathbf{Z}(\mu) = \begin{pmatrix} \Theta(\sqrt{\mu}) & \mathcal{O}(\sqrt{\mu}) \\ \mathcal{O}(\sqrt{\mu}) & \Theta(1) \end{pmatrix} . \tag{12}$$

Denote $\mathbf{L}(\mu) := \mathbf{L}_{\mathbf{X}(\mu)} \in L_{++}^n$ the lower Cholesky factor of the matrices $\mathbf{X}(\mu)$ and $\mathbf{U}(\mu) := \mathbf{U}_{\mathbf{S}(\mu)} \in U_{++}^n$ the upper Cholesky factor of the matrices $\mathbf{S}(\mu)$ (which exist and are uniquely determined). It holds

$$\mathbf{X}(\mu) = \mathbf{L}(\mu)\mathbf{L}^T(\mu), \qquad \mathbf{S}(\mu) = \mathbf{U}(\mu)\mathbf{U}^T(\mu),$$

where we denote $\mathbf{L}^T(\mu) := (\mathbf{L}(\mu))^T$ and $\mathbf{U}^T(\mu) := (\mathbf{U}(\mu))^T$. Assume that any lower triangular matrix $\mathbf{L}$ and upper triangular matrix $\mathbf{U}$ is partitioned in the following way:

$$\mathbf{L} = \begin{pmatrix} \mathbf{L}_B & 0 \\ \mathbf{L}_V^T & \mathbf{L}_N \end{pmatrix}, \qquad \mathbf{U} = \begin{pmatrix} \mathbf{U}_B & \mathbf{U}_V \\ 0 & \mathbf{U}_N \end{pmatrix}.$$

Then the associated blocks satisfy the following equalities:

$$\begin{aligned}
\mathbf{X}_B(\mu) &= \mathbf{L}_B(\mu)\mathbf{L}_B^T(\mu),\\
\mathbf{S}_B(\mu) &= \mathbf{U}_B(\mu)\mathbf{U}_B^T(\mu) + \mathbf{U}_V(\mu)\mathbf{U}_V^T(\mu),\\
\mathbf{X}_V(\mu) &= \mathbf{L}_B(\mu)\mathbf{L}_V(\mu),\\
\mathbf{S}_V(\mu) &= \mathbf{U}_V(\mu)\mathbf{U}_N^T(\mu),\\
\mathbf{X}_N(\mu) &= \mathbf{L}_V^T(\mu)\mathbf{L}_V(\mu) + \mathbf{L}_N(\mu)\mathbf{L}_N^T(\mu),\\
\mathbf{S}_N(\mu) &= \mathbf{U}_N(\mu)\mathbf{U}_N^T(\mu).
\end{aligned} \tag{13}$$

The asymptotic behavior of the Cholesky factors is stated in the following proposition.

Proposition 2.4. *It holds*

$$\begin{aligned}
\mathbf{L}(\mu) = \mathbf{L}_{\mathbf{X}(\mu)} &= \begin{pmatrix} \Theta(1) & 0 \\ \mathcal{O}(\sqrt{\mu}) & \Theta(\sqrt{\mu}) \end{pmatrix},\\
\mathbf{U}(\mu) = \mathbf{U}_{\mathbf{S}(\mu)} &= \begin{pmatrix} \Theta(\sqrt{\mu}) & \mathcal{O}(\sqrt{\mu}) \\ 0 & \Theta(1) \end{pmatrix}
\end{aligned} \tag{14}$$

Let $\rho := \sqrt{\mu}$. In the following we introduce the normalized matrices $\widetilde{\mathbf{X}}(\rho)$, $\widetilde{\mathbf{S}}(\rho)$, $\widetilde{\mathbf{Y}}(\rho)$, $\widetilde{\mathbf{Z}}(\rho)$, $\widetilde{\mathbf{L}}(\rho)$, $\widetilde{\mathbf{U}}(\rho)$ which will be useful in the further analysis.

$$\begin{aligned} \widetilde{\mathbf{X}}(\rho) &:= \begin{pmatrix} \mathbf{X}_B(\rho^2) & \mathbf{X}_V(\rho^2)/\rho \\ \mathbf{X}_V^T(\rho^2)/\rho & \mathbf{X}_N(\rho^2)/\rho^2 \end{pmatrix}, \\ \widetilde{\mathbf{S}}(\rho) &:= \begin{pmatrix} \mathbf{S}_B(\rho^2)/\rho^2 & \mathbf{S}_V(\rho^2)/\rho \\ \mathbf{S}_V^T(\rho^2)/\rho & \mathbf{S}_N(\rho^2) \end{pmatrix} \end{aligned} \tag{15}$$

$$\begin{aligned} \widetilde{\mathbf{Y}}(\rho) &:= \begin{pmatrix} \mathbf{Y}_B(\rho^2) & \mathbf{Y}_V(\rho^2)/\rho \\ \mathbf{Y}_V^T(\rho^2)/\rho & \mathbf{Y}_N(\rho^2)/\rho \end{pmatrix}, \\ \widetilde{\mathbf{Z}}(\rho) &:= \begin{pmatrix} \mathbf{Z}_B(\rho^2)/\rho & \mathbf{Z}_V(\rho^2)/\rho \\ \mathbf{Z}_V^T(\rho^2)/\rho & \mathbf{Z}_N(\rho^2) \end{pmatrix} \end{aligned} \tag{16}$$

$$\begin{aligned} \widetilde{\mathbf{L}}(\rho) &:= \begin{pmatrix} \mathbf{L}_B(\rho^2) & 0 \\ \mathbf{L}_V^T(\rho^2)/\rho & \mathbf{L}_N(\rho^2)/\rho \end{pmatrix}, \\ \widetilde{\mathbf{U}}(\rho) &:= \begin{pmatrix} \mathbf{U}_B(\rho^2)/\rho & \mathbf{U}_V(\rho^2)/\rho \\ 0 & \mathbf{U}_N(\rho^2) \end{pmatrix} \end{aligned} \tag{17}$$

Note that from the statements in Proposition 2.2, Proposition 2.3 and Proposition 2.4 it follows that the normalized matrices satisfy:

$$\widetilde{\mathbf{X}}(\rho) = \widetilde{\mathbf{S}}(\rho) = \widetilde{\mathbf{Y}}(\rho) = \widetilde{\mathbf{Z}}(\rho) = \widetilde{\mathbf{L}}(\rho) = \widetilde{\mathbf{U}}(\rho) = \mathcal{O}(1),$$

moreover, the diagonal blocks of all normalized matrices exhibit the following behavior:

$$\widetilde{\mathbf{X}}_B(\rho) = \widetilde{\mathbf{S}}_B(\rho) = \widetilde{\mathbf{Y}}_B(\rho) = \widetilde{\mathbf{Z}}(\rho)_B = \widetilde{\mathbf{L}}_B(\rho) = \widetilde{\mathbf{U}}(\rho)_B = \Theta(1),$$

$$\widetilde{\mathbf{X}}_N(\rho) = \widetilde{\mathbf{S}}_N(\rho) = \widetilde{\mathbf{Y}}_N(\rho) = \widetilde{\mathbf{Z}}_N(\rho) = \widetilde{\mathbf{L}}_N(\rho) = \widetilde{\mathbf{U}}_N(\rho) = \Theta(1).$$

Define $\widetilde{y}(\rho) = y(\mu) = \mathcal{O}(1)$ (see Proposition 2.1). From the asymptotic behavior stated above it follows that for any sequence $\{\rho_k\} \to 0$, the matrix sequences $\widetilde{\mathbf{X}}(\rho_k)$, $\widetilde{\mathbf{S}}(\rho_k)$ and the vector $\widetilde{y}(\rho_k)$ are bounded, hence there exists a convergent subsequence and we may assume that the limit

$$\lim_{k\to\infty} (\widetilde{\mathbf{X}}(\rho_k), \widetilde{y}(\rho_k), \widetilde{\mathbf{S}}(\rho_k)) = (\widetilde{\mathbf{X}}^*, \widetilde{y}^*, \widetilde{\mathbf{S}}^*) \tag{18}$$

exists (though the limit point is not necessary unique). Moreover, from Proposition 2.2 it follows that the matrices $\widetilde{\mathbf{X}}^*_B, \widetilde{\mathbf{X}}^*_N, \widetilde{\mathbf{S}}^*_B, \widetilde{\mathbf{S}}^*_N$ are positive definite.

3. Analyticity of the paths at the boundary point

The aim of this section is to prove the main result of this paper which is stated in the following theorem

Theorem 3.1. *The weighted paths* $(\mathbf{X}(\mu), y(\mu), \mathbf{S}(\mu))$ *associated with symmetrization maps defined in* (6), (7) *are analytic functions of* $\sqrt{\mu}$ *for all* $\mu \geq 0$.

3.1. Feasibility conditions

The first step in proving Theorem 3.1 is the transformation the feasibility conditions to an equivalent system with a special property which is stated in the following theorem.

Theorem 3.2. *There exists a map*

$$\Psi : S^n \times R^m \times S^n \times R \to S^n \times R^m,$$

such that for any $\rho > 0$*, it holds*

$$\Psi(\widetilde{\mathbf{X}}(\rho), \widetilde{y}(\rho), \widetilde{\mathbf{S}}(\rho), \rho) = 0$$

if and only if $(\mathbf{X}(\mu), y(\mu), \mathbf{S}(\mu), \mu)$ *satisfies the feasibility conditions* (3), (4)*, that is*

$$\mathbf{A}^i \bullet \mathbf{X}(\mu) = b_i + \mu \triangle b_i, \;\; i = 1, \ldots, m,$$

$$\sum_{i=1}^{m} \mathbf{A}^i y_i(\mu) + \mathbf{S}(\mu) = \mathbf{C} + \mu \triangle \mathbf{C}.$$

Moreover, the condition

$$D\Psi(\widetilde{\mathbf{X}}^*, \widetilde{y}^*, \widetilde{\mathbf{S}}^*, 0)[\triangle\widetilde{\mathbf{X}}, \triangle\widetilde{y}, \triangle\widetilde{\mathbf{S}}] = 0,$$

where $(\widetilde{\mathbf{X}}^*, \widetilde{y}^*, \widetilde{\mathbf{S}}^*)$ *is the limit point from* (18)*,* $D\Psi(\widetilde{\mathbf{X}}^*, \widetilde{y}^*, \widetilde{\mathbf{S}}^*, 0)$ *is the (partial) Fréchet derivative of the map* Ψ *with respect to variables* $(\widetilde{\mathbf{X}}, \widetilde{y}, \widetilde{\mathbf{S}})$ *at the point* $(\widetilde{\mathbf{X}}^*, \widetilde{y}^*, \widetilde{\mathbf{S}}^*, 0)$*, implies* $\triangle\widetilde{\mathbf{X}} \bullet \triangle\widetilde{\mathbf{S}} = 0$*.*

The proof of the above theorem, including the construction of the map Ψ can be found in all details in Section 3.2 and Section 3.3 of [**10**] or in Section 4.2.1 and Section 4.2.2 of [**18**], therefore it is omitted. A different approach transformation of the feasibility conditions was used in [**11**] or [**15**].

3.2. Nonsingularity of Fréchet derivatives

Consider the symmetrization map $\Phi_1(\mathbf{X}, \mathbf{S}) = (\mathbf{X}^{\frac{1}{2}}\mathbf{S}^{\frac{1}{2}} + \mathbf{X}^{\frac{1}{2}}\mathbf{S}^{\frac{1}{2}})/2$. In this case, the last condition in the system (3)-(5) is of the form

$$(\mathbf{X}^{\frac{1}{2}}\mathbf{S}^{\frac{1}{2}} + \mathbf{X}^{\frac{1}{2}}\mathbf{S}^{\frac{1}{2}})/2 = \sqrt{\mu}\mathbf{W}$$

and can be equivalently rewritten as

$$\mathbf{YZ} + \mathbf{ZY} = 2\sqrt{\mu}\mathbf{W},$$

$$\mathbf{Y}^2 = \mathbf{X},$$

$$\mathbf{Z}^2 = \mathbf{S}.$$

Let $\mathcal{U}^n_{BN}$ be the vector space of all upper block triangular matrices with symmetric diagonal blocks of dimensions $|B| \times |B|$ and $|N| \times |N|$. Let L be the linear map[1]

$$L : \mathcal{U}^n_{BN} \to R^{n\times n}, \quad L : \left(\begin{bmatrix} \mathbf{M}_B & \mathbf{M}_V \\ 0 & \mathbf{M}_N \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ \mathbf{M}_V^T & 0 \end{bmatrix}.$$

[1]The idea of defining this map was used by Lu and Monteiro in [**11**].

Define

$$\widetilde{\mathbf{U}}_Y(\rho) := \begin{pmatrix} \mathbf{Y}_B(\rho^2) & \mathbf{Y}_V(\rho^2)/\rho \\ 0 & \mathbf{Y}_N(\rho^2)/\rho \end{pmatrix}, \quad \widetilde{\mathbf{U}}_Y(\rho) := \begin{pmatrix} \mathbf{Z}_B(\rho^2)/\rho & \mathbf{Z}_V(\rho^2)/\rho \\ 0 & \mathbf{Z}_N(\rho^2) \end{pmatrix}.$$

Lemma 3.1. *For any $\rho = \sqrt{\mu} > 0$, the systems*

$$\begin{aligned} \mathbf{Y}(\mu)\mathbf{Z}(\mu) + \mathbf{Z}(\mu)\mathbf{Y}(\mu) &= 2\sqrt{\mu}\mathbf{W} \\ \mathbf{Y}(\mu)^2 &= \mathbf{X}(\mu) \\ \mathbf{Z}(\mu)^2 &= \mathbf{S}(\mu). \end{aligned}$$

and

$$\begin{aligned} [\widetilde{\mathbf{U}}_Y(\rho) + \rho L(\widetilde{\mathbf{U}}_Y(\rho))][\widetilde{\mathbf{U}}_Z(\rho) + \rho L(\widetilde{\mathbf{U}}_Z(\rho))] & \\ +[\widetilde{\mathbf{U}}_Z(\rho) + \rho L(\widetilde{\mathbf{U}}_Z(\rho))]^T[\widetilde{\mathbf{U}}_Y(\rho) + \rho L(\widetilde{\mathbf{U}}_Y(\rho))]^T &= 2\mathbf{W} \\ [\widetilde{\mathbf{U}}_Y(\rho) + \rho L(\widetilde{\mathbf{U}}_Y(\rho))]^T[\widetilde{\mathbf{U}}_Z(\rho) + \rho L(\widetilde{\mathbf{U}}_Z(\rho))] &= \widetilde{\mathbf{X}}(\rho) \\ [\widetilde{\mathbf{U}}_Z(\rho) + \rho L(\widetilde{\mathbf{U}}_Z(\rho))][\widetilde{\mathbf{U}}_Y(\rho) + \rho L(\widetilde{\mathbf{U}}_Y(\rho))]^T &= \widetilde{\mathbf{S}}(\rho) \end{aligned}$$

are equivalent.

Proof. Follows from simple computation. □

From the asymptotic behavior stated in Section 2.2 it follows that the sequence

$$(\widetilde{\mathbf{X}}(\rho_k), \widetilde{\mathbf{U}}_Y(\rho_k), \widetilde{y}(\rho_k), \widetilde{\mathbf{S}}(\rho_k), \widetilde{\mathbf{U}}_Z(\rho_k))$$

is bounded for any $\{\rho_k\} \to 0$, hence there exists a convergent subsequence and we may assume that the following limit

$$\lim_{k\to\infty}(\widetilde{\mathbf{X}}(\rho_k), \widetilde{\mathbf{U}}_Y(\rho_k), \widetilde{y}(\rho_k), \widetilde{\mathbf{S}}(\rho_k), \widetilde{\mathbf{U}}_Z(\rho_k)) = (\widetilde{\mathbf{X}}^*, \widetilde{\mathbf{U}}_Y^*, \widetilde{y}^*, \widetilde{\mathbf{S}}^*, \widetilde{\mathbf{U}}_Z^*)$$

exists. Define the map $\widetilde{F}^1$ as follows

$$\widetilde{F}^1(\widetilde{\mathbf{X}}, \widetilde{\mathbf{U}}_Y, \widetilde{y}, \widetilde{\mathbf{S}}, \widetilde{\mathbf{U}}_Z, \rho) = \begin{bmatrix} \Psi(\widetilde{\mathbf{X}}, \widetilde{y}, \widetilde{\mathbf{S}}, \rho) \\ (\widetilde{\mathbf{U}}_Y + \rho L(\widetilde{\mathbf{U}}_Y))(\widetilde{\mathbf{U}}_Z + \rho L(\widetilde{\mathbf{U}}_Z)) + (\widetilde{\mathbf{U}}_Z + \rho L(\widetilde{\mathbf{U}}_Z))^T(\widetilde{\mathbf{U}}_Y + \rho L(\widetilde{\mathbf{U}}_Y))^T - 2\mathbf{W} \\ (\widetilde{\mathbf{U}}_Y + \rho L(\widetilde{\mathbf{U}}_Y))^T(\widetilde{\mathbf{U}}_Z + \rho L(\widetilde{\mathbf{U}}_Z)) - \widetilde{\mathbf{X}} \\ (\widetilde{\mathbf{U}}_Z + \rho L(\widetilde{\mathbf{U}}_Z))(\widetilde{\mathbf{U}}_Y + \rho L(\widetilde{\mathbf{U}}_Y))^T - \widetilde{\mathbf{S}} \end{bmatrix}.$$

From Theorem 3.2 and Lemma 3.1 it follows that for any $\rho = \sqrt{\mu} > 0$ the system $\widetilde{F}^1 = 0$ is equivalent to the system (3)–(5) in the sense that

$$\widetilde{F}^1(\widetilde{\mathbf{X}}(\rho), \widetilde{\mathbf{U}}_Y(\rho), \widetilde{y}(\rho), \widetilde{\mathbf{S}}(\rho), \widetilde{\mathbf{U}}_Z(\rho), \rho) = 0.$$

Moreover,

$$\widetilde{F}^1(\widetilde{\mathbf{X}}^*, \widetilde{\mathbf{U}}_Y^*, \widetilde{y}^*, \widetilde{\mathbf{S}}^*, \widetilde{\mathbf{U}}_Z^*, 0) = 0.$$

The Fréchet derivative of the map $\widetilde{F}^1$ at the point $(\widetilde{\mathbf{X}}^*, \widetilde{\mathbf{U}}_Y^*, \widetilde{y}^*, \widetilde{\mathbf{S}}^*, \widetilde{\mathbf{U}}_Z^*, 0)$ with respect to the variables $(\widetilde{\mathbf{X}}, \widetilde{\mathbf{U}}_Y, \widetilde{y}, \widetilde{\mathbf{S}}, \widetilde{\mathbf{U}}_Z)$ is the linear map

$$
\begin{aligned}
&D\widetilde{F}^1(\widetilde{\mathbf{X}}^*, \widetilde{\mathbf{U}}_Y^*, \widetilde{y}^*, \widetilde{\mathbf{S}}^*, \widetilde{\mathbf{U}}_Z^*, 0)[\triangle\widetilde{\mathbf{X}}, \triangle\widetilde{\mathbf{U}}_Y, \triangle\widetilde{y}, \triangle\widetilde{\mathbf{S}}, \triangle\widetilde{\mathbf{U}}_Z] \\
&= \left[\begin{array}{l}
D\Psi(\widetilde{\mathbf{X}}^*, \widetilde{y}^*, \widetilde{\mathbf{S}}^*, 0)[\triangle\widetilde{\mathbf{X}}, \triangle\widetilde{y}, \triangle\widetilde{\mathbf{S}}] \\
\triangle\widetilde{\mathbf{U}}_Y\widetilde{\mathbf{U}}_Z^* + \widetilde{\mathbf{U}}_Y^*\triangle\widetilde{\mathbf{U}}_Z + (\triangle\widetilde{\mathbf{U}}_Z)^T(\widetilde{\mathbf{U}}_Y^*)^T + (\widetilde{\mathbf{U}}_Z^*)^T(\triangle\widetilde{\mathbf{U}}_Y)^T \\
(\triangle\widetilde{\mathbf{U}}_Y)^T\widetilde{\mathbf{U}}_Y^* + (\widetilde{\mathbf{U}}_Y^*)^T\triangle\widetilde{\mathbf{U}}_Y - \triangle\widetilde{\mathbf{X}} \\
\triangle\widetilde{\mathbf{U}}_Z(\widetilde{\mathbf{U}}_Z^*)^T + \widetilde{\mathbf{U}}_Z^*(\triangle\widetilde{\mathbf{U}}_Z)^T - \triangle\widetilde{\mathbf{S}}
\end{array}\right].
\end{aligned}
$$

Our goal now is to prove that

$$D\widetilde{F}^1(\widetilde{\mathbf{X}}^*, \widetilde{\mathbf{U}}_Y^*, \widetilde{y}^*, \widetilde{\mathbf{S}}^*, \widetilde{\mathbf{U}}_Z^*, 0)[\triangle\widetilde{\mathbf{X}}, \triangle\widetilde{\mathbf{U}}_Y, \triangle\widetilde{y}, \triangle\widetilde{\mathbf{S}}, \triangle\widetilde{\mathbf{U}}_Z]$$

is a nonsingular linear map. Fot this aim we state several auxiliary lemmas. First, denote

$$\mathcal{U}_{++}^n = \left\{\mathbf{M} \in \mathcal{U}_{BN}^n;\ \mathbf{M}_B \succ 0, \mathbf{M}_N \succ 0\right\}.$$

Lemma 3.2.

a) *If* $\mathbf{M} \in \mathcal{U}_{++}^n$, *then* $\mathbf{M}^{-1} \in \mathcal{U}_{++}^n$.

b) *If* $\mathbf{M} \in \mathcal{U}_{++}^n$ *and* $\mathbf{H} \in \mathcal{U}_{BN}^n$ *are such that* $\mathbf{MH} + \mathbf{H}^T\mathbf{M}^T = \mathbf{W}$ *for some* $\mathbf{W} \in S^n$, *then*

$$\|\mathbf{MH}\|_F \le \frac{\|\mathbf{W}\|_F}{\sqrt{2}}.$$

Proof. a) The statement follows from properties of block matrices and positive definiteness.
b) It holds

$$
\begin{aligned}
\mathrm{tr}(\mathbf{MHMH}) &= \mathrm{tr}(\mathbf{M}_B\mathbf{H}_B\mathbf{M}_B\mathbf{H}_B) + \mathrm{tr}(\mathbf{M}_N\mathbf{H}_N\mathbf{M}_N\mathbf{H}_N) \\
&= \mathrm{tr}(\mathbf{M}_B^{\frac{1}{2}}\mathbf{H}_B\mathbf{M}_B\mathbf{H}_B\mathbf{M}_B^{\frac{1}{2}}) + \mathrm{tr}(\mathbf{M}_N^{\frac{1}{2}}\mathbf{H}_N\mathbf{M}_N\mathbf{H}_N\mathbf{M}_N^{\frac{1}{2}}) \ge 0.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|\mathbf{W}\|_F^2 &= (\mathbf{MH} + \mathbf{H}^T\mathbf{M}^T) \bullet (\mathbf{MH} + \mathbf{H}^T\mathbf{M}^T) \\
&= 2\mathrm{tr}(\mathbf{MHH}^T\mathbf{M}^T) + 2\mathrm{tr}(\mathbf{MHMH}) \ge 2\|\mathbf{MH}\|_F^2.
\end{aligned}
$$

□

The next two lemmas contain simple, but usefull properties of matrix norms.

Lemma 3.3. *If* $\mathbf{A} \in S^n$ *and* $\mathbf{B} \in R^{n\times n}$, *then* $\|\mathbf{AB}\|_F \le \|\mathbf{A}\|_2\|\mathbf{B}\|_F$.

Lemma 3.4 (Lemma 8 of [**13**]). *Let* $\mathbf{B} \in R^{n\times n}$ *be a matrix with real eigenvalues and let* $\beta \in (0, \frac{1}{\sqrt{2}})$. *Then if* $\|\frac{\mathbf{B}+\mathbf{B}^T}{2} - \mathbf{I}\|_F \le \beta$, *then*

(a) $\|\mathbf{B} - \mathbf{I}\|_F \le \sqrt{2}\beta$;

(b) $\|\mathbf{B}^{-1}\|_2 \le \dfrac{1}{1 - \sqrt{2}\beta}$.

Proof. The following lemma is proved using similar techniques to those used in the proof of Proposition 4 of [**13**]. (See also Lemma 3.2.3 and Lemma 3.2.4 of [**18**].)

Lemma 3.5. *Let* $\mathbf{U}, \mathbf{V} \in \mathcal{U}^n_{++}$ *be given matrices and* $\gamma \in (0, \frac{1}{3\sqrt{2}})$. *If there exists* $\mu > 0$ *such that* $\|(\mathbf{UV} + \mathbf{V}^T\mathbf{U}^T)/2 - \mu\mathbf{I}\|_F \leq \gamma\mu$, *then for* $\triangle\mathbf{U}, \triangle\mathbf{V} \in \mathcal{U}^n_{BN}$ *and* $\triangle\mathbf{X}, \triangle\mathbf{S} \in S^n$ *the following implication holds*

$$\left.\begin{array}{rcl} \triangle\mathbf{UV} + \mathbf{U}\triangle\mathbf{V} + \triangle\mathbf{V}^T\mathbf{U}^T + \mathbf{V}^T\triangle\mathbf{U}^T & = & 0 \\ \triangle\mathbf{U}^T\mathbf{U} + \mathbf{U}^T\triangle\mathbf{U} & = & \triangle\mathbf{X} \\ \triangle\mathbf{VV}^T + \mathbf{V}\triangle\mathbf{V}^T & = & \triangle\mathbf{S} \\ \triangle\mathbf{X} \bullet \triangle\mathbf{S} & = & 0 \end{array}\right\}$$
$$\Longrightarrow \quad \triangle\mathbf{U} = \triangle\mathbf{V} = \triangle\mathbf{X} = \triangle\mathbf{S} = 0.$$

Assume that

$$\triangle\mathbf{UV} + \mathbf{U}\triangle\mathbf{V} + \triangle\mathbf{V}^T\mathbf{U}^T + \mathbf{V}^T\triangle\mathbf{U}^T = 0, \tag{19}$$
$$\triangle\mathbf{U}^T\mathbf{U} + \mathbf{U}^T\triangle\mathbf{U} = \triangle\mathbf{X}, \tag{20}$$
$$\triangle\mathbf{VV}^T + \mathbf{V}\triangle\mathbf{V}^T = \triangle\mathbf{S}, \tag{21}$$
$$\triangle\mathbf{X} \bullet \triangle\mathbf{S} = 0. \tag{22}$$

Obviously, the equations (20), (21) are equivalent to

$$\mathbf{U}^{-T}\triangle\mathbf{U}^T + \triangle\mathbf{UU}^{-1} = \mathbf{U}^{-T}\triangle\mathbf{XU}^{-1}, \quad \triangle\mathbf{V}^T\mathbf{V}^{-T} + \mathbf{V}^{-1}\triangle\mathbf{V} = \mathbf{V}^{-1}\triangle\mathbf{SV}^{-T}.$$

From Lemma 3.2 it follows that

$$\|\mathbf{U}^{-T}\triangle\mathbf{U}^T\|_F = \|\triangle\mathbf{UU}^{-1}\|_F \leq \frac{\|\mathbf{U}^{-T}\triangle\mathbf{XU}^{-1}\|_F}{\sqrt{2}}, \tag{23}$$
$$\|\triangle\mathbf{V}^T\mathbf{V}^{-T}\|_F = \|\mathbf{V}^{-1}\triangle\mathbf{V}\|_F \leq \frac{\|\mathbf{V}^{-1}\triangle\mathbf{SV}^{-T}\|_F}{\sqrt{2}}. \tag{24}$$

Define

$$\triangle\bar{\mathbf{X}} := \mathbf{V}^T\triangle\mathbf{XU}^{-1}, \quad \triangle\bar{\mathbf{S}} := \mathbf{V}^{-1}\triangle\mathbf{SU}^T.$$

It can be easily seen that the condition (22) implies $\triangle\bar{\mathbf{X}} \bullet \triangle\bar{\mathbf{S}} = 0$ and hence

$$\|\triangle\bar{\mathbf{X}} + \triangle\bar{\mathbf{S}}\|_F^2 = \|\triangle\bar{\mathbf{X}}\|_F^2 + \|\triangle\bar{\mathbf{S}}\|_F^2. \tag{25}$$

From (20), (21) it follows, that the matrices $\triangle\bar{\mathbf{X}}$, $\triangle\bar{\mathbf{S}}$ can be also expressed as

$$\triangle\bar{\mathbf{X}} = \mathbf{V}^T\mathbf{U}^T\triangle\mathbf{UU}^{-1} + \mathbf{V}^T\triangle\mathbf{U}^T, \quad \triangle\bar{\mathbf{S}} = \triangle\mathbf{V}^T\mathbf{U}^T + \mathbf{V}^{-1}\triangle\mathbf{VV}^T\mathbf{U}^T.$$

Therefore

$$\begin{aligned} \triangle\bar{\mathbf{X}} + \triangle\bar{\mathbf{S}} &= \mathbf{V}^T\mathbf{U}^T\triangle\mathbf{UU}^{-1} + \mathbf{V}^T\triangle\mathbf{U}^T + \triangle\mathbf{V}^T\mathbf{U}^T + \mathbf{V}^{-1}\triangle\mathbf{VV}^T\mathbf{U}^T \\ &= \mathbf{V}^T\mathbf{U}^T\triangle\mathbf{UU}^{-1} + \mathbf{V}^{-1}\triangle\mathbf{VV}^T\mathbf{U}^T - \triangle\mathbf{U}(\mathbf{U}^{-1}\mathbf{U})\mathbf{V} - \mathbf{U}(\mathbf{VV}^{-1})\triangle\mathbf{V} \\ &= (\mathbf{V}^T\mathbf{U}^T - \mu\mathbf{I})\triangle\mathbf{UU}^{-1} + \triangle\mathbf{UU}^{-1}(\mu\mathbf{I} - \mathbf{UV}) \\ &\quad + (\mu\mathbf{I} - \mathbf{UV})\mathbf{V}^{-1}\triangle\mathbf{V} + \mathbf{V}^{-1}\triangle\mathbf{V}(\mathbf{V}^T\mathbf{U}^T - \mu\mathbf{I}) \end{aligned}$$

and, by using (25), we obtain

$$
\begin{aligned}
&(\|\triangle\bar{\mathbf{X}}\|_F^2+\|\triangle\bar{\mathbf{S}}\|_F^2)^{\frac{1}{2}}\\
&=\|(\mathbf{V}^T\mathbf{U}^T-\mu\mathbf{I})\triangle\mathbf{U}\mathbf{U}^{-1}+\triangle\mathbf{U}\mathbf{U}^{-1}(\mu\mathbf{I}-\mathbf{U}\mathbf{V})\\
&\qquad+(\mu\mathbf{I}-\mathbf{U}\mathbf{V})\mathbf{V}^{-1}\triangle\mathbf{V}+\mathbf{V}^{-1}\triangle\mathbf{V}(\mathbf{V}^T\mathbf{U}^T-\mu\mathbf{I})\|_F\\
&\leq 2\|\mu\mathbf{I}-\mathbf{U}\mathbf{V}\|_F(\|\triangle\mathbf{U}\mathbf{U}^{-1}\|_F+\|\mathbf{V}^{-1}\triangle\mathbf{V}\|_F)\\
&\leq 2\sqrt{2}\gamma\mu(\|\triangle\mathbf{U}\mathbf{U}^{-1}\|_F+\|\mathbf{V}^{-1}\triangle\mathbf{V}\|_F)\\
&\leq 2\gamma\mu(\|\mathbf{U}^{-T}\triangle\mathbf{X}\mathbf{U}^{-1}\|_F+\|\mathbf{V}^{-1}\triangle\mathbf{S}\mathbf{V}^{-T}\|_F)\\
&\leq 2\gamma\mu\|\mathbf{V}^{-1}\mathbf{U}^{-1}\|_2(\|\mathbf{U}^{-T}\triangle\mathbf{X}\mathbf{V}\|_F+\|\mathbf{U}\triangle\mathbf{S}\mathbf{V}^{-T}\|_F)\\
&\leq \frac{2\gamma}{1-\sqrt{2}\gamma}(\|\mathbf{U}^{-T}\triangle\mathbf{X}\mathbf{V}\|_F+\|\mathbf{U}\triangle\mathbf{S}\mathbf{V}^{-T}\|_F)\\
&\leq \frac{2\gamma}{1-\sqrt{2}\gamma}(\|\triangle\bar{\mathbf{X}}\|_F^2+\|\triangle\bar{\mathbf{S}}\|_F^2)^{\frac{1}{2}},
\end{aligned}
\tag{26}
$$

where the inequalities follow from properties of matrix norms, Lemma 3.4a), (23), (24), Lemma 3.3 and Lemma 3.4b). Since $\gamma\in(0,\frac{1}{3\sqrt{2}})$, we have $\frac{2\gamma}{1-\sqrt{2}\gamma}<1$ which together with (26) imply $(\|\triangle\bar{\mathbf{X}}\|_F^2+\|\triangle\bar{\mathbf{S}}\|_F^2)^{\frac{1}{2}}=0$ and therefore also $\triangle\mathbf{X}=\triangle\mathbf{S}=0$. This fact, (23) and (24) give $\triangle\mathbf{U}=\triangle\mathbf{V}=0$. ☐

Proposition 3.1. *$D\widetilde{F}^1(\widetilde{\mathbf{X}}^*,\widetilde{\mathbf{U}}_Y^*,\widetilde{y}^*,\widetilde{\mathbf{S}}^*,\widetilde{\mathbf{U}}_Z^*,0)$ is a nonsingular linear map.*

Proof. Assume $D\widetilde{F}^1(\widetilde{\mathbf{X}}^*,\widetilde{\mathbf{U}}_Y^*,\widetilde{y}^*,\widetilde{\mathbf{S}}^*,\widetilde{\mathbf{U}}_Z^*,0)[\triangle\widetilde{\mathbf{X}},\triangle\widetilde{\mathbf{U}}_Y,\triangle\widetilde{y},\triangle\widetilde{\mathbf{S}},\triangle\widetilde{\mathbf{U}}_Z]=0$. Theorem 3.2 gives $\triangle\widetilde{\mathbf{X}}\bullet\triangle\widetilde{\mathbf{S}}=0$. It holds $\widetilde{\mathbf{U}}_Y^*\widetilde{\mathbf{U}}_Z^*+(\widetilde{\mathbf{U}}_Z^*)^T(\widetilde{\mathbf{U}}_Y^*)^T=2\mathbf{W}$ and from the asymptotic behavior it follows that $\widetilde{\mathbf{U}}_Y^*,\widetilde{\mathbf{U}}_Z^*\in\mathcal{U}_{++}^n$. Since $\mathbf{W}\in\mathcal{M}_{\frac{1}{3\sqrt{2}}}$, the assumptions of Lemma 3.5 are satisfied. Therefore $\triangle\widetilde{\mathbf{X}}=\triangle\widetilde{\mathbf{U}}_Y=\triangle\widetilde{\mathbf{S}}=\triangle\widetilde{\mathbf{U}}_Z=0$. Assumption (A1) yields $\triangle\widetilde{y}=0$. ☐

Analogously, we can prove a similar result for the symmetrization map $\Phi_2(\mathbf{X},\mathbf{S})=({\mathbf{U}_\mathbf{S}}^T\mathbf{L}_\mathbf{X}+{\mathbf{L}_\mathbf{X}}^T\mathbf{U}_\mathbf{S})/2$. In this case, the last condition in the system (3)–(5)

$$({\mathbf{U}_\mathbf{S}}^T\mathbf{L}_\mathbf{X}+{\mathbf{L}_\mathbf{X}}^T\mathbf{U}_\mathbf{S})/2=\sqrt{\mu}\mathbf{W}$$

can be equivalently rewritten as

$$
\begin{aligned}
\mathbf{U}^T\mathbf{L}+\mathbf{L}^T\mathbf{U}&=2\sqrt{\mu}\mathbf{W},\\
\mathbf{L}\mathbf{L}^T&=\mathbf{X},\\
\mathbf{U}\mathbf{U}^T&=\mathbf{S}.
\end{aligned}
$$

The following lemma can be proved by simple computation.

Lemma 3.6. *For any $\rho=\sqrt{\mu}>0$, the systems*

$$
\begin{aligned}
\mathbf{U}(\mu)^T\mathbf{L}(\mu)+\mathbf{L}(\mu)^T\mathbf{U}(\mu)&=2\sqrt{\mu}\mathbf{W}\\
\mathbf{L}(\mu)\mathbf{L}(\mu)^T&=\mathbf{X}(\mu)\\
\mathbf{U}(\mu)\mathbf{U}(\mu)^T&=\mathbf{S}(\mu).
\end{aligned}
$$

and

$$\begin{aligned} \widetilde{\mathbf{U}}(\rho)^T\widetilde{\mathbf{L}}(\rho) + \widetilde{\mathbf{L}}(\rho)^T\widetilde{\mathbf{U}}(\rho) &= 2\mathbf{W} \\ \widetilde{\mathbf{L}}(\rho)\widetilde{\mathbf{L}}(\rho)^T &= \widetilde{\mathbf{X}}(\rho) \\ \widetilde{\mathbf{U}}(\rho)\widetilde{\mathbf{U}}(\rho)^T &= \widetilde{\mathbf{S}}(\rho) \end{aligned} \tag{27}$$

are equivalent.

From the asymptotic behavior stated in Subsection 2.2 it follows that, for any sequence $\{\rho_k\} \to 0$ the sequence $(\widetilde{\mathbf{X}}(\rho_k), \widetilde{\mathbf{L}}(\rho_k), \widetilde{y}(\rho_k), \widetilde{\mathbf{S}}(\rho_k), \widetilde{\mathbf{U}}(\rho_k))$ is bounded, hence there exists a convergent subsequence and we may assume that the limit

$$\lim_{k\to\infty} (\widetilde{\mathbf{X}}(\rho_k), \widetilde{\mathbf{L}}(\rho_k), \widetilde{y}(\rho_k), \widetilde{\mathbf{S}}(\rho_k), \widetilde{\mathbf{U}}(\rho_k)) = (\widetilde{\mathbf{X}}^*, \widetilde{\mathbf{L}}^*, \widetilde{y}^*, \widetilde{\mathbf{S}}^*, \widetilde{\mathbf{U}}^*)$$

exists. By inserting $\rho = \rho_k$ in the system (27) and taking the limit $\{\rho_k\} \to 0$, we obtain

$$\begin{gathered} (\widetilde{\mathbf{U}}^*)^T\widetilde{\mathbf{L}}^* + (\widetilde{\mathbf{L}}^*)^T\widetilde{\mathbf{U}}^* = 2\mathbf{W} \\ \widetilde{\mathbf{L}}^*(\widetilde{\mathbf{L}}^*)^T = \widetilde{\mathbf{X}}^*, \qquad \widetilde{\mathbf{U}}^*(\widetilde{\mathbf{U}}^*)^T = \widetilde{\mathbf{S}}^*. \end{gathered} \tag{28}$$

Define the map $\widetilde{F}^2$ as follows

$$\widetilde{F}^2(\widetilde{\mathbf{X}}, \widetilde{\mathbf{L}}, \widetilde{y}, \widetilde{\mathbf{S}}, \widetilde{\mathbf{U}}, \rho) = \begin{bmatrix} \Psi(\widetilde{\mathbf{X}}, \widetilde{y}, \widetilde{\mathbf{S}}, \rho) \\ \widetilde{\mathbf{U}}^T\widetilde{\mathbf{L}} + \widetilde{\mathbf{L}}^T\widetilde{\mathbf{U}} - 2\mathbf{W} \\ \widetilde{\mathbf{L}}\widetilde{\mathbf{L}}^T - \widetilde{\mathbf{X}}, \\ \widetilde{\mathbf{U}}\widetilde{\mathbf{U}}^T - \widetilde{\mathbf{S}} \end{bmatrix}.$$

From Theorem 3.2 and Lemma 3.6 it follows that, for any $\rho = \sqrt{\mu} > 0$, the system $\widetilde{F}^2 = 0$ is equivalent with the system (3)–(5) in the sense that

$$\widetilde{F}^2(\widetilde{\mathbf{X}}(\rho), \widetilde{\mathbf{L}}(\rho), \widetilde{y}(\rho), \widetilde{\mathbf{S}}(\rho), \widetilde{\mathbf{U}}(\rho), \rho) = 0,$$

and moreover,

$$\widetilde{F}^2(\widetilde{\mathbf{X}}^*, \widetilde{\mathbf{L}}^*, \widetilde{y}^*, \widetilde{\mathbf{S}}^*, \widetilde{\mathbf{U}}^*, 0) = 0.$$

The Fréchet derivative of the map $\widetilde{F}^2$ at the point $(\widetilde{\mathbf{X}}^*, \widetilde{\mathbf{L}}^*, \widetilde{y}^*, \widetilde{\mathbf{S}}^*, \widetilde{\mathbf{U}}^*, 0)$ with respect to the variables $(\widetilde{\mathbf{X}}, \widetilde{\mathbf{L}}, \widetilde{y}, \widetilde{\mathbf{S}}, \widetilde{\mathbf{U}})$ is the linear map

$$\begin{aligned} &D\widetilde{F}^2(\widetilde{\mathbf{X}}^*, \widetilde{\mathbf{L}}^*, \widetilde{y}^*, \widetilde{\mathbf{S}}^*, \widetilde{\mathbf{U}}^*, 0)[\triangle\widetilde{\mathbf{X}}, \triangle\widetilde{\mathbf{L}}, \triangle\widetilde{y}, \triangle\widetilde{\mathbf{S}}, \triangle\widetilde{\mathbf{U}}] \\ &\quad = \begin{bmatrix} D\Psi(\widetilde{\mathbf{X}}^*, \widetilde{y}^*, \widetilde{\mathbf{S}}^*, 0)[\triangle\widetilde{\mathbf{X}}, \triangle\widetilde{y}, \triangle\widetilde{\mathbf{S}}] \\ (\triangle\widetilde{\mathbf{U}})^T\widetilde{\mathbf{L}}^* + (\widetilde{\mathbf{U}}^*)^T\triangle\widetilde{\mathbf{L}} + (\triangle\widetilde{\mathbf{L}})^T\widetilde{\mathbf{U}}^* + (\widetilde{\mathbf{L}}^*)^T\triangle\widetilde{\mathbf{U}} \\ \triangle\widetilde{\mathbf{L}}(\widetilde{\mathbf{L}}^*)^T + \widetilde{\mathbf{L}}^*(\triangle\widetilde{\mathbf{L}})^T - \triangle\widetilde{\mathbf{X}} \\ \triangle\widetilde{\mathbf{U}}(\widetilde{\mathbf{U}}^*)^T + \widetilde{\mathbf{U}}^*(\triangle\widetilde{\mathbf{U}})^T - \triangle\widetilde{\mathbf{S}} \end{bmatrix}. \end{aligned}$$

The nonsingularity result follows from the next lemma. For the proof see also Proposition 5 of [**13**] or Lemma 3.2.4 of [**18**].

Lemma 3.7. *Let* $\mathbf{X}, \mathbf{S} \in S^n_{++}$ *be given matrices and* $\gamma \in (0, \frac{1}{3\sqrt{2}})$. *If there exists* $\mu > 0$ *such that* $\|(\mathbf{U_S}^T\mathbf{L_X} + \mathbf{L_X}^T\mathbf{U_S})/2 - \mu\mathbf{I}\|_F \leq \gamma\mu$ *(where* $\mathbf{L_X}$ *is the* *lower Cholesky factor of* $\mathbf{X}$ *and* $\mathbf{U_S}$ *is the upper Cholesky factor of* $\mathbf{S}$*), then for* $\triangle\mathbf{L} \in L^n$*,* $\triangle\mathbf{U} \in U^n$ *and* $\triangle\mathbf{X}, \triangle\mathbf{S} \in S^n$ *the following implication holds*

$$\left.\begin{array}{rcl} \triangle\mathbf{L}^T\mathbf{U_S} + \mathbf{U_S}^T\triangle\mathbf{L} + \triangle\mathbf{U}^T\mathbf{L_X} + \mathbf{L_X}^T\triangle\mathbf{U} & = & 0 \\ \triangle\mathbf{L}\mathbf{L_X}^T + \mathbf{L_X}\triangle\mathbf{L^T} & = & \triangle\mathbf{X} \\ \triangle\mathbf{U}\mathbf{U_S}^T + \mathbf{U_S}\triangle\mathbf{U}^T & = & \triangle\mathbf{S} \\ \triangle\mathbf{X} \bullet \triangle\mathbf{S} & = & 0 \end{array}\right\}$$

$$\Longrightarrow \quad \triangle\mathbf{L} = \triangle\mathbf{U} = \triangle\mathbf{X} = \triangle\mathbf{S} = 0.$$

Proposition 3.2. $D\widetilde{F}^2(\widetilde{\mathbf{X}}^*, \widetilde{\mathbf{L}}^*, \widetilde{y}^*, \widetilde{\mathbf{S}}^*, \widetilde{\mathbf{U}}^*, 0)$ *is a nonsingular linear map.*

Proof. Assume $D\widetilde{F}^2(\widetilde{\mathbf{X}}^*, \widetilde{\mathbf{L}}^*, \widetilde{y}^*, \widetilde{\mathbf{S}}^*, \widetilde{\mathbf{U}}^*, 0)[\triangle\widetilde{\mathbf{X}}, \triangle\widetilde{\mathbf{L}}, \triangle\widetilde{y}, \triangle\widetilde{\mathbf{S}}, \triangle\widetilde{\mathbf{U}}] = 0$. Theorem 3.2 implies $\triangle\widetilde{\mathbf{X}} \bullet \triangle\widetilde{\mathbf{S}} = 0$. From the asymptotic behavior stated in Subsection 2.2 it follows that $\widetilde{\mathbf{L}}^* \in L^n_{++}$ and $\widetilde{\mathbf{U}}^* \in U^n_{++}$. The rest follows from (28), Lemma 3.7 and Assumption (A1). □

3.3. Analyticity of the weighted paths as a function of $\sqrt{\mu}$ at $\mu = 0$

Now we are ready to prove Theorem 3.1. The idea of the proof is analogous to the proof of Proposition 4.2.2 of [**18**] or Proposition 6.1 of [**10**].

Proof. We will only consider the weighted path associated with the symmetrization Φ_1. The proof for the path associated with the Φ_2 is the same. Recall that

$$\widetilde{F}^1 : S^n \times \mathcal{U}^n_{BN} \times R^m \times S^n \times \mathcal{U}^n_{BN} \to R^m \times S^n \times S^n \times S^n \times S^n$$

is for an analytic function of $(\widetilde{\mathbf{X}}, \widetilde{\mathbf{U}}_Y, \widetilde{y}, \widetilde{\mathbf{S}}, \widetilde{\mathbf{U}}_Z, \rho)$ such that

1. there exists $(\widetilde{\mathbf{X}}^*, \widetilde{\mathbf{U}}^*_Y, \widetilde{y}^*, \widetilde{\mathbf{S}}^*, \widetilde{\mathbf{U}}^*_Z, 0)$ such that
$$\widetilde{F}^1(\widetilde{\mathbf{X}}^*, \widetilde{\mathbf{U}}^*_Y, \widetilde{y}^*, \widetilde{\mathbf{S}}^*, \widetilde{\mathbf{U}}^*_Z, 0) = 0;$$
2. the Fréchet derivative of the map $\widetilde{F}^2$ with respect to $(\widetilde{\mathbf{X}}, \widetilde{\mathbf{U}}_Y, \widetilde{y}, \widetilde{\mathbf{S}}, \widetilde{\mathbf{U}}_Z)$ is nonsingular at the point $(\widetilde{\mathbf{X}}^*, \widetilde{\mathbf{U}}^*_Y, \widetilde{y}^*, \widetilde{\mathbf{S}}^*, \widetilde{\mathbf{U}}^*_Z, 0)$ (see Proposition 3.1).

Now we can apply the (analytic version of) implicit function theorem (see [**5**]) and obtain that there exist: a neighborhood $\mathcal{I}$ of $\rho = 0$, a neighborhood $\mathcal{U}$ of $(\widetilde{\mathbf{X}}^*, \widetilde{\mathbf{U}}^*_Y, \widetilde{y}^*, \widetilde{\mathbf{S}}^*, \widetilde{\mathbf{U}}^*_Z)$ and an analytic function

$$(\hat{\mathbf{X}}, \hat{\mathbf{U}}_Y, \hat{y}, \hat{\mathbf{S}}, \hat{\mathbf{U}}_Z) : \mathcal{I} \to \mathcal{U}$$

such that $(\hat{\mathbf{X}}, \hat{\mathbf{U}}_Y, \hat{y}, \hat{\mathbf{S}}, \hat{\mathbf{U}}_Z)(0) = (\widetilde{\mathbf{X}}^*, \widetilde{\mathbf{U}}^*_Y, \widetilde{y}^*, \widetilde{\mathbf{S}}^*, \widetilde{\mathbf{U}}^*_Z)$ and

$$\widetilde{F}^1((\hat{\mathbf{X}}, \hat{\mathbf{U}}_Y, \hat{y}, \hat{\mathbf{S}}, \hat{\mathbf{U}}_Z)(\rho)) = 0 \tag{29}$$

for all $\rho \in \mathcal{I}$. There exists $\bar{k} > 0$ such that for all $k \geq \bar{k}$ it holds $\rho_k \in \mathcal{I}$ and $(\widetilde{\mathbf{X}}(\rho_k), \widetilde{\mathbf{U}}_Y(\rho_k), \widetilde{y}(\rho_k), \widetilde{\mathbf{S}}(\rho_k), \widetilde{\mathbf{U}}_Z(\rho_k)) \in \mathcal{U}$. Since $(\widetilde{\mathbf{X}}(\rho), \widetilde{\mathbf{U}}_Y(\rho), \widetilde{y}(\rho), \widetilde{\mathbf{S}}(\rho), \widetilde{\mathbf{U}}_Z(\rho))$ and $(\hat{\mathbf{X}}, \hat{\mathbf{U}}_Y, \hat{y}, \hat{\mathbf{S}}, \hat{\mathbf{U}}_Z)(\rho)$ are solutions of (29) for $\rho > 0$, from the uniqueness of the positive definite solutions it follows that

$$(\widetilde{\mathbf{X}}(\rho), \widetilde{\mathbf{U}}_Y(\rho), \widetilde{y}(\rho), \widetilde{\mathbf{S}}(\rho), \widetilde{\mathbf{U}}_Z(\rho)) = (\hat{\mathbf{X}}, \hat{\mathbf{U}}_Y, \hat{y}, \hat{\mathbf{S}}, \hat{\mathbf{U}}_Z)(\rho)$$

for all $\rho \in \mathcal{I} \cap (0, \infty)$. Thus the path function $(\widetilde{\mathbf{X}}(\rho), \widetilde{\mathbf{U}}_Y(\rho), \widetilde{y}(\rho), \widetilde{\mathbf{S}}(\rho), \widetilde{\mathbf{U}}_Z(\rho))$ is analytically extendable to $\rho = 0$ by prescription $(\widetilde{\mathbf{X}}(0), \widetilde{\mathbf{U}}_Y(0), \widetilde{y}(0), \widetilde{\mathbf{S}}(0), \widetilde{\mathbf{U}}_Z(0)) =$

$(\widetilde{\mathbf{X}}^*, \widetilde{\mathbf{U}}_Y^*, \widetilde{y}^*, \widetilde{\mathbf{S}}^*, \widetilde{\mathbf{U}}_Z^*)$. Therefore also the function $(\widetilde{\mathbf{X}}(\rho), \widetilde{y}(\rho), \widetilde{\mathbf{S}}(\rho))$ is analytically extendable to $\rho = 0$. □

4. Conclusion

Note that contrary to the weighted paths associated with the symmetrization maps Φ_{AHO}, Φ_{SR}, Φ_{CH}, the paths studied in this paper are parameterized by $\sqrt{\mu}$ in the symmetrization condition (5). This parameterization causes that both the types of paths associated with the symmetrizations (6), (7) are for $(\mathbf{W}, \triangle b, \triangle\mathbf{C}) = (\mathbf{I}, 0, 0)$ identical with the central path, and moreover, these paths possess similar asymptotic behavior like to the paths associated with symmetrizations Φ_{AHO}, Φ_{SR}, Φ_{CH}. We can also observe that the paths studied in this paper are analytic at the boundary of the same order as the condition (5). This property is satisfied only for the paths associated with Φ_{AHO} – the analyticity at the boundary point of the paths associated with Φ_{SR} and Φ_{CH} depends on the structure of the weight matrix (see [**10, 11, 12, 15**]).

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