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THE INVERSE OF THE PASCAL LOWER TRIANGULAR MATRIX MODULO p

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Abstract. Let $L(n)_p$ be the Pascal lower triangular matrix with coefficients $\binom{i}{j}$ (mod p), $0 \le i, j < n$. In this paper, we found the inverse of $L(n)_p$ modulo p. In fact, we generalize a result due to David Callan [**4**].

1. Introduction

Consider the infinite unipotent lower triangular matrix

$$L(\infty) = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ \vdots & & & & \ddots \end{pmatrix} = \exp \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & 2 & 0 & & \\ & 0 & 3 & 0 & \\ & & & & \ddots \end{pmatrix}$$

with coefficients $L(\infty)_{i,j} = \binom{i}{j}$, $i, j \ge 0$, where, as usual, we use the convention $\binom{i}{j} = 0$ if $i < j$. We denote by $L(n)$ the $n \times n$ principal submatrix with coefficients $L(n)_{i,j}$, $0 \le i, j < n$ obtained by considering the first n rows and columns of $L(\infty)$. Given a prime p, we define $L(n)_p$ with coefficients $(L(n)_p)_{i,j} \in \{0, 1, \dots, p-1\}$ as the reduction modulo p of $L(n)$ by setting

$$(L(n)_p)_{i,j} = \binom{i}{j} \ (\mathrm{mod}\ p) \in \{0, 1, \dots, p-1\}.$$

For instance, the matrices $L(5)_2$, $L(6)_3$ and $L(7)_5$ are given as follows:

$$L(4)_2 = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 0 & 1 & & \\ 1 & 1 & 1 & 1 & \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad L(5)_3 = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 0 & 0 & 1 & & \\ 1 & 1 & 0 & 1 & 1 & \\ 1 & 2 & 1 & 1 & 2 & 1 \end{pmatrix}$$

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$$L(6)_5 = \begin{pmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ 1 & 3 & 3 & 1 & & & \\ 1 & 4 & 1 & 4 & 1 & & \\ 1 & 0 & 0 & 0 & 0 & 1 & \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

For a prime p and a positive integer n, we denote by $s_p(n)$ the sum of the digits in the base-p representation of the integer n, that is, $s_p(n) = \sum_{k\geq 0} n_k$ when writing $n = \sum_{k\geq 0} n_k p^k$ in base p. The *Thue-Morse sequence*

$$\mathbf{t} = \Big\{t(n) = s_2(n) \pmod 2\Big\}_{n=0}^{\infty} = 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0 \ldots,$$

records the parity of the sum of the binary digits of $n = \sum_{k\geq 0} n_k 2^k$. It can also be defined recursively by $t(0) = 0$, $t(2n) = t(n)$, $t(2n+1) = \overline{t(n)}$, for all $n \geq 0$, where, for $x \in \{0,1\}$, we define $\overline{x} = 1-x$. The sequence $\mathbf{t}$ has appeared in various fields of mathematics, see, for instance, [**1**]. Replacing 0 by a and 1 by b yields the Thue-Morse sequence on the alphabet $\mathcal{A} = \{a,b\}$ (called the ± 1 Thue-Morse sequence if $a = 1$ and $b = -1$)

$$\mathbf{t}(a,b) = a\ b\ b\ a\ b\ a\ a\ b\ b\ a\ a\ b\ a\ b\ b\ a\ \ldots$$

In [**4**], David Callan showed that the sequence $\mathbf{t}$ is related to the matrix $L(\infty)_2$. In fact, the following result is due to Callan.

Callan Theorem ([**4**])**.** *The inverse matrix of $L(\infty)_2$ is a $(0,\pm 1)$-matrix. It has the same pattern of zeroes as $L(\infty)_2$ and the nonzero entries in each column form the ± 1 Thue-Morse sequence.*

In order to prove his result, Callan defined the lower triangular matrix $L_2(x)$ with entries $L_2(x)_{i,j}$ by

$$L_2(x)_{i,j} = \binom{i}{j} x^{s_2(i-j)} \pmod 2 \qquad \text{for each } i,j \geq 0,$$

and then he showed that $L_2(x) + L_2(y) = L_2(x+y)$. It is worth mentioning that, Roland Bacher and Robin Chapman have obtained the same result observing that the $2^k \times 2^k$ upper left submatrix of $L_2(x)$ is the k-fold Kronecker product of $\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ (see [**2**], [**3**]). Here, we are going to generalize Callan Theorem. Following Callan [**4**], we present the following definition.

Definition 1. Let x be an indeterminate. Define the infinite lower triangular matrix $L_p(x)$ with coefficients $L_p(x)_{i,j}$ by setting

$$L_p(x)_{i,j} = \binom{i}{j} x^{s_p(i-j)} \pmod p.$$

In particular, we have $L_p(1) = L(\infty)_p$.

Then the matrices $L_2(x)$ and $L_3(x)$, for example, are given by

$$L_2(x) = \begin{pmatrix} 1 & & & & & & & & \\ x & 1 & & & & & & & \\ x & 0 & 1 & & & & & & \\ x^2 & x & x & 1 & & & & & \\ x & 0 & 0 & 0 & 1 & & & & \\ x^2 & x & 0 & 0 & x & 1 & & & \\ x^2 & 0 & x & 0 & x & 0 & 1 & & \\ x^3 & x^2 & x^2 & x & x^2 & x & x & 1 & \\ \vdots & & & & & & & & \ddots \end{pmatrix}$$

and

$$L_3(x) = \begin{pmatrix} 1 & & & & & & & & \\ x & 1 & & & & & & & \\ x^2 & 2x & 1 & & & & & & \\ x & 0 & 0 & 1 & & & & & \\ x^2 & x & 0 & x & 1 & & & & \\ x^3 & 2x^2 & x & x^2 & 2x & 1 & & & \\ x^2 & 0 & 0 & 2x & 0 & 0 & 1 & & \\ x^3 & x^2 & 0 & 2x^2 & 2x & 0 & x & 1 & \\ \vdots & & & & & & & & \ddots \end{pmatrix}.$$

Indeed, the purpose of this paper is to prove the following theorem.

Main Theorem. *Let p be a prime and let x and y be indeterminates. Then there holds*

$$L_p(x) \cdot L_p(y) \equiv L_p(x+y) \pmod{p}. \tag{1}$$

In particular, we conclude that $L_p(1)^{-1} \equiv L_p(-1) \pmod{p}$.

It is worth mentioning that the idea in the proof of this Theorem follows that one of Callan [**4**].

As an immediate consequence of Main Theorem, we have the following.

Corollary 1. If n is a positive integer, then we have $L_p(x)^n \equiv L_p(nx) \pmod{p}$.

Proof. By an easy induction on n. □

Corollary 2. If r is a rational number, then we have $L_p(r) \equiv L_p(1)^r \pmod{p}$.

Proof. Let $r = \frac{m}{n}$, with m, n positive integers. Then, by Corollary 1, we obtain

$$L_p(r)^n \equiv L_p\left(\frac{m}{n}\right)^n \equiv L_p(m) \equiv L_p(1)^m \pmod{p}.$$

For negative r, it now suffices to show that $L_p(1)^{-1} \equiv L_p(-1) \pmod{p}$, and this follows

$$L_p(-1)L_p(1) \equiv L_p(-1+1) \equiv L_p(0) \equiv I \pmod{p},$$

by Main Theorem. This completes the proof of the corollary. □

Remark 1. Note that the main result of this paper can also be obtained by the Kronecker product method attributed to Roland Bacher: the $p^k \times p^k$ upper left submatrix of $L_p(x)$ is the k-fold Kronecker product of the upper left $p \times p$ submatrix of $L_p(x)$.

2. Preliminaries

In this section, we collect a number of results that we will need in the proof of the Main theorem. We start with a well-known result due to Lucas. In fact, Lucas discovered an easy method to determine the value of $\binom{n}{m}$ (mod p).

Lemma 1 (Lucas Theorem [**5**])**.** *Let p be a prime number and m, n be non-negative integers. Suppose*

$$m = \sum_{k\geq 0} m_k p^k \qquad \textit{and} \qquad n = \sum_{k\geq 0} n_k p^k,$$

are written in base p, that is, $m_k, n_k \in \{0, 1, \ldots, p-1\}$ for all k. Then we have

$$\binom{n}{m} \equiv \binom{n_0}{m_0}\binom{n_1}{m_1}\cdots\binom{n_d}{m_d} \pmod{p}.$$

In 1852 Kummer showed that the power of prime p that divides the binomial coefficient $\binom{i}{j}$ is given by the number of 'carries' when we add j and $i-j$ in base p.

Lemma 2 (Kummer Theorem [**6**])**.** *If p is a prime number, then its exponent in the canonical expansion of the binomial coefficient $\binom{i}{j}$ into prime factors is equal to the number of carries required when adding the numbers j and $i - j$ in base p.*

Proof. Note that the identity

$$\binom{i}{j} = \frac{i!}{j!(i-j)!}$$

implies that

$$e_p\left(\binom{i}{j}\right) = e_p(i!) - e_p(j!) - e_p((i-j)!),$$

where $e_p(k)$ is the exponent of p in the prime factorization of k. It is not difficult to see that

$$e_p(k!) = \left\lfloor \frac{k}{p} \right\rfloor + \left\lfloor \frac{k}{p^2} \right\rfloor + \cdots,$$

because among the numbers $1, 2, \ldots, k$, there are exactly $\lfloor \frac{k}{p} \rfloor$ numbers divisible by p, exactly $\lfloor \frac{k}{p^2} \rfloor$ numbers divisible by p^2, and so on. Thus,

$$e_p\Big(\binom{i}{j}\Big) = \sum_{l\geq 0} \Big(\Big\lfloor \frac{i}{p^l} \Big\rfloor - \Big\lfloor \frac{j}{p^l} \Big\rfloor - \Big\lfloor \frac{i-j}{p^l} \Big\rfloor\Big).$$

Now, it suffices to note that in this sum, the lth summand is either 1 or 0 depending on whether or not there is a carry from the $(l-1)$th digit. □

Definition 2. Let p be a prime and i, j be non-negative integers. Suppose

$$i = \sum_{k\geq 0} i_k p^k \qquad \text{and} \qquad j = \sum_{k\geq 0} j_k p^k,$$

are written in base p. We say i is *p-free of j* if

$$0 \leq i_k + j_k \leq p-1, \quad \text{for all } k.$$

Lemma 3. *Let p be a prime number and let i and j be positive integers with $i \geq j$. Suppose that $i = \sum_{k\geq 0} i_k p^k$ and $j = \sum_{k\geq 0} j_k p^k$ are written in base p. Then, the following four statements are equivalent:*

(a) *$i - j$ is p-free of j.*
(b) *for every $k \geq 0$, $i_k \geq j_k$.*
(c) *There exists l between i and j such that $i - l$ is p-free of l and $l - j$ is p-free of j.*
(d) $0 \not\equiv \binom{i}{j} \pmod p$.

Proof. Before starting the proof we give an easy observation

$$(i-j)_k = \begin{cases} i_k - j_k & \text{if} \quad i_k \geq j_k \\ p + i_k - j_k & \text{if} \quad i_k < j_k. \end{cases} \tag{2}$$

$(a) \Rightarrow (b)$ Assume the contrary that there exists k such that $i_k < j_k$. But then, by Eq. (2), we have

$$(i-j)_k + j_k = p + i_k - j_k + j_k = p + i_k > p - 1,$$

which contradicts our assumption, i.e., $i - j$ is p-free of j.

$(b) \Rightarrow (a)$ We can easily see that

$$(i-j)_k + j_k = i_k - j_k + j_k = i_k \leq p - 1,$$

and so by definition, we conclude the result.

$(a) \Rightarrow (c)$ If $i - j$ is p-free of j, then by part (b), we have $i_k \geq j_k$ for every k. Now, for every k, we choose l_k such that $i_k \geq l_k \geq j_k$, and we put $l = \sum_{k\geq 0} l_k p^k$. It is evident that $j \leq l \leq i$. Moreover, by Eq. (2), we observe that

$$(i-l)_k + l_k = i_k - l_k + l_k = i_k \leq p - 1,$$

and also

$$(l-j)_k + j_k = l_k - j_k + j_k = l_k \leq i_k \leq p - 1$$

which implies that $i - l$ is p-free of l and $l - j$ is p-free of j by definition.

$(c) \Rightarrow (a)$ Assume that there exists $j \leq l \leq i$ such that $i - l$ is p-free of l and $l - j$ is p-free of j. Put $l = \sum_{k\geq 0} l_k p^k$, where $l_k \in \{0, 1, \ldots, p-1\}$. Then, by part (a), we obtain $j_k \leq l_k \leq i_k$ for every k. Now, by Eq. (2), it follows that

$$(i-j)_k + j_k = i_k - j_k + j_k = i_k \leq p - 1,$$

and so $i - j$ is p-free of j by definition.

$(d) \Leftrightarrow (a)$ This follows immediately from Kummer Theorem.

This completes the proof of the lemma. □

Remark 2. Note that, if $i \geq j$ and $i-j$ is p-free of j, then we have $s_p(i-j) = s_p(i) - s_p(j)$.

Lemma 4. *Let p be a prime and n, r be positive integers. Then we have*

$$\sum_{\substack{0 \leq t \leq n \\ s_p(t)=r}} \binom{n}{t} \equiv \binom{s_p(n)}{r} \pmod{p}.$$

Proof. We write $n = \sum_{k=0}^{d} n_k p^k$ in base p, so that $0 \leq n_k \leq p-1$ for each k. Now, we consider the following equation

$$(1+X)^{s_p(n)} = (1+X)^{n_0}(1+X)^{n_1} \cdots (1+X)^{n_d} \tag{3}$$

and compare the coefficient of X^r modulo p in both sides of this equation. Evidently, the coefficient of X^r on the left-hand side of Eq. (3) is equal to $\binom{s_p(n)}{r} \pmod{p}$. On the other hand, the coefficient of X^r on the right-hand side of Eq. (3) is equal to

$$\sum_{r_0+r_1+\cdots+r_d=r} \binom{n_0}{r_0}\binom{n_1}{r_1}\cdots\binom{n_d}{r_d} \pmod{p}. \tag{4}$$

But, by Lucas Theorem, the sum in Eq. (4) is congruent to

$$\sum_{\substack{0 \leq t \leq n \\ s_p(t)=r}} \binom{n}{t} \pmod{p}.$$

This completes the proof of the lemma. □

3. Proof of the Main Theorem

Proof. For the proof of the Eq. (1) we compute the (i, j)-th entry of $L_p(x) \cdot L_p(y)$, that is,

$$(L_p(x) \cdot L_p(y))_{i,j} = \sum_t L_p(x)_{i,t} L_p(y)_{t,j}.$$

First of all, since the matrices $L_p(x)$ and $L_p(y)$ are lower triangular matrices, thus $L_p(x) \cdot L_p(y)$ is also a lower triangular matrix. Furthermore, it is easy to see the product of row i of $L_p(x)$ with column i of $L_p(y)$ is always 1, since every pair of entries except entry i is either 0 in the row or 0 in the column, and the product at entry i is $1 \times 1 = 1$ for $i \geq 1$. Now, we must show that the product of row i of $L_p(x)$ with column j of $L_p(y)$ when $i > j$ is always $L_p(x+y)_{i,j}$. Therefore, from now on we assume that $i > j$. In this case, the (i, j)-th entry of $L_p(x) \cdot L_p(y)$ is equal to

$$(L_p(x) \cdot L_p(y))_{i,j} = \sum_{t=j}^{i} L_p(x)_{i,t} L_p(y)_{t,j}.$$

We now consider two cases separately:

CASE 1. *$i-j$ is not p-free of j.*

In this case, by Lemma 3, there does not exist t between j and i such that $i-t$ is p-free of t and $t-j$ is p-free of j. Hence for every t between j and i, we have $L_p(x)_{i,t}=0$ or $L_p(y)_{t,j}=0$, and so

$$(L_p(x)\cdot L_p(y))_{i,j}=\sum_{t=j}^{i}0=0=L_p(x+y)_{i,j}.$$

CASE 2. *$i-j$ is p-free of j.*

In this case, by Lemma 4, we have $\binom{i}{j} \pmod p \not\equiv 0$. First, we notice that

$$\binom{i}{t}\binom{t}{j}=\binom{i}{j}\binom{i-j}{t-j},\qquad \text{for } i\geq t\geq j. \tag{5}$$

Now, we calculate the sum in question

$$\begin{aligned}(L_p(x)\cdot L_p(y))_{i,j} &\equiv \sum_{t=j}^{i}\binom{i}{t}\binom{t}{j}x^{s_p(i-t)}y^{s_p(t-j)} \pmod p\\ &\equiv \sum_{t=j}^{i}\binom{i}{j}\binom{i-j}{t-j}x^{s_p(i-t)}y^{s_p(t-j)} \pmod p \qquad \text{(by Eq. (5))}\\ &= \sum_{t=0}^{i-j}\binom{i}{j}\binom{i-j}{t}x^{s_p(i-j-t)}y^{s_p(t)} \pmod p\end{aligned}$$

If $i-j-t$ is not p-free of t, then, by Lemma 3, we obtain that $0\equiv\binom{i-j}{t} \pmod p$. Hence, we may restrict the last sum to $0\leq t\leq i-j$ such that $i-j-t$ is p-free of t. But then, by Remark 2, we have $s_p(i-j-t)=s_p(i-j)-s_p(t)$. Thus we obtain

$$\begin{aligned}(L_p(x)\cdot L_p(y))_{i,j} &= \binom{i}{j}\sum_{t=0}^{i-j}\binom{i-j}{t}x^{s_p(i-j)-s_p(t)}y^{s_p(t)} \pmod p\\ &= \binom{i}{j}\sum_{r=0}^{s_p(i-j)}\Big\{\Big(\sum_{\substack{0\leq t\leq i-j\\ s_p(t)=r}}\binom{i-j}{t}\Big)x^{s_p(i-j)-r}y^r\Big\} \pmod p\\ &\equiv \binom{i}{j}\sum_{r=0}^{s_p(i-j)}\binom{s_p(i-j)}{r}_p x^{s_p(i-j)-r}y^r \pmod p \quad \text{(by Lemma 4)}\\ &= \binom{i}{j}(x+y)^{s_p(i-j)} \pmod p\\ &= L_p(x+y)_{i,j}\end{aligned}$$

as desired. □

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