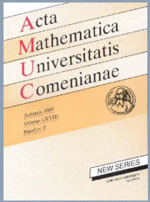

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ON THE COMPUTATION OF MULTIPLICITY BY THE REDUCTION OF DIMENSION

E. BOĎA AND D. JAŠKOVÁ

Abstract. In this short note we describe one method for the computation of the Samuel multiplicity of the polynomial ideals and prove a formula for the multiplicity of the ideal $(\alpha_i x_i^{a_i} - \beta_{i+1} x_{i+1}^{b_{i+1}};\ i = 1,\dots,n) \cdot R$ in R (with the convention $x_{n+1} = x_1$, $\beta_{n+1} = \beta_1$, $b_{n+1} = b_1$), where $(R, m) = k\,[x_1, x_2, \dots, x_n]_{(x_1, x_2, \dots, x_n)}$ is a local polynomial ring over an algebraic closed field k.

Let (A, m) be a Noetherian local ring with $\dim A = d$. For any m-primary ideal Q in A the A-module A/Q^n is of the finite length for all $n \in \mathbb{N}$. For large n this length function becomes a polynomial (Hilbert-Samuel polynomial) which can be written as

$$L(A/Q^n) = e_0(Q, A)\frac{n^d}{d!} + \text{terms of lower degree.}$$

The coefficient $e_0(Q, A)$ is called the Samuel multiplicity (or simply) multiplicity of Q in A. We present one method how to count this multiplicity when Q is generated by a system of parameters in a local polynomial ring.

Let $P = k[x_1, \dots, x_n]$ be a polynomial ring over an algebraic closed field k. Let $f_1, \dots, f_{n-r}$ denote a system of polynomials in P such that algebraic variety $V(f_1, \dots, f_{n-r})$ is of dimension r,

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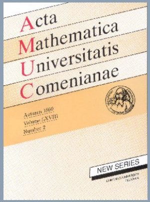

$0 \leq r < n$. We say that the set of polynomials $\{u_i(s_1, \ldots, s_r) \in k[s_1, \ldots, s_r],\ i = 1, \ldots, n\}$ represents the polynomial parametrization of W if the image of the map

$$k^r \to E^n$$

given by

$$(a_1, a_2, \ldots, a_r) \longmapsto (u_1(a_1, \ldots, a_r), \ldots, u_n(a_1, \ldots, a_r))$$

is $V(f_1, \ldots, f_{n-r})$.

Now we can formulate the main theorem of this note.

Theorem 1. *Let $P = k[x_1, \ldots, x_n]$ be a polynomial ring over an algebraic closed field k and $(R, m) = k[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}$ the localization of P with respect to maximal ideal $(x_1, \ldots, x_n) \cdot P$. Let $f_1, \ldots, f_n$ denote a system of polynomials in P such that $(f_1, \ldots, f_n) \cdot R$ is an m-primary ideal in R. Let W be an algebraic variety in E^n defined by the equations $f_1(x_1, \ldots, x_n) = \ldots = f_{n-r}(x_1, \ldots, x_n) = 0$ with $\dim W = r$ and the polynomial parametrization $\{u_i(s_1, \ldots, s_r) \in k[s_1, \ldots, s_r],\ i = 1, \ldots, n\}$. Suppose that the polynomial ring $k\,[s_1, \ldots, s_r]$ is a finite $k[u_1, \ldots, u_n]$-module. Let d denote the dimension of the field $k(s_1, \ldots, s_r)$ as a vector space over the field $k(u_1, \ldots, u_n)$. With this hypothesis we have*

$$e_0((f_1, \ldots, f_n) \cdot R, R) \cdot d = e_0((F_{n-r+1}, \ldots, F_n) \cdot S, S)$$

where $F_i = f_i(u_1(s_1, \ldots, s_r), \ldots, u_n(s_1, \ldots, s_r))$ for $i = n - r + 1, \ldots, n$ and $S = k\,[s_1, \ldots, s_r]_{(s_1, \ldots, s_r)}$.

Proof. From our construction we have the monomorphism

$$k[x_1, \ldots, x_n]/(f_1, \ldots, f_{n-r}) \cdot k[x_1, \ldots, x_n] \cong k\,[u_1 \ldots, u_n] \hookrightarrow k\,[s_1, \ldots, s_r]$$

and hence the local monomorphism

$$R/(f_1, \ldots, f_{n-r}) \cdot R \cong k\,[u_1, \ldots, u_n]_{(u_1, \ldots, u_n)} \hookrightarrow k\,[s_1 \ldots, s_r]_{(s_1, \ldots, s_r)}\,.$$

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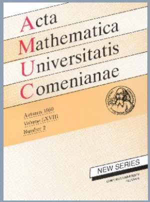

As the module $k\,[s_1,\dots,s_r]_{(s_1,\dots,s_r)}$ is finite over the ring $k\,[u_1,\dots,u_n]_{(u_1,\dots,u_n)}$, the additivity formula applied to the multiplicity $e_0((f_1,\dots,f_n)\cdot R,R)$ provides the equality

$$\begin{aligned} &e_0((f_1,\dots,f_n)\cdot R/(f_1,\dots,f_{n-r})\cdot R, R/(f_1,\dots,f_{n-r})\cdot R)\cdot d\\ &= e_0((F_{n-r+1},\dots,F_n)\cdot S,S) \end{aligned}$$

(cf. [3, Theorem 14.7]). As the ideal $(f_1,\dots,f_n)\cdot R$ is generated by a system of parameters, we have

$$e_0((f_1,\dots,f_n)\cdot R,R)\cdot d = e_0((F_{n-r+1},\dots,F_n)\cdot S,S),$$

(cf. [4, Chap.7, Theorem 18]) which completes the proof. $\square$

Let us shift to the ideal $(\alpha_i x_i^{a_i}-\beta_{i+1}x_{i+1}^{b_{i+1}}; i=1,\dots,n)\cdot R$ in the local polynomial ring $(R,m)=k\,[x_1,x_2,\dots,x_n]_{(x_1,x_2,\dots,x_n)}$. As the mentioned ideal satisfies the condition of the above formulated Theorem 1, we can prove the formula for its multiplicity. We start with $n=2$.

Lemma 2. *Let* $(\alpha x^a-\beta y^b,\gamma y^c-\delta x^d)\cdot A$ *be a parameter ideal in the local ring* $(A,m)=k\,[x,y]_{(x,y)}$ $(a,b,c,d\in\mathbb{N};\ \ \alpha,\beta,\gamma,\delta\in k)$. *Then*

$$e_0\left((\alpha x^a-\beta y^b,\gamma y^c-\delta x^d)\cdot A,A\right)=\min\{ac,bd\}.$$

Proof. After dividing the polynomials of the basis by α resp. γ, we can assume that $\alpha=\gamma=1$. If $\gcd(a,b)=r$, $a=\bar{a}r$, $b=\bar{b}r$, then

$$x^a-\beta y^b=\prod_{i=1}^{r}(x^{\bar{a}}-\xi_i y^{\bar{b}})$$

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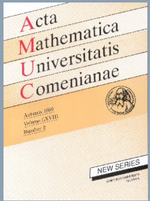

for certain $\xi_i \in k$ (k being algebraically closed). As

$$e_0\left((x^a - \beta y^b, y^c - \delta x^d)\cdot A, A\right) = \sum_{i=1}^{r} e_0\left((x^{\overline{a}} - \xi_i y^{\overline{b}}, y^c - \delta x^d)\cdot A, A\right)$$

(see [4, Chap. VII, Theorem 7]), we can assume that $a,\ b$ are relatively prime with $k \cdot a - l \cdot b = 1$ for certain $k, l \in N$. Then the equations

$$x = \beta^k s^b$$
$$y = \beta^l s^a$$

represent the polynomial parametrization of the curve V given by $x^a - \beta y^b = 0$. In addition, $k(\beta^k s^b, \beta^l s^a) = k(s)$. Now Theorem 1 provides the following equalities

$$\begin{aligned} e_0\left((x^a - \beta y^b, y^c - \delta x^d)\cdot A, A\right) &= e_0((\beta^{l\cdot c} s^{a\cdot c} - \delta \beta^{k\cdot d} s^{b\cdot d})\cdot k\left[s\right]_{(s)}, k\left[s\right]_{(s)}) \\ &= \min\{ac, bd\} \end{aligned}$$

which completes the proof. □

And now we formulate the general result.

Theorem 3. *Let $I = (\alpha_i x_i^{a_i} - \beta_{i+1} x_{i+1}^{b_{i+1}}; i = 1, \ldots, n) \cdot R$ be a parameter ideal in R (with the convention $x_{n+1} = x_1$, $\beta_{n+1} = \beta_1$, $b_{n+1} = b_1$), where $(R, m) = k\left[x_1, x_2, \ldots, x_n\right]_{(x_1, x_2, \ldots, x_n)}$ is a local polynomial ring over an algebraic closed field k. Then*

$$e_0(I, R) = \min\left\{\prod_{i=1}^{n} a_i, \prod_{i=1}^{n} b_i\right\}.$$

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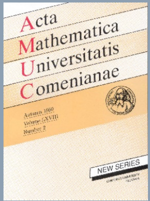

Proof. We use induction on $n \geq 2$. For $n = 2$ the assertion is the above Lemma 2. Let now

$$I = (\alpha_i x_i^{a_i} - \beta_{i+1} x_{i+1}^{b_{i+1}}; i = 1, \dots, n) \cdot k\,[x_1, x_2, \dots, x_n]_{(x_1, x_2, \dots, x_n)}\,, \qquad n > 2.$$

As in Lemma 2 we can assume that the first polynomial is of the form $x_1^{a_1} - \beta_2 x_2^{b_2}$ with a_1, b_1 being relatively prime with $k \cdot a_1 - l \cdot b_2 = 1$ for certain $k, l \in \mathbb{N}$. So the polynomial parametrization of the hypersurface $V(x_1^{a_1} - \beta_2 x_2^{b_2})$ in $\mathbb{E}^n$ has the following form

$$\begin{aligned} x_1 &= \beta_2^k s_1^{b_2} \\ x_2 &= \beta_2^l s_1^{a_1} \\ x_i &= s_{i-1} \qquad \text{for } i = 3, \dots, n. \end{aligned}$$

As $k(\beta_2^k s_1^{b_2}, \beta_2^l s_1^{a_1}, s_2, \dots, s_{n-1}) = k(s_1, \dots, s_{n-1})$, the induction hypothesis and the Theorem 1 imply

$$\begin{aligned} e_0(I, R) = e_0((&\alpha_2 \beta_2^{l \cdot a_2} s_1^{a_1 \cdot a_2} - \beta_3 s_2^{b_3}, \alpha_3 s_2^{a_3} - \beta_4 s_3^{b_4}, \dots\dots, \alpha_{n-1} s_{n-2}^{a_{n-1}} - \beta_n s_{n-1}^{b_n}, \\ &\dots, \alpha_n s_{n-1}^{a_n} - \beta_1 \beta_2^{k \cdot b_1} s_1^{b_2 \cdot b_1}) \cdot k\,[s_1 \dots, s_{n-1}]_{(s_1, \dots, s_{n-1})}\,, k\,[s_1 \dots, s_{n-1}]_{(s_1, \dots, s_{n-1})}) \\ = \min\{&a_1 \cdot a_2 \dots a_n, b_1 \cdot b_2 \dots b_n\}, \end{aligned}$$

which completes the proof. □

Finally, we illustrate the previous results by an example.

Example 4. Let $I = (x^3 - y^4, x^5 - z^7, y^6 - z^8) \cdot C\,[x, y, z]_{(x,y,z)}$ be a parameter ideal in the ring $C\,[x, y, z]_{(x,y,z)}$. As $\gcd(3, 4) = \gcd(5, 7) = 1$, we can take the curve W given by the equations

$$x^3 - y^4 = x^5 - z^7 = 0$$

and the parametrization

$$x = s^{28}$$
$$y = s^{21}$$
$$z = s^{20}.$$

Then the Theorem 1 applied to our ideal I and the variety W provides the equality

$$\begin{aligned} &e_0(x^3 - y^4, x^5 - z^7, y^6 - z^8) \cdot C\left[x, y, z\right]_{(x,y,z)}, C\left[x, y, z\right]_{(x,y,z)} \\ &= e_0((s^{6\cdot 21} - s^{8\cdot 20}) \cdot C\left[s\right]_{(s)}, C\left[s\right]_{(s)}) = 126. \end{aligned}$$

On the other hand, we can take the polynomial

$$y^6 - z^8 = (y^3)^2 - (z^4)^2 = (y^3 - z^4)(y^3 + z^4)$$

and the surface V given by $y^3 - z^4 = 0$, resp. parametrically

$$x = s$$
$$y = t^4$$
$$z = t^3$$

and compute

$$\begin{aligned} &e_0((x^3 - y^4, x^5 - z^7, y^6 - z^8) \cdot C\left[x, y, z\right]_{(x,y,z)}, C\left[x, y, z\right]_{(x,y,z)}) \\ &= 2 \cdot e_0((y^3 - z^4, x^3 - y^4, x^5 - z^7) \cdot C\left[x, y, z\right]_{(x,y,z)}, C\left[x, y, z\right]_{(x,y,z)}) \\ &= 2 \cdot e_0((s^3 - t^{16}, s^5 - t^{21}) \cdot C\left[s, t\right]_{(s,t)}, C\left[s, t\right]_{(s,t)}) = 2 \cdot \min\{3 \cdot 21, 5 \cdot 16\} = 126. \end{aligned}$$

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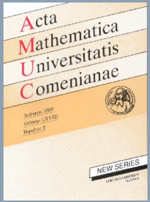

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E. Boďa, KAZDM, Facultuty of Mathematics, Physics and Informatics, Mlynská dolina, 842 48 Bratislava, Slovakia, *e-mail*: `eduard.boda@fmph.uniba.sk`

D. Jašková, Faculty of Mechatronics, University of Trenčín, 911 06 Trenčín, Slovakia, *e-mail*: `jaskova@tnuni.sk`

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