



SOME FAMILIES OF p -VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

M. K. AOUF

ABSTRACT. We introduce two subclasses $T^*(p, \alpha, j)$ and $C(p, \alpha, j)$ ($0 \leq \alpha < p - j + 1$, $1 \leq j \leq p$, $p \in N = \{1, 2, \dots\}$) of p -valent starlike and p -valent convex functions with negative coefficients. In this paper we obtain coefficient inequalities, distortion theorems, extreme points and integral operators for functions belonging to the classes $T^*(p, \alpha, j)$ and $C(p, \alpha, j)$. We also determine the radii of close-to-convexity and convexity for the functions belonging to the class $T^*(p, \alpha, j)$. Also we obtain several results for the modified Hadamard products of functions belonging to the classes $T^*(p, \alpha, j)$ and $C(p, \alpha, j)$.

1. INTRODUCTION

Let $A(p)$ denote the class of functions of the form:

$$(1.1) \quad f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in N = \{1, 2, \dots\})$$

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which are analytic and p -valent in the unit disc $U = \{z : |z| < 1\}$. A function $f(z) \in A(p)$ is called p -valent starlike of order α ($0 \leq \alpha < p$) if $f(z)$ satisfies the conditions

$$(1.2) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in U)$$

and

$$(1.3) \quad \int_0^{2\pi} \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} d\theta = 2p\pi \quad (z \in U).$$

We denote by $S(p, \alpha)$ the class of p -valent starlike functions of order α . Also a function $f(z) \in A(p)$ is called p -valent convex of order α ($0 \leq \alpha < p$) if $f(z)$ satisfies the following conditions

$$(1.4) \quad \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (z \in U)$$

and

$$(1.5) \quad \int_0^{2\pi} \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} d\theta = 2p\pi \quad (z \in U).$$

We denote by $K(p, \alpha)$ the class of p -valent convex functions of order α . We note that

$$(1.6) \quad f(z) \in K(p, \alpha) \quad \text{if and only if} \quad \frac{zf'(z)}{p} \in S(p, \alpha) \quad (0 \leq \alpha < p).$$

The class $S(p, \alpha)$ was introduced by Patil and Thakare [3] and the class $K(p, \alpha)$ was introduced by Owa [1].



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For $0 \leq \alpha < p - j + 1$, $1 \leq j \leq p$ and $p \in \mathbb{N}$, we say $f(z) \in A(p)$ is in the class $S(p, \alpha, j)$ if it satisfies the following inequality:

$$(1.7) \quad \operatorname{Re} \left\{ \frac{z f^{(j)}(z)}{f^{(j-1)}(z)} \right\} > \alpha \quad (z \in U).$$

Also for $0 \leq \alpha < p - j + 1$, $1 \leq j \leq p$ and $p \in \mathbb{N}$, we say $f(z) \in A(p)$ is in the class $K(p, \alpha, j)$ if it satisfies the following inequality:

$$(1.8) \quad \operatorname{Re} \left\{ 1 + \frac{z f^{(j+1)}(z)}{f^{(j)}(z)} \right\} > \alpha \quad (z \in U).$$

It follows from (1.7) and (1.8) that:

$$(1.9) \quad f(z) \in K(p, \alpha, j) \quad \text{if and only if} \quad \frac{z f^{(j)}(z)}{p - j + 1} \in S(p, \alpha, j).$$

The classes $S(p, \alpha, j)$ and $K(p, \alpha, j)$ were studied by Srivastava et al. [6] (see also Nunokawa [2]). We note that $S(p, \alpha, 1) = S(p, \alpha)$ and $K(p, \alpha, 1) = K(p, \alpha)$.

Let $T(p)$ denote the subclass of $A(p)$ consisting of functions of the form:

$$(1.10) \quad f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k \quad (a_k \geq 0; p \in \mathbb{N}).$$

We denote by $T^*(p, \alpha, j)$ and $C(p, \alpha, j)$ the classes obtained by taking intersections, respectively, of the classes $S(p, \alpha, j)$ and $K(p, \alpha, j)$ with $T(p)$, that is

$$T^*(p, \alpha, j) = S(p, \alpha, j) \cap T(p)$$

and

$$C(p, \alpha, j) = K(p, \alpha, j) \cap T(p).$$



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We note that:

$$\begin{aligned} \text{(i)} \quad & T^*(p, \alpha, 1) = T^*(p, \alpha) \quad \text{and} \quad C(p, \alpha, 1) = C(p, \alpha) \quad (\text{Owa [1]}); \\ \text{(ii)} \quad & T^*(1, \alpha, 1) = T^*(\alpha) \quad \text{and} \quad C(1, \alpha, 1) = C(\alpha) \quad (\text{Silverman [5]}). \end{aligned}$$

In this paper we obtain coefficient inequalities, distortion theorems, extreme points and integral operators for functions belonging to the classes $T^*(p, \alpha, j)$ and $C(p, \alpha, j)$. We also determine the radii of close-to-convexity and convexity for the functions belonging to the class $T^*(p, \alpha, j)$. Also we obtain several results for the modified Hadamard products of functions belonging to the classes $T^*(p, \alpha, j)$ and $C(p, \alpha, j)$.

2. COEFFICIENT ESTIMATES

Theorem 1. *Let the function $f(z)$ be defined by (1.10). Then $f(z) \in T^*(p, \alpha, j)$ if and only if*

$$(2.1) \quad \sum_{k=p+1}^{\infty} \frac{\delta(k, j-1)}{\delta(p, j-1)} (k-j+1-\alpha) a_k \leq (p-j+1-\alpha),$$

where

$$(2.2) \quad \delta(p, j) = \frac{p!}{(p-j)!} = \begin{cases} p(p-1)\dots(p-j+1) & (j \neq 0), \\ 1 & (j = 0). \end{cases}$$



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Proof. Assume that the inequality (2.1) holds true. Then we obtain

$$\begin{aligned}
 (2.3) \quad \left| \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} - (p-j+1) \right| &= \left| \frac{\sum_{k=p+1}^{\infty} \frac{k!(k-p)}{(k-j+1)!} a_k z^{k-p}}{\frac{p!}{(p-j+1)!} - \sum_{k=p+1}^{\infty} \frac{k!}{(k-j+1)!} a_k z^{k-p}} \right| \\
 &\leq \frac{\sum_{k=p+1}^{\infty} \frac{k!(k-p)}{(k-j+1)!} a_k}{\frac{p!}{(p-j+1)!} - \sum_{k=p+1}^{\infty} \frac{k!}{(k-j+1)!} a_k} \\
 &\leq p-j+1-\alpha.
 \end{aligned}$$

This shows that the values of $\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}$ lie in a circle which is centered at $w = (p-j+1)$ and whose radius is $p-j+1-\alpha$. Hence $f(z)$ satisfies the condition (1.7).

Conversely, assume that the function $f(z)$ defined by (1.10) is in the class $T^*(p, \alpha, j)$. Then we have

$$(2.4) \quad \operatorname{Re} \left\{ \frac{zf^{(j)}(z)}{f^{(j-1)}(z)} \right\} \operatorname{Re} \left\{ \frac{\frac{p!}{(p-j)!} - \sum_{k=p+1}^{\infty} \frac{k!}{(k-j)!} a_k z^{k-p}}{\frac{p!}{(p-j+1)!} - \sum_{k=p+1}^{\infty} \frac{k!}{(k-j+1)!} a_k z^{k-p}} \right\} > \alpha$$

for $0 \leq \alpha < p-j+1$, $1 \leq j \leq p$, $p \in \mathbb{N}$ and $z \in U$. Choose values of z on the real axis so that $\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}$ is real. Upon clearing the denominator in (2.4) and letting $z \rightarrow 1^-$ through real values,



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we can see that

$$(2.5) \quad \frac{p!}{(p-j)!} - \sum_{k=p+1}^{\infty} \frac{k!}{(k-j)!} a_k \geq \alpha \left(\frac{p!}{(p-j+1)!} - \sum_{k=p+1}^{\infty} \frac{k!}{(k-j+1)!} a_k \right).$$

Thus we have the required inequality (2.1). □

Corollary 1. *Let the function $f(z)$ defined by (1.10) be in the class $T^*(p, \alpha, j)$. Then we have*

$$(2.6) \quad a_k \leq \frac{\delta(p, j-1)(p-j+1-\alpha)}{\delta(k, j-1)(k-j+1-\alpha)} \quad (k \geq p+1; p \in N).$$

The result is sharp for the function $f(z)$ given by

$$(2.7) \quad f(z) = z^p - \frac{\delta(p, j-1)(p-j+1-\alpha)}{\delta(k, j-1)(k-j+1-\alpha)} z^k \quad (k \geq p+1; p \in N).$$

Theorem 2. *Let the function $f(z)$ be defined by (1.10). Then $f(z) \in C(p, \alpha, j)$ if and only if*

$$(2.8) \quad \sum_{k=p+1}^{\infty} \frac{\delta(k, j)}{\delta(p, j)} (k-j+1-\alpha) a_k \leq (p-j+1-\alpha).$$

Proof. Since $f(z) \in C(p, \alpha, j)$ if and only if $\frac{zf^{(j)}(z)}{p-j+1} \in T^*(p, \alpha, j)$, we have the theorem by replacing a_k with $\left(\frac{k-j+1}{p-j+1}\right) a_k$ ($k \geq p+1$) in Theorem 1. □

Corollary 2. *Let the function $f(z)$ defined by (1.10) be in the class $C(p, \alpha, j)$. Then we have*

$$(2.9) \quad a_k \leq \frac{\delta(p, j)(p-j+1-\alpha)}{\delta(k, j)(k-j+1-\alpha)} \quad (k \geq p+1; p \in N).$$

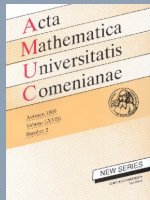


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The result is sharp for the function $f(z)$ given by

$$(2.10) \quad f(z) = z^p - \frac{\delta(p, j)(p - j + 1 - \alpha)}{\delta(k, j)(k - j + 1 - \alpha)} z^k \quad (k \geq p + 1; p \in N).$$

3. EXTREME POINTS

From Theorem 1 and Theorem 2, we see that both $T^*(p, \alpha, j)$ and $C(p, \alpha, j)$ are closed under convex linear combinations, which enables us to determine the extreme points for these classes.

Theorem 3. *Let*

$$(3.1) \quad f_p(z) = z^p$$

and

$$(3.2) \quad f_k(z) = z^p - \frac{\delta(p, j - 1)(p - j + 1 - \alpha)}{\delta(k, j - 1)(k - j + 1 - \alpha)} z^k \quad (k \geq p + 1; p \in N).$$

Then $f(z) \in T^*(p, \alpha, j)$ if and only if it can be expressed in the form

$$(3.3) \quad f(z) = \sum_{k=p}^{\infty} \lambda_k f_k(z),$$

where $\lambda_k \geq 0$ ($k \geq p$) and $\sum_{k=p}^{\infty} \lambda_k = 1$.

Proof. Suppose that

$$(3.4) \quad f(z) = \sum_{k=p}^{\infty} \lambda_k f_k(z) = z^p - \sum_{k=p+1}^{\infty} \frac{\delta(p, j - 1)(p - j + 1 - \alpha)}{\delta(k, j - 1)(k - j + 1 - \alpha)} \lambda_k z^k.$$

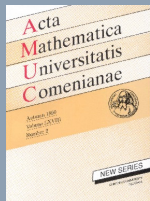


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Then it follows that

$$\begin{aligned}
 (3.5) \quad & \sum_{k=p+1}^{\infty} \frac{\delta(k, j-1)(k-j+1-\alpha)}{\delta(p, j-1)(p-j+1-\alpha)} \cdot \frac{\delta(p, j-1)(p-j+1-\alpha)}{\delta(k, j-1)(k-j+1-\alpha)} \lambda_k \\
 & = \sum_{k=p+1}^{\infty} \lambda_k = 1 - \lambda_p \leq 1.
 \end{aligned}$$

Therefore, by Theorem 1, $f(z) \in T^*(p, \alpha, j)$.

Conversely, assume that the function $f(z)$ defined by (1.10) belongs to the class $T^*(p, \alpha, j)$. Then

$$(3.6) \quad a_k \leq \frac{\delta(p, j-1)(p-j+1-\alpha)}{\delta(k, j-1)(k-j+1-\alpha)} \quad (k \geq p+1; k \in N).$$

Setting

$$(3.7) \quad \lambda_k = \frac{\delta(k, j-1)(k-j+1-\alpha)}{\delta(p, j-1)(p-j+1-\alpha)} a_k \quad (k \geq p+1; k \in N)$$

and

$$(3.8) \quad \lambda_p = 1 - \sum_{k=p+1}^{\infty} \lambda_k,$$

we see that $f(z)$ can be expressed in the form (3.3). This completes the proof of Theorem 3. \square

Corollary 3. *The extreme points of the class $T^*(p, \alpha, j)$ are the functions $f_p(z) = z^p$ and*

$$f_k(z) = z^p - \frac{\delta(p, j-1)(p-j+1-\alpha)}{\delta(k, j-1)(k-j+1-\alpha)} z^k \quad (k \geq p+1; k \in N).$$



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Similarly, we have

Theorem 4. *Let*

$$(3.9) \quad f_p(z) = z^p$$

and

$$(3.10) \quad f_k(z) = z^p - \frac{\delta(p, j)(p - j + 1 - \alpha)}{\delta(k, j)(k - j + 1 - \alpha)} z^k \quad (k \geq p + 1; \quad p \in N).$$

Then $f(z) \in C(p, \alpha, j)$ if and only if it can be expressed in the form

$$(3.11) \quad f(z) = \sum_{k=p}^{\infty} \lambda_k f_k(z),$$

where $\lambda_k \geq 0$ ($k \geq p$) and $\sum_{k=p}^{\infty} \lambda_k = 1$.

Corollary 4. *The extreme points of the class $C(p, \alpha, j)$ are the functions $f_p(z) = z^p$ and*

$$f_k(z) = z^p - \frac{\delta(p, j)(p - j + 1 - \alpha)}{\delta(k, j)(k - j + 1 - \alpha)} z^k \quad (k \geq p + 1; \quad p \in N).$$

4. DISTORTION THEOREMS

Theorem 5. *Let the function $f(z)$ defined by (1.10) be in the class $T^*(p, \alpha, j)$. Then, for $|z| = r < 1$,*

$$(4.1) \quad r^p - \frac{(p - j + 1 - \alpha)(p - j + 2)}{(p - j + 2 - \alpha)(p + 1)} r^{p+1} \leq |f(z)| \leq r^p + \frac{(p - j + 1 - \alpha)(p - j + 2)}{(p - j + 2 - \alpha)(p + 1)} r^{p+1}$$



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and

$$(4.2) \quad pr^{p-1} - \frac{(p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)}r^p \leq |f'(z)| \leq pr^{p-1} + \frac{(p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)}r^p.$$

The equalities in (4.1) and (4.2) are attained for the function $f(z)$ given by

$$(4.3) \quad f(z) = z^p - \frac{(p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)(p+1)}z^{p+1} \quad (z = \pm r).$$

Proof. Since $f(z) \in T^*(p, \alpha, j)$, in view of Theorem 1, we have

$$\frac{\delta(p+1, j-1)(p-j+2-\alpha)}{\delta(p, j-1)} \sum_{k=p+1}^{\infty} a_k \leq \sum_{k=p+1}^{\infty} \frac{\delta(k, j-1)}{\delta(p, j-1)} (k-j+1-\alpha)a_k$$

$$\leq (p-j+1-\alpha),$$

which evidently yields

$$(4.4) \quad \sum_{k=p+1}^{\infty} a_k \leq \frac{(p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)(p+1)}.$$

Consequently, for $|z| = r < 1$, we obtain

$$|f(z)| \leq r^p + r^{p+1} \sum_{k=p+1}^{\infty} a_k \leq r^p + \frac{(p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)(p+1)}r^{p+1}$$

and

$$|f(z)| \geq r^p - r^{p+1} \sum_{k=p+1}^{\infty} a_k \geq r^p - \frac{(p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)(p+1)}r^{p+1},$$



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which prove the assertion (4.1) of Theorem 5.

Also from Theorem 1, it follows that

$$(4.5) \quad \sum_{k=p+1}^{\infty} ka_k \leq \frac{(p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)}.$$

Consequently, for $|z| = r < 1$, we have

$$\begin{aligned} |f'(z)| &\leq pr^{p-1} + \sum_{k=p+1}^{\infty} ka_k r^{k-1} \leq pr^{p-1} + r^p \sum_{k=p+1}^{\infty} ka_k \\ &\leq pr^{p-1} + \frac{(p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)} r^p \end{aligned}$$

and

$$\begin{aligned} |f'(z)| &\geq pr^{p-1} - \sum_{k=p+1}^{\infty} ka_k r^{k-1} \geq pr^{p-1} - r^p \sum_{k=p+1}^{\infty} ka_k \\ &\geq pr^{p-1} - \frac{(p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)} r^p, \end{aligned}$$

which prove the assertion (4.2) of Theorem 5. □

Finally, it is easy to see that the bounds in (4.1) and (4.2) are attained for the function $f(z)$ given already by (4.3).

Corollary 5. *Let the function $f(z)$ defined by (1.10) be in the class $T^*(p, \alpha, j)$. Then the unit disc U is mapped onto a domain that contains the disc*

$$(4.6) \quad |w| < \frac{(p-j+2-\alpha)(p+1) - (p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)(p+1)}.$$



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The result is sharp, with the extremal function $f(z)$ given by (4.3).

Theorem 6. Let the function $f(z)$ defined by (1.10) be in the class $C(p, \alpha, j)$. Then, for $|z| = r < 1$,

$$(4.7) \quad r^p - \frac{(p-j+1-\alpha)(p-j+1)}{(p-j+2-\alpha)(p+1)}r^{p+1} \leq |f(z)| \leq r^p + \frac{(p-j+1-\alpha)(p-j+1)}{(p-j+2-\alpha)(p+1)}r^{p+1}$$

and

$$(4.8) \quad pr^{p-1} - \frac{(p-j+1-\alpha)(p-j+1)}{(p-j+2-\alpha)}r^p \leq |f'(z)| \leq pr^{p-1} + \frac{(p-j+1-\alpha)(p-j+1)}{(p-j+2-\alpha)}r^p.$$

The results are sharp.

Proof. The proof of Theorem 6 is obtained by using the same technique as in the proof of Theorem 5 with the aid of Theorem 2. Further we can show that the bounds of Theorem 6 are sharp for the function $f(z)$ defined by

$$(4.9) \quad f(z) = z^p - \frac{(p-j+1-\alpha)(p-j+1)}{(p-j+2-\alpha)(p+1)}z^{p+1}.$$

□

Corollary 6. Let the function $f(z)$ defined by (1.10) be in the class $C(p, \alpha, j)$. Then the unit disc U is mapped onto a domain that contains the disc

$$(4.10) \quad |w| < \frac{(p-j+2-\alpha)(p+1) - (p-j+1-\alpha)(p-j+1)}{(p-j+2-\alpha)(p+1)}.$$

The result is sharp, with the extremal function $f(z)$ given by (4.9).



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5. INTEGRAL OPERATORS

Theorem 7. Let the function $f(z)$ defined by (1.10) be in the class $T^*(p, \alpha, j)$, and let c be a real number such that $c > -p$. Then the function $F(z)$ defined by

$$(5.1) \quad F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t) dt$$

also belongs to the class $T^*(p, \alpha, j)$.

Proof. From the representation of $F(z)$, it follows that

$$(5.2) \quad F(z) = z^p - \sum_{k=p+1}^{\infty} b_k z^k,$$

where

$$b_k = \left(\frac{c+p}{c+k} \right) a_k.$$

Therefore

$$\begin{aligned} & \sum_{k=p+1}^{\infty} \frac{\delta(k, j-1)(k-j+1-\alpha)}{\delta(p, j-1)} b_k \\ &= \sum_{k=p+1}^{\infty} \frac{\delta(k, j-1)(k-j+1-\alpha)}{\delta(p, j-1)} \left(\frac{c+p}{c+k} \right) a_k \\ &\leq \sum_{k=p+1}^{\infty} \frac{\delta(k, j-1)(k-j+1-\alpha)}{\delta(p, j-1)} a_k \leq (p-j+1-\alpha), \end{aligned}$$



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since $f(z) \in T^*(p, \alpha, j)$. Hence, by Theorem 1, $f(z) \in T^*(p, \alpha, j)$. □

Corollary 7. *Under the same conditions as Theorem 7, a similar proof shows that the function $F(z)$ defined by (5.1) is in the class $C(p, \alpha, j)$, whenever $f(z)$ is in the class $C(p, \alpha, j)$.*

6. RADII OF CLOSE-TO-CONVEXITY AND CONVEXITY FOR THE CLASS $T^*(p, \alpha, j)$

Theorem 8. *Let the function $f(z)$ defined by (1.10) be in the class $T^*(p, \alpha, j)$, then $f(z)$ is p -valently close-to-convex of order ϕ ($0 \leq \phi < p$) in $|z| < r_1$, where*

$$(6.1) \quad r_1 = \inf_k \left\{ \frac{\delta(k, j-1)(k-j+1-\alpha)}{\delta(p, j-1)(p-j+1-\alpha)} \left(\frac{p-\phi}{k} \right) \right\}^{\frac{1}{k-p}} \quad (k \geq p+1).$$

The result is sharp, with the extremal function $f(z)$ given by (2.7).

Proof. We must show that $\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \phi$ for $|z| < r_1$. We have

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq \sum_{k=p+1}^{\infty} k a_k |z|^{k-p}.$$

Thus $\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \phi$ if

$$(6.2) \quad \sum_{k=p+1}^{\infty} \left(\frac{k}{p-\phi} \right) a_k |z|^{k-p} \leq 1.$$

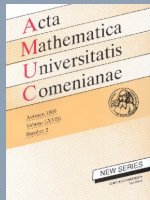


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Hence, by Theorem 1, (6.2) will be true if

$$\left(\frac{k}{p-\phi}\right) |z|^{k-p} \leq \frac{\delta(k, j-1)(k-j+1-\alpha)}{\delta(p, j-1)(p-j+1-\alpha)}$$

or if

$$(6.3) \quad |z| \leq \left\{ \frac{\delta(k, j-1)(k-j+1-\alpha)}{\delta(p, j-1)(p-j+1-\alpha)} \left(\frac{p-\phi}{k}\right) \right\}^{\frac{1}{k-p}} \quad (k \geq p+1).$$

The theorem follows easily from (6.3). □

Theorem 9. Let the function $f(z)$ defined by (1.10) be in the class $T^*(p, \alpha, j)$ then $f(z)$ is p -valently convex of order ϕ ($0 \leq \phi < p$) in $|z| < r_2$, where

$$(6.4) \quad r_2 = \inf_k \left\{ \frac{\delta(p, j-1)(k-j+1-\alpha)}{\delta(k, j-1)(p-j+1-\alpha)} \cdot \left(\frac{p(p-\phi)}{k(k-\phi)}\right) \right\}^{\frac{1}{k-p}} \quad (k \geq p+1).$$

The result is sharp, with the extremal function $f(z)$ given by (2.7).

Proof. It is sufficient to show that

$$\left| 1 + \frac{z f''(z)}{f'(z)} - p \right| \leq p - \phi \quad \text{for} \quad |z| < r_2.$$

We have

$$\left| 1 + \frac{z f''(z)}{f'(z)} - p \right| \leq \frac{\sum_{k=p+1}^{\infty} k(k-p) a_k |z|^{k-p}}{p - \sum_{k=p+1}^{\infty} k a_k |z|^{k-p}}.$$

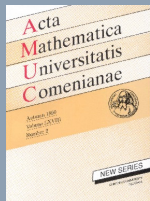


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Thus $\left|1 + \frac{zf''(z)}{f'(z)} - p\right| \leq p - \phi$ if

$$(6.5) \quad \sum_{k=p+1}^{\infty} \frac{k(k-\phi)}{p(p-\phi)} a_k |z|^{k-p} \leq 1$$

Hence, by Theorem 1, (6.5) will be true if

$$\frac{k(k-\phi)}{p(p-\phi)} |z|^{k-p} \leq \frac{\delta(k, j-1)(k-j+1-\alpha)}{\delta(p, j-1)(p-j+1-\alpha)}.$$

or if

$$(6.6) \quad |z| \leq \left\{ \frac{\delta(k, j-1)(k-j+1-\alpha)}{\delta(p, j-1)(p-j+1-\alpha)} \cdot \left(\frac{p(p-\phi)}{k(k-\phi)} \right) \right\}^{\frac{1}{k-p}} \quad (k \geq p+1).$$

The theorem follows easily from (6.6). □

7. MODIFIED HADAMARD PRODUCTS

Let the functions $f_v(z)$ ($v = 1, 2$) be defined by

$$(7.1) \quad f_v(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,v} z^k \quad (a_{k,v} \geq 0; \quad v = 1, 2).$$

Then the modified Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ is defined by

$$(7.2) \quad (f_1 * f_2)(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,1} a_{k,2} z^k.$$



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Theorem 10. Let the functions $f_v(z)$ ($v = 1, 2$) defined by (7.1) be in the class $T^*(p, \alpha, j)$. Then $(f_1 * f_2)(z) \in T^*(p, \gamma, j)$, where

$$(7.3) \quad \gamma = (p - j + 1) - \frac{(p - j + 1 - \alpha)^2(p - j + 2)}{(p - j + 2 - \alpha)^2(p + 1) - (p - j + 1 - \alpha)^2(p - j + 2)}.$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silveanu [4], we need to find the largest γ such that

$$(7.4) \quad \sum_{k=p+1}^{\infty} \frac{\delta(k, j - 1)(k - j + 1 - \gamma)}{\delta(p, j - 1)(p - j + 1 - \gamma)} a_{k,1} a_{k,2} \leq 1.$$

Since

$$(7.5) \quad \sum_{k=p+1}^{\infty} \frac{\delta(k, j - 1)(k - j + 1 - \alpha)}{\delta(p, j - 1)(p - j + 1 - \alpha)} a_{k,1} \leq 1$$

and

$$(7.6) \quad \sum_{k=p+1}^{\infty} \frac{\delta(k, j - 1)(k - j + 1 - \alpha)}{\delta(p, j - 1)(p - j + 1 - \alpha)} a_{k,2} \leq 1,$$

by the Cauchy-Schwarz inequality, we have

$$(7.7) \quad \sum_{k=p+1}^{\infty} \frac{\delta(k, j - 1)(k - j + 1 - \alpha)}{\delta(p, j - 1)(p - j + 1 - \alpha)} \sqrt{a_{k,1} a_{k,2}} \leq 1.$$



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Thus it is sufficient to show that

$$(7.8) \quad \frac{(k-j+1-\gamma)}{(p-j+1-\gamma)} a_{k,1} a_{k,2} \leq \frac{(k-j+1-\alpha)}{(p-j+1-\alpha)} \sqrt{a_{k,1} a_{k,2}} \quad (k \geq p+1),$$

that is

$$(7.9) \quad \sqrt{a_{k,1} a_{k,2}} \leq \frac{(k-j+1-\alpha)(p-j+1-\gamma)}{(k-j+1-\gamma)(p-j+1-\alpha)}.$$

Note that

$$(7.10) \quad \sqrt{a_{k,1} a_{k,2}} \leq \frac{\delta(p, j-1)(p-j+1-\alpha)}{\delta(k, j-1)(k-j+1-\alpha)} \quad (k \geq p+1).$$

Consequently, we need only to prove that

$$(7.11) \quad \frac{\delta(p, j-1)(p-j+1-\alpha)}{\delta(k, j-1)(k-j+1-\alpha)} \leq \frac{(k-j+1-\alpha)(p-j+1-\gamma)}{(k-j+1-\gamma)(p-j+1-\alpha)} \quad (k \geq p+1)$$

or, equivalently, that

$$(7.12) \quad \gamma \leq (p-j+1) - \frac{\delta(p, j-1)(p-j+1-\alpha)^2(k-p)}{\delta(k, j-1)(k-j+1-\alpha)^2 - \delta(p, j-1)(p-j+1-\alpha)^2} \quad (k \geq p+1).$$

Since

$$(7.13) \quad D(k) = (p-j+1) - \frac{\delta(p, j-1)(p-j+1-\alpha)^2(k-p)}{\delta(k, j-1)(k-j+1-\alpha)^2 - \delta(p, j-1)(p-j+1-\alpha)^2}$$



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is an increasing function of k ($k \geq p + 1$), letting $k = p + 1$ in (7.13) we obtain

$$(7.14) \quad \gamma \leq D(p+1) = (p-j+1) - \frac{(p-j+1-\alpha)^2(p-j+2)}{(p-j+2-\alpha)^2(p+1) - (p-j+1-\alpha)^2(p-j+2)},$$

which completes the proof Theorem 10. □

Finally, by taking the functions

$$(7.15) \quad f_v(z) = z^p - \frac{(p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)(p+1)} z^{p+1} \quad (v = 1, 2; \quad p \in \mathbb{N})$$

we can see that the result is sharp.

Corollary 8. *Let the functions $f_v(z)$ ($v = 1, 2$) be the same as in Theorem 10, we have*

$$(7.16) \quad h(z) = z^p - \sum_{k=p+1}^{\infty} \sqrt{a_{k,1}a_{k,2}} z^k$$

belongs to the class $T^(p, \alpha, j)$.*

The result follows from the inequality (7.7). It is sharp for the same functions as in Theorem 10.

Corollary 9. *Let the functions $f_\nu(z)$ ($\nu = 1, 2$) defined by (7.1) be in the class $C(p, \alpha, j)$. Then $(f_1 * f_2)(z) \in C(p, \lambda, j)$ where*

$$(7.17) \quad \lambda = (p-j+1) - \frac{(p-j+1-\alpha)^2(p-j+1)}{(p-j+2-\alpha)^2(p+1) - (p-j+1-\alpha)^2(p-j+1)}.$$

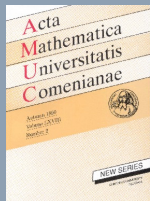


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The result is sharp for the functions

$$(7.18) \quad f_\nu(z) = z^p - \frac{(p-j+1-\alpha)(p-j+1)}{(p-j+2-\alpha)(p+1)} z^{p+1} \quad (\nu = 1, 2 \quad p \in N).$$

Using arguments similar to those in the proof of Theorem 10, we obtain the following result.

Theorem 11. Let the function $f_1(z)$ defined by (7.1) be in the class $T^*(p, \alpha, j)$ and the function $f_2(z)$ defined by (7.1) be in the class $T^*(p, \tau, j)$, then $(f_1 * f_2)(z) \in T^*(p, \zeta, j)$, where

$$(7.19) \quad \zeta = (p-j+1) - \frac{(p-j+1-\alpha)(p-j+1-\tau)(p-j+2)}{(p-j+2-\alpha)(p-j+2-\tau)(p+1) - (p-j+1-\alpha)(p-j+1-\tau)(p-j+2)}.$$

The result is the best possible for the functions

$$(7.20) \quad f_1(z) = z^p - \frac{(p-j+1-\alpha)(p-j+2)}{(p-j+2-\alpha)(p+1)} z^{p+1} \quad (p \in N)$$

and

$$(7.21) \quad f_2(z) = z^p - \frac{(p-j+1-\tau)(p-j+2)}{(p-j+2-\tau)(p+1)} z^{p+1} \quad (p \in N).$$



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Corollary 10. Let the function $f_1(z)$ defined by (7.1) be in the class $C(p, \alpha, j)$ and the function $f_2(z)$ defined by (7.1) be in the class $C(p, \tau, j)$, then $(f_1 * f_2)(z) \in C(p, \theta, j)$, where

$$\theta = (p - j + 1) - \frac{(p - j + 1 - \alpha)(p - j + 1 - \tau)(p - j + 1)}{(p - j + 2 - \alpha)(p - j + 2 - \tau)(p + 1) - (p - j + 1 - \alpha)(p - j + 1 - \tau)(p - j + 1)}.$$

(7.22)

The result is sharp for the functions

$$(7.23) \quad f_1(z) = z^p - \frac{(p - j + 1 - \alpha)(p - j + 1)}{(p - j + 2 - \alpha)(p + 1)} z^{p+1} \quad (p \in N)$$

and

$$(7.24) \quad f_2(z) = z^p - \frac{(p - j + 1 - \tau)(p - j + 1)}{(p - j + 2 - \tau)(p + 1)} z^{p+1} \quad (p \in N).$$

Theorem 12. Let the functions $f_\nu(z)$ ($\nu = 1, 2$) defined by (7.1) be in the class $T^*(p, \alpha, j)$. Then the function

$$(7.25) \quad h(z) = z^p - \sum_{k=p+1}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k$$

belongs to the class $T^*(p, \varphi, j)$, where

$$(7.26) \quad \varphi = (p - j + 1) - \frac{2(p - j + 1 - \alpha)^2(p - j + 2)}{(p - j + 2 - \alpha)^2(p + 1) - 2(p - j + 1 - \alpha)^2(p - j + 2)}.$$

The result is sharp for the functions $f_\nu(z)$ ($\nu = 1, 2$) defined by (7.15).



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Proof. By virtue of Theorem 1, we obtain

$$(7.27) \quad \sum_{k=p+1}^{\infty} \left[\frac{\delta(k, j-1)(k-j+1-\alpha)}{\delta(p, j-1)(p-j+1-\alpha)} \right]^2 a_{k,\nu}^2 \leq \left[\sum_{k=p+1}^{\infty} \frac{\delta(k, j-1)(k-j+1-\alpha)}{\delta(p, j-1)(p-j+1-\alpha)} a_{k,\nu} \right]^2 \leq 1 \quad (\nu = 1, 2).$$

It follows from (7.27) for $\nu = 1$ and $\nu = 2$ that

$$(7.28) \quad \sum_{k=p+1}^{\infty} \frac{1}{2} \left[\frac{\delta(k, j-1)(k-j+1-\alpha)}{\delta(p, j-1)(p-j+1-\alpha)} \right]^2 (a_{k,1}^2 + a_{k,2}^2) \leq 1.$$

Therefore, we need to find the largest φ such that

$$(7.29) \quad \frac{\delta(k, j-1)(k-j+1-\varphi)}{\delta(p, j-1)(p-j+1-\varphi)} \leq \frac{1}{2} \left[\frac{\delta(k, j-1)(k-j+1-\alpha)}{\delta(p, j-1)(p-j+1-\alpha)} \right]^2 \quad (k \geq p+1)$$

that is,

$$(7.30) \quad \varphi \leq (p-j+1) - \frac{2\delta(p, j-1)(k-j+1-\alpha)^2(k-p)}{\delta(k, j-1)(k-j+1-\alpha)^2 - 2\delta(p, j-1)(p-j+1-\alpha)^2} \quad (k \geq p+1).$$

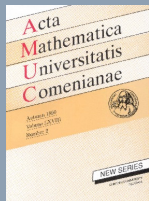


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Since

$$\Psi(k) = (p - j + 1) - \frac{2\delta(p, j - 1)(p - j + 1 - \alpha)^2(k - p)}{\delta(k, j - 1)(k - j + 1 - \alpha)^2 - 2\delta(p, j - 1)(p - j + 1 - \alpha)^2}$$

is an increasing function of k ($k \geq p + 1$), we readily have

$$\varphi \leq \Psi(p + 1) = (p - j + 1) - \frac{2(p - j + 1 - \alpha)^2(p - j + 2)}{(p - j + 2 - \alpha)^2(p + 1) - 2(p - j + 1 - \alpha)^2(p - j + 2)},$$

and Theorem 12 follows at once. □

Corollary 11. *Let the functions $f_\nu(z)$ ($\nu = 1, 2$) defined by (7.1) be in the class $C(p, \alpha, j)$. Then the function $h(z)$ defined by (7.25) belongs to the class $C(p, \xi, j)$, where*

$$(7.31) \quad \xi = (p - j + 1) - \frac{2(p - j + 1 - \alpha)^2(p - j + 1)}{(p - j + 2 - \alpha)^2(p + 1) - 2(p - j + 1 - \alpha)^2(p - j + 1)}.$$

The result is sharp for the functions $f_\nu(z)$ ($\nu = 1, 2$) defined by (7.18).

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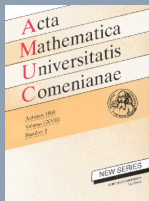


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M. K. Aouf, Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt, *e-mail*: mkaouf127@yahoo.com



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