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ON THE HILBERT INEQUALITY

ZHOU YU AND GAO MINGZHE

Abstract. In this paper it is shown that the Hilbert inequality for double series can be improved by introducing a weight function of the form $\frac{\sqrt{n}}{n+1}\left(\frac{\sqrt{n}-1}{\sqrt{n}+1}-\frac{\ln n}{\pi}\right)$, where $n \in N$. A similar result for the Hilbert integral inequality is also given. As applications, some sharp results of Hardy-Littlewood's theorem and Widder's theorem are obtained.

1. Introduction

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of complex numbers. It is all-round known that the inequality

$$\left|\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m\bar{b}_n}{m+n}\right|^2 \le \pi^2\sum_{n=1}^{\infty}|a_n|^2\sum_{n=1}^{\infty}|b_n|^2 \tag{1.1}$$

is called the Hilbert inequality for double series, where $\sum\limits_{n=1}^{\infty}|a_n|^2 < +\infty$ and $\sum\limits_{n=1}^{\infty}|b_n|^2 < +\infty$, and that the constant factor π^2 in (1.1) is the best possible. The equality in (1.1) holds if and only if $\{a_n\}$, or $\{b_n\}$ is a zero-sequence (see [**?**]). The corresponding integral form of (1.1) is that

$$\left|\int_0^{\infty}\int_0^{\infty}\frac{f(x)\bar{g}(y)}{x+y}\mathrm{d}x\mathrm{d}y\right|^2 \le \pi^2\left(\int_0^{\infty}|f(x)|^2\,\mathrm{d}x\right)\left(\int_0^{\infty}|g\left(x\right)|^2\,\mathrm{d}x\right) \tag{1.2}$$

where $\int\limits_0^{\infty}|f\left(x\right)|^2\mathrm{d}x < +\infty$ and $\int\limits_0^{\infty}|g\left(x\right)|^2\mathrm{d}x < +\infty$, and that the constant factor π^2 in (1.2) is also the best possible. The equality in (1.2) holds if and only if $f\left(x\right) = 0$, or $g\left(x\right) = 0$. Recently, various improvements and extensions of (1.1) and (1.2) appeared in a great deal of papers (see [**?**]). The purpose of the present paper is to build the Hilbert inequality with the weights by means of a monotonic function

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of the form $\frac{\sqrt{x}}{1+\sqrt{x}}$, thereby new refinements of (1.1) and (1.2) are established, and then to give some of their important applications.

For convenience, we need the following lemmas.

Lemma 1.1. *Let $n \in \mathbb{N}$. Then*

$$\int_0^\infty \frac{\mathrm{d}x}{(n+x^2)(1+x)} = \frac{1}{n+1}\left(\frac{\pi}{2\sqrt{n}} + \frac{1}{2}\ln n\right) \tag{1.3}$$

Proof. Let a, e and f be real numbers. Then

$$\begin{aligned}&\int \frac{\mathrm{d}x}{(a^2+x^2)(e+fx)}\\ &\quad= \frac{1}{e^2+a^2f^2}\left\{f\ln|e+fx| - \frac{1}{2}\ln(a^2+x^2) + \frac{e}{a}\arctan\frac{x}{a}\right\} + C\end{aligned}$$

where C is an arbitrary constant. This result has been given in the papers (see [**3**]–[**4**]). Based on this indefinite integral it is easy to deduce that the equality (1.3) is true. □

Lemma 1.2. *Let $n \in \mathbb{N}$, $x \in (0, +\infty)$. Define two functions by*

$$f(x) = \left(\frac{1}{x+n}\left(\frac{n}{x}\right)^{\frac{1}{2}}\right)\left(1 - \left(\frac{\sqrt{x}}{1+\sqrt{x}} - \frac{\sqrt{n}}{1+\sqrt{n}}\right)\right)$$
$$g(x) = \left(\frac{1}{x+n}\left(\frac{n}{x}\right)^{\frac{1}{2}}\right)\left(1 + \left(\frac{\sqrt{x}}{1+\sqrt{x}} - \frac{\sqrt{n}}{1+\sqrt{n}}\right)\right),$$

then $f(x)$ and $g(x)$ are monotonously decreasing in (0, + ∞), and

$$\int_0^\infty f(x)\,\mathrm{d}x = \pi - \pi\omega(n) \tag{1.4}$$

$$\int_0^\infty g(x)\,\mathrm{d}x = \pi + \pi\omega(n) \tag{1.5}$$

where the weight function ω is defined by

$$\omega(n) = \frac{\sqrt{n}}{n+1}\left(\frac{\sqrt{n}-1}{\sqrt{n}+1} - \frac{\ln n}{\pi}\right) \tag{1.6}$$

Proof. At first, notice that $1 - \frac{\sqrt{x}}{1+\sqrt{x}} = \frac{1}{1+\sqrt{x}}$, hence we can write $f(x)$ in form $f(x) = f_1(x) + f_2(x)$, where

$$f_1(x) = \left(\frac{1}{(x+n)\sqrt{x}}\right)\left(\frac{n}{1+\sqrt{n}}\right), \qquad f_2(x) = \frac{\sqrt{n}}{(x+n)(1+\sqrt{x})\sqrt{x}}.$$

It is obvious that $f_1(x)$ and $f_2(x)$ are monotonously decreasing in $(0, +\infty)$. Hence $f(x)$ is monotonously decreasing in $(0, +\infty)$. Next, notice that $1 - \frac{\sqrt{n}}{1+\sqrt{n}} = \frac{1}{1+\sqrt{n}}$, we can write $g(x)$ in form $g(x) = g_1(x) + g_2(x)$, where

$$g_1(x) = \frac{\sqrt{n}}{(1+\sqrt{n})(x+n)\sqrt{x}}, \qquad g_2(x) = \frac{\sqrt{n}}{(x+n)(1+\sqrt{x})}.$$

It is obvious that $g_1(x)$ and $g_2(x)$ are monotonously decreasing in $(0, +\infty)$. Hence $g(x)$ is also monotonously decreasing in $(0, +\infty)$. Further we need only to compute two integrals.

$$\begin{aligned}
\int_0^\infty f(x)\,\mathrm{d}x &= \int_0^\infty \left(\frac{1}{x+n}\left(\frac{n}{x}\right)^{\frac{1}{2}}\right)\left(1+\frac{\sqrt{n}}{1+\sqrt{n}}-\frac{\sqrt{x}}{1+\sqrt{x}}\right)\mathrm{d}x \\
&= \left(1+\frac{\sqrt{n}}{1+\sqrt{n}}\right)\int_0^\infty \left(\frac{1}{x+n}\left(\frac{n}{x}\right)^{\frac{1}{2}}\right)\mathrm{d}x - \int_0^\infty \left(\frac{1}{x+n}\left(\frac{n}{x}\right)^{\frac{1}{2}}\right)\left(\frac{\sqrt{x}}{1+\sqrt{x}}\right)\mathrm{d}x \\
&= \left(1+\frac{\sqrt{n}}{1+\sqrt{n}}\right)\pi - \int_0^\infty \left(\frac{1}{x+n}\left(\frac{n}{x}\right)^{\frac{1}{2}}\right)\left(\frac{\sqrt{x}}{1+\sqrt{x}}\right)\mathrm{d}x \\
&= \pi - \left\{2\sqrt{n}\left(\int_0^\infty \frac{1}{(n+t^2)}\mathrm{d}t - \int_0^\infty \frac{1}{(n+t^2)(1+t)}\mathrm{d}t\right) - \frac{\sqrt{n}\,\pi}{1+\sqrt{n}}\right\} \\
&= \pi - \left\{\pi - 2\sqrt{n}\int_0^\infty \frac{1}{(n+t^2)(1+t)}\mathrm{d}t - \frac{\sqrt{n}\,\pi}{1+\sqrt{n}}\right\}
\end{aligned}$$

By Lemma 1.1, we obtain

$$\int_0^\infty f(x)\,\mathrm{d}x = \pi - \left\{\pi - \left(\frac{\pi}{n+1}+\frac{\sqrt{n}\ln n}{n+1}\right) - \frac{\sqrt{n}\,\pi}{1+\sqrt{n}}\right\} \tag{1.7}$$

The equality (1.4) follows from (1.7) at once after some simple computations and simplifications.

Similarly, the equality (1.5) can be obtained.

□

2. Main Results

First, we establish a new refinement of (1.1).

Theorem 2.1. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two sequences of complex numbers. If* $\sum\limits_{n=1}^{\infty}|a_n|^2 < +\infty$ *and* $\sum\limits_{n=1}^{\infty}|b_n|^2 < +\infty$*, then*

$$\left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m\bar{b}_n}{m+n}\right|^4 \le \pi^4\left\{\left(\sum_{n=1}^{\infty}|a_n|^2\right)^2-\left(\sum_{n=1}^{\infty}\omega(n)|a_n|^2\right)^2\right\}$$
$$\times\left\{\left(\sum_{n=1}^{\infty}|b_n|^2\right)^2-\left(\sum_{n=1}^{\infty}\omega(n)|b_n|^2\right)^2\right\} \tag{2.1}$$

where the weight function $\omega(n)$ *is defined by* (1.6).

Proof. Let $c(x)$ be a real function and satisfy the condition $1-c(n)+c(m)\ge 0$, $(n,m \in N)$. Firstly we suppose that $b_n = a_n$. Applying Cauchy's inequality we have

$$\begin{aligned}\left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m\bar{a}_n}{m+n}\right|^2 &= \left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m\bar{a}_n}{m+n}(1-c(n)+c(m))\right|^2\\
&= \left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{a_m(1-c(n)+c(m))^{1/2}}{(m+n)^{1/2}}\left(\frac{m}{n}\right)^{1/4}\right)\right.\\
&\qquad\left.\times\left(\frac{\bar{a}_n(1-c(n)+c(m))^{1/2}}{(m+n)^{1/2}}\left(\frac{n}{m}\right)^{1/4}\right)\right|^2\\
&\le J_1J_2\end{aligned} \tag{2.2}$$

where
$$J_1 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{|a_m|^2}{m+n}\left(\frac{m}{n}\right)^{\frac12}(1-c(n)+c(m))$$
$$J_2 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{|\bar{a}_n|^2}{m+n}\left(\frac{n}{m}\right)^{\frac12}(1-c(n)+c(m))$$

We can write the double series J_1 in the following form:

$$J_1 = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{1}{m+n}\left(\frac{n}{m}\right)^{\frac12}(1-c(m)+c(n))\right)|a_n|^2.$$

Let $c(x) = \frac{\sqrt{x}}{1+\sqrt{x}}$. It is obvious that $1-\frac{\sqrt{x}}{1+\sqrt{x}}+\frac{\sqrt{n}}{1+\sqrt{n}}\ge 0$. It is known from Lemma 1.2 that the function $f(x)$ is monotonously decreasing. Hence we have

$$\begin{aligned}J_1 &= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{1}{m+n}\left(\frac{n}{m}\right)^{\frac12}\left(1-\frac{\sqrt{m}}{1+\sqrt{m}}+\frac{\sqrt{n}}{1+\sqrt{n}}\right)\right)|a_n|^2\\
&\le \sum_{n=1}^{\infty}\left\{\int_0^{\infty}\left(\frac{1}{x+n}\left(\frac{n}{x}\right)^{\frac12}\right)\left(1-\left(\frac{\sqrt{x}}{1\,+\,\sqrt{x}}-\frac{\sqrt{n}}{1\,+\,\sqrt{n}}\right)\right)\mathrm{d}x\right\}|a_n|^2\\
&= \pi\sum_{n=1}^{\infty}|a_n|^2-\pi\sum_{n=1}^{\infty}\omega(n)|a_n|^2\end{aligned}$$

where the weight function $\omega(n)$ is defined by (1.6).

Similarly,

$$J_2 \le \sum_{n=1}^{\infty} \left\{ \int_0^{\infty} \frac{1}{x+n} \left(\frac{n}{x}\right)^{\frac{1}{2}} \left(1 + \left(\frac{\sqrt{x}}{1\ +\ \sqrt{x}} - \frac{\sqrt{n}}{1\ +\ \sqrt{n}}\right)\right) \mathrm{d}x \right\} |\bar{a}_n|^2$$
$$= \pi \sum_{n=1}^{\infty} |a_n|^2 + \pi \sum_{n=1}^{\infty} \omega(n) |a_n|^2.$$

Whence $J_1 J_2 \le \pi^2 \left\{ \left(\sum\limits_{n=1}^{\infty} |a_n|^2 \right)^2 - \left(\sum\limits_{n=1}^{\infty} \omega(n) |a_n|^2 \right)^2 \right\}$.

Consequently, we have

$$\left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m \bar{a}_n}{m+n} \right|^2 \le \pi^2 \left\{ \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^2 - \left(\sum_{n=1}^{\infty} \omega(n) |a_n|^2 \right)^2 \right\} \tag{2.3}$$

where the weight function $\omega(n)$ is defined by (1.6).

If $b_n \ne a_n$, then we can apply Schwarz's inequality to estimate the right-hand side of (2.1) as follows:

$$\left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m \bar{b}_n}{m+n} \right|^4 = \left\{ \left| \int_0^1 \left(\sum_{m=1}^{\infty} a_m t^{m-\frac{1}{2}} \right) \left(\sum_{n=1}^{\infty} \bar{b}_n t^{n-\frac{1}{2}} \right) \mathrm{d}t \right|^2 \right\}^2$$
$$\le \left| \int_0^1 \left(\sum_{m=1}^{\infty} |a_m| \, t^{m-\frac{1}{2}} \right)^2 \mathrm{d}t \int_0^1 \left(\sum_{n=1}^{\infty} |b_n| \, t^{n-\frac{1}{2}} \right)^2 \mathrm{d}t \right|^2$$
$$= \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m \bar{a}_n}{m+n} \right|^2 \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{b_m \bar{b}_n}{m+n} \right|^2 \tag{2.4}$$

And then by using the relation (2.3), from (2.4) and the inequality (2.1), we obtain at once. □

Similarly, we can establish a new refinement of (1.2).

Theorem 2.2. *Let $f(x)$ and $g(x)$ be two functions in complex number field. If $\int\limits_0^{\infty} |f(x)|^2 \,\mathrm{d}x < +\infty$, $\int\limits_0^{\infty} |g(x)|^2 \,\mathrm{d}x < +\infty$, then*

$$\left| \int_0^{\infty} \int_0^{\infty} \frac{f(x) \overline{g}(x)}{x+y} \mathrm{d}x \mathrm{d}y \right|^4 \le \pi^4 \left\{ \left(\int_0^{\infty} |f(x)|^2 \,\mathrm{d}x \right)^2 - \left(\int_0^{\infty} \omega(x) |f(x)|^2 \,\mathrm{d}x \right)^2 \right\}$$
$$\times \left\{ \left(\int_0^{\infty} |g(x)|^2 \,\mathrm{d}x \right)^2 - \left(\int_0^{\infty} \omega(x) |g(x)|^2 \,\mathrm{d}x \right)^2 \right\} \tag{2.5}$$

where the weight function ω *is defined by*

$$\omega(x) = \begin{cases} 0 & x = 0 \\ \dfrac{\sqrt{x}}{x+1}\left(\dfrac{\sqrt{x}-1}{\sqrt{x}+1} - \dfrac{\ln x}{\pi}\right) & x > 0 \end{cases} \tag{2.6}$$

Its proof is similar to that of Theorem 2.1, it is omitted here.

For the convenience of the applications, we list the following result.

Corollary 2.3. *Let* $f(x)$ *be a function in complex number field. If* $\int_0^\infty |f(x)|^2 \mathrm{d}x < +\infty$, *then*

$$\begin{aligned} &\left|\int_0^\infty\int_0^\infty \frac{f(x)\overline{f}(y)}{x+y}\mathrm{d}x\mathrm{d}y\right|^2 \\ &\le \pi^2\left\{\left(\int_0^\infty |f(x)|^2\mathrm{d}x\right)^2 - \left(\int_0^\infty \omega(x)|f(x)|^2\mathrm{d}x\right)^2\right\} \end{aligned} \tag{2.7}$$

where the weight function ω *is defined by* (2.6).

3. Applications

As applications, we shall give some new refinements of Hardy-Littlewood's theorem and Widder's theorem.

Let $f(x) \in L^2(0,1)$ and $f(x) \neq 0$ for all x. Define a sequence $\{a_n\}$ by $a_n = \int_0^1 x^n f(x)\mathrm{d}x$, $n = 0,\ 1,\ 2,\ \ldots$. Hardy-Littlewood ([**1**]) proved that

$$\sum_{n=0}^\infty a_n^2 < \pi \int_0^1 f^2(x)\mathrm{d}x, \tag{3.1}$$

where π is the best constant that the inequality (3.1) keeps valid.

Theorem 3.1. *Let* $f(x) \in L^2(0,1)$ *and* $f(x) \neq 0$ *for all* x. *Define a sequence* $\{a_n\}$ *by* $a_n = \int_0^1 x^{n-1/2} f(x)\mathrm{d}x \quad n = 1, 2, \ldots$. *Then*

$$\left(\sum_{n=1}^\infty a_n^2\right)^2 \le \pi\left\{\left(\sum_{n=1}^\infty a_n^2\right)^2 - \left(\sum_{n=1}^\infty \omega(n)a_n^2\right)^2\right\}^{\frac{1}{2}} \int_0^1 f^2(x)\mathrm{d}x \tag{3.2}$$

where $\omega(n)$ *is defined by* (1.6).

Proof. By our assumptions, we may write a_n^2 in the form

$$a_n^2 = \int_0^1 a_n x^{n-1/2} f(x)\mathrm{d}x.$$

Applying Cauchy-Schwarz's inequality we estimate the right hand side of (3.2) as follows

$$
\begin{aligned}
\left(\sum_{n=1}^{\infty} a_n^2\right)^2 &= \left(\sum_{n=1}^{\infty} \int_0^1 a_n x^{n-1/2} f(x)\mathrm{d}x\right)^2 = \left\{\int_0^1 \left(\sum_{n=1}^{\infty} a_n x^{n-1/2}\right) f(x)\mathrm{d}x\right\}^2 \\
&\le \int_0^1 \left(\sum_{n=1}^{\infty} a_n x^{n-1/2}\right)^2 \mathrm{d}x \int_0^1 f^2(x)\mathrm{d}x \\
&= \int_0^1 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_m a_n x^{m+n-1}\mathrm{d}x \int_0^1 f^2(x)\mathrm{d}x \\
&= \left(\sum_{m=}^{\infty}\sum_{n=1}^{\infty} \frac{a_m a_n}{m+n}\right) \int_0^1 f^2(x)\mathrm{d}x \qquad (3.3)
\end{aligned}
$$

It is known from (2.3) and (3.3) that the inequality (3.2) is valid. Therefore the theorem is proved. □

Let $a_n \ge 0$ $(n = 0, 1, 2, \ldots.)$, $A(x) = \sum\limits_{n=0}^{\infty} a_n x^n$, $A^*(x) = \sum\limits_{n=0}^{\infty} \frac{a_n x^n}{n!}$. Then

$$
\int_0^1 A^2(x)\mathrm{d}x \le \pi \int_0^{\infty} \left(\mathrm{e}^{-x} A^*(x)\right)^2 \mathrm{d}x \qquad (3.4)
$$

This is Widder's theorem (see [**1**]).

Theorem 3.2. *With the assumptions as the above-mentioned, it yields*

$$
\begin{aligned}
&\left(\int_0^1 A^2(x)\,\mathrm{d}x\right)^2 \\
&\le \pi^2 \left\{\left(\int_0^{\infty} \left(\mathrm{e}^{-x} A^*(x)\right)^2 \mathrm{d}x\right)^2 - \left(\int_0^{\infty} \omega(x) \left(\mathrm{e}^{-x} A^*(x)\right)^2 \mathrm{d}x\right)^2\right\} \qquad (3.5)
\end{aligned}
$$

where $\omega(x)$ *is defined by* (2.6).

Proof. At first we have the following relation:

$$
\begin{aligned}
\int_0^{\infty} \mathrm{e}^{-t} A^*(tx)\mathrm{d}t &= \int_0^{\infty} \mathrm{e}^{-t} \sum_{n=0}^{\infty} \frac{a_n (xt)^n}{n!}\mathrm{d}t \\
&= \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \int_0^{\infty} t^n \mathrm{e}^{-t}\,\mathrm{d}t = \sum_{n=0}^{\infty} a_n x^n = A(x)
\end{aligned}
$$

Let $tx = s$. Then we have

$$\int_0^1 A^2(x)\mathrm{d}x = \int_0^1 \left\{ \int_0^\infty \mathrm{e}^{-t} A^*(tx)\,\mathrm{d}t \right\}^2 \mathrm{d}x = \int_0^i \left(\int_0^\infty \mathrm{e}^{-\frac{s}{x}} A^*(s)\,\mathrm{d}s \right)^2 \frac{1}{x^2}\mathrm{d}x$$

$$= \int_1^\infty \left(\int_0^\infty \mathrm{e}^{-sy} A^*(s)\mathrm{d}s \right)^2 \mathrm{d}y = \int_0^\infty \left(\int_0^\infty \mathrm{e}^{-s(u+1)} A^*(s)\,\mathrm{d}s \right)^2 \mathrm{d}u$$

$$= \int_0^\infty \left(\int_0^\infty \mathrm{e}^{-su} f(s)\,\mathrm{d}s \right)^2 \mathrm{d}u = \int_0^\infty \int_0^\infty \frac{f(s)\,f(t)}{s\,+\,t}\mathrm{d}s\mathrm{d}t \tag{3.6}$$

where $f(x) = \mathrm{e}^{-x} A^*(x)$. By Corollary 2.3, the inequality (3.5) follows from (3.6) at once. □

References

1. Hardy G. H., Littlewood J. E. and Polya G., *Inequalities*, Cambridge Univ. Press, Cambridge, U.K., 1952.
2. Gao Mingzhe and Hsu Lizhi, *A survey of various refinements and generalizations of Hilbert's inequalities*, Journal of Mathematical Research and Exposition **25(2)** (2005), 227–243.
3. Zwillinger D. et al., *CRC Standard Mathematical Tables and Formulae*, CRC Press, 1988.
4. Gradshteyn I. S. and Ryzhik I. M., *Table of Integrals*, Series, and Products, Academic Press, 2000.

Zhou Yu, Department of Mathematics and Computer Science Normal College, Jishou University Jishou Hunan 416000, P. R. China, *e-mail*: `hong2990@163.com`

Gao Mingzhe, Department of Mathematics and Computer Science Normal College, Jishou University Jishou Hunan 416000, P. R. China, *e-mail*: `mingzhegao@163.com`