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A MEAN VALUE PROPERTY OF HARMONIC FUNCTIONS ON PROLATE ELLIPSOIDS OF REVOLUTION

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Abstract. We establish a mean value property for harmonic functions on the interior of a prolate ellipsoid of revolution. This property connects their boundary values with those on the interfocal segment.

1. Introduction

Let D be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$, let be f a continuous real-valued function on its boundary ∂D. The classical Dirichlet problem consists in the determination of a harmonic function H_f on D which can be continuously extended into ∂D by f. If ∂D is sufficiently smooth, the Dirichlet solution H_f possesses an integral representation of the form

$$H_f(z) = \frac{1}{\mu(\partial D)} \int_{\partial D} P_D(z,x) f(x) \mathrm{d}\mu(x),$$

where P_D is the so-called Poisson kernel of D and μ is an adequate measure on ∂D (see [**2**, Theorem 21,VI]). The Poisson kernel can be explicitly given only in some few cases. However, it often may be worth to try to find an explicit *connection* between distinguished interior points z and the boundary values $f(x)$. In the case of a ball such a connection is given by the mean value property of harmonic functions, when z is the centre of the ball. (Here, of course, the Poisson kernel is easily written down). Generally, it is reasonable to expect that whenever similar connections exist, they always have to do with the geometric properties of the domain D.

In this work we study the domain class of prolate balls, that is, interiors of prolate ellipsoids of revolution (the latter are also called "prolate spheroids") in $\mathbb{R}^n$ $(n \geq 3)$. The two-dimensional case of elliptic discs (i. e. interiors of ellipses) has been studied in the context of complex analysis before. The Poisson kernel for such discs can be explicitly given in terms of an infinite series, which takes a closed form under the use of the Jacobi zeta function ([**3**]). In the course of the derivation of this Poisson kernel it is observed that there exists a purely elementary "mean value property" connecting the boundary values of the harmonic function with

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those on the interfocal segment of the ellipse. For the elliptic disc $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$ $(a > b)$ with foci at $(-c, 0)$ and $(c, 0)$ $(c = \sqrt{a^2 - b^2})$ and a harmonic function h on an open neighbourhood of it the property states (see [**3**]):

$$\frac{1}{\pi}\int_{-c}^{c} \frac{h(x,0)}{\sqrt{c^2 - x^2}}\,\mathrm{d}x \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} h(a\cos s, b\sin s)\mathrm{d}s\,. \tag{1}$$

This property can be traced back at least up to the 1960's.

The goal of this work is a generalization of (1) to higher dimensions. (This is achieved at equation (11).) As already mentioned, we restrict ourselves to prolate ellipsoids of revolution, because they still have well-defined foci. We are led to the result by imposing the Dirichlet boundary condition on the generic solution of Laplace's equation obtained by separation of variables.

2. The Mean Value Property

Let $E_R \subseteq \mathbb{R}^n$ $(n \geq 3)$ be the normalized prolate ball

$$\frac{x_1^2}{\cosh^2 R} + \frac{x_2^2 + \ldots + x_n^2}{\sinh^2 R} < 1$$

with foci at $(-1, 0, \ldots, 0)$ and $(1, 0, \ldots, 0)$. A parametrization is given by the mapping

$$\begin{aligned}\alpha : (r, s_1, \ldots, s_{n-1}) \mapsto (&\cosh r\cos s_1, \sinh r\sin s_1\cos s_2, \sinh r\sin s_1\sin s_2\cos s_3, \ldots,\\ &\sinh r\sin s_1 \ldots \sin s_{n-2}\cos s_{n-1}, \sinh r\sin s_1 \ldots \sin s_{n-2}\sin s_{n-1})\end{aligned}$$

for $r \in [0, R[$, $s_1, \ldots, s_{n-2} \in [0, \pi]$ and $s_{n-1} \in]-\pi, \pi]$. The computation of the Laplacian in these elliptic coordinates requires the coefficients of the metric tensor

$$g_{11} = \left|\frac{\partial\alpha}{\partial r}\right|^2, \; g_{1j} = \left\langle \frac{\partial\alpha}{\partial r}, \frac{\partial\alpha}{\partial s_{j-1}} \right\rangle, \; g_{ij} = \left\langle \frac{\partial\alpha}{\partial s_{i-1}}, \frac{\partial\alpha}{\partial s_{j-1}} \right\rangle$$

for $2 \leq i, j \leq n$. We compute: $g_{11} = \sinh^2 r + \sin^2 s_1 = g_{22}$; $g_{kk} = \sinh^2 r \sin^2 s_1 \ldots \sin^2 s_{k-2}$ for $3 \leq k \leq n$; $g_{ij} = 0$ for $i \neq j$. The Laplacian of a function u is given by

$$\Delta u = \frac{1}{\sqrt{\bar{g}}}\sum_{k=1}^{n} \partial_k \left(\sum_{j=1}^{n} g^{jk}\sqrt{\bar{g}}\,\partial_j u \right), \tag{2}$$

where g^{jk} are the coefficients of the inverse matrix (here equal to $g_{jk}^{-1}\delta_{jk}$), $\bar{g} = \det(g_{ij})_{i,j}$ and ∂_j denotes the partial derivative with respect to the j-th coordinate. Thus,

$$\begin{aligned}\Delta u &= \frac{1}{\sqrt{\bar{g}}}\sum_{k=1}^{n} \partial_k \left(g^{kk}\sqrt{\bar{g}}\partial_k u\right)\\ &= g^{11}\frac{\partial^2 u}{\partial r^2} + \sum_{k=2}^{n} g^{kk}\frac{\partial^2 u}{\partial s_{k-1}^2} + \frac{\frac{\partial}{\partial r}\left(g^{11}\sqrt{\bar{g}}\right)}{\sqrt{\bar{g}}}\cdot\frac{\partial u}{\partial r} + \sum_{k=2}^{n} \frac{\frac{\partial}{\partial s_{k-1}}\left(g^{kk}\sqrt{\bar{g}}\right)}{\sqrt{\bar{g}}}\cdot\frac{\partial u}{\partial s_{k-1}}\,.\end{aligned}$$

For the mean value property it suffices to restrict ourselves to functions u which are invariant with respect to rotations about the x_1-axis. Such functions do not depend on $s_2, \ldots, s_{n-1}$, so in this case we have

$$\begin{aligned}\Delta u = \; & g^{11}\frac{\partial^2 u}{\partial r^2} + g^{22}\frac{\partial^2 u}{\partial s_1^2} + \frac{\frac{\partial}{\partial r}\left(g^{11}\sqrt{\bar g}\right)}{\sqrt{\bar g}} \cdot \frac{\partial u}{\partial r} + \frac{\frac{\partial}{\partial s_1}\left(g^{22}\sqrt{\bar g}\right)}{\sqrt{\bar g}} \cdot \frac{\partial}{\partial s_1} \\ = \; & \frac{1}{\sinh^2 r + \sin^2 s_1}\left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial s_1^2}\right) + \frac{(n-2)\coth r}{\sinh^2 r + \sin^2 s_1} \cdot \frac{\partial u}{\partial r} \\ & + \frac{(n-2)\cot s_1}{\sinh^2 r + \sin^2 s_1} \cdot \frac{\partial u}{\partial s_1}\end{aligned}$$

after the computations, $\bar g$ being equal to

$$\left(\sinh^2 r + \sin^2 s_1\right)^2 \left(\sinh^2 r \sin^2 s_1\right)^{n-2} \left(\sin^2 s_2\right)^{n-3} \ldots \left(\sin^2 s_{n-2}\right)^1 .$$

Thus, the harmonic functions of r and s_1 are characterized by the equation

$$\frac{\partial^2 u}{\partial r^2} + (n-2)\coth r \cdot \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial s_1^2} + (n-2)\cot s_1 \cdot \frac{\partial u}{\partial s_1} = 0 . \tag{3}$$

Now let $f : \partial E_R \to \mathbb{R}$ be a continuous function. First we assume that f is invariant with respect to rotations about the x_1-axis. For shortness we write $f(s_1)$ instead of $f(\alpha(R, s_1, \ldots, s_{n-1}))$. We shall solve the Dirichlet problem for E_R by separation of variables, so it is necessary to determine all harmonic functions of the form $u(r, s_1) = U(r)V(s_1)$. Equation (3) implies:

$$\frac{U''}{U} + (n-2)\coth r \cdot \frac{U'}{U} = -\frac{V''}{V} - (n-2)\cot s_1 \cdot \frac{V'}{V} =: \lambda \in \mathbb{R} . \tag{4}$$

After substituting $x = \frac{1-\cos s_1}{2} = \sin^2 \frac{s_1}{2}$, the second equation becomes

$$x(1-x)\tilde V'' + \left[\frac{n-1}{2} - (n-1)x\right]\tilde V' + \lambda \tilde V = 0 , \;\; \tilde V(x) = V(s_1). \tag{5}$$

It can be shown that this equation has a bounded solution for $0 \le x \le 1$ if and only if λ is of the form $k(k+n-2)$ with $k \in \mathbb{N} \cup \{0\}$ (see [**4**], [**5**, p.11] or, for another way of solving, [**1**, Intro.3.1]). Then, (5) becomes a *hypergeometric* differential equation

$$x(1-x)\tilde V'' + [c - (a+b+1)x]\tilde V' - ab\tilde V = 0$$

with $a = -k$, $b = k+n-2$ and $c = \frac{n-1}{2}$. The solution which is regular at $x = 0$ and takes there the value 1 is given by the function

$$F(a, b; c; x) := \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j} \cdot \frac{x^j}{j!} ,$$

where $(\eta)_0 := 1$, $(\eta)_{j+1} := (\eta)_j(\eta + j)$ for $j \in \mathbb{N} \cup \{0\}$, the so-called Pochhammer symbol (classical notations). Thus, (5) implies

$$\tilde V = \tilde V_k(x) = F\left(-k, k+n-2; \frac{n-1}{2}; x\right) .$$

For solutions of the second part of (4) we choose the functions

$$V = V_k(s_1) := \tilde{V}_k\left(\frac{1-\cos s_1}{2}\right)\cdot\binom{k+n-3}{k} = C_k^{\frac{n-2}{2}}(\cos s_1)\,,$$

where $C_k^{\frac{n-2}{2}}$ is the so-called Gegenbauer (or "ultraspherical") polynomial of degree k:

$$C_k^{\frac{n-2}{2}}(x) := \binom{k+n-3}{k}\cdot F\left(-k, k+n-2; \frac{n-1}{2}; \frac{1-x}{2}\right).$$

The first part of (4) being equal to $\lambda = k(k+n-2)$ leads after the substitution $z = -\sinh^2\frac{r}{2}$, $Q(z) = U(r)$ to the hypergeometric differential equation

$$z(1-z)Q'' + \left[\frac{n-1}{2} - (n-1)z\right]Q' + k(k+n-2)Q = 0\,, \tag{6}$$

the same one as in (5). For solutions we take the functions

$$Q = Q_k(z) = \binom{k+n-3}{k}F\left(-k, k+n-2; \frac{n-1}{2}; z\right) = C_k^{\frac{n-2}{2}}(1-2z)\,,$$

which lead to

$$U = U_k(r) = C_k^{\frac{n-2}{2}}(\cosh r)\,.$$

For the functions

$$u = u_k(r, s_1) = U_k(r)V_k(s_1) = C_k^{\frac{n-2}{2}}(\cosh r)C_k^{\frac{n-2}{2}}(\cos s_1)\,,\quad k \in \mathbb{N}\cup\{0\},$$

to be harmonic, it remains to show that they are everywhere smooth, since the parametrization α is not diffeomorphic on the interfocal segment. To this end we recall that the Gegenbauer polynomial $C_k^{\frac{n-2}{2}}(x)$ has the parity of x^k.[1] Therefore, there are polynomials P_k and Q_k such that

$$C_{2k}^{\frac{n-2}{2}}(\cosh r)C_{2k}^{\frac{n-2}{2}}(\cos s_1) = P_k(\cosh^2 r)P_k(\cos^2 s_1)\,,$$

$$C_{2k+1}^{\frac{n-2}{2}}(\cosh r)C_{2k+1}^{\frac{n-2}{2}}(\cos s_1) = \cosh r\cos s_1\cdot Q_k(\cosh^2 r)Q_k(\cos^2 s_1)\,.$$

According to the fundamental theorem on symmetric polynomials, the right sides can be written as polynomials in $\cosh r\cos s_1$ and $\cosh^2 r + \cos^2 s_1$. Since these expressions are recognized as the smooth functions x_1 and $1 + x_1^2 + \ldots + x_n^2$, everything is established.

The Dirichlet solution H_f is now assumed of the form

$$H_f(r, s_1) = \sum_{k=0}^{\infty} a_k C_k^{\frac{n-2}{2}}(\cosh r)C_k^{\frac{n-2}{2}}(\cos s_1) \tag{7}$$

(the left side is an abbreviation for $H_f(\alpha(r, s_1, \ldots, s_{n-1}))$. For $r = R$ it should hold:

$$\sum_{k=0}^{\infty} a_k C_k^{\frac{n-2}{2}}(\cosh R)C_k^{\frac{n-2}{2}}(\cos s_1) = f(s_1)\,. \tag{8}$$

[1] For the basic facts about Gegenbauer polynomials see for instance [**7**].

The Gegenbauer polynomials $C_k^{\frac{n-2}{2}}(x)$ form an orthogonal system in $L^2\left([-1,1]\,;\,(1-x^2)^{\frac{n-3}{2}}\right)$. In fact,

$$\int_{-1}^{1} C_k^{\frac{n-2}{2}}(x) C_l^{\frac{n-2}{2}}(x)(1-x^2)^{\frac{n-3}{2}}\,\mathrm{d}x = \frac{2^{3-n}\pi\Gamma(k+n-2)}{k!\left(k+\frac{n-2}{2}\right)\Gamma\left(\frac{n-2}{2}\right)^2}\delta_{kl} \tag{9}$$

([**7**, p. 179]). Since $C_k^{\frac{n-2}{2}}$ has degree k, it follows from the approximation theorem of Weierstraß that the system $\left(C_k^{\frac{n-2}{2}}\right)_k$ is complete. For (8) being the Fourier expansion of f it must therefore hold:

$$a_k = \frac{1}{C_k^{\frac{n-2}{2}}(\cosh R)} \cdot \frac{k!\left(k+\frac{n-2}{2}\right)\Gamma\left(\frac{n-2}{2}\right)^2}{2^{3-n}\pi\Gamma(k+n-2)} \cdot \int_0^{\pi} f(s_1) C_k^{\frac{n-2}{2}}(\cos s_1)\sin^{n-2} s_1\,\mathrm{d}s_1$$

for $k \in \mathbb{N} \cup \{0\}$. For the moment we assume that f is a polynomial in $\cos s_1$. In this case, only finite number of the a_k are nonzero and the right side of (7) is a finite sum which presents the solution to the Dirichlet problem for E_R. From (7) and (9) it follows:

$$\int_0^{\pi} H_f(0,s_1)\sin^{n-2} s_1 \mathrm{d}s_1 = a_0 C_0^{\frac{n-2}{2}}(1) \cdot \frac{2^{3-n}\pi\Gamma(n-2)}{\frac{n-2}{2}\Gamma\left(\frac{n-2}{2}\right)^2} = \int_0^{\pi} f(s_1)\sin^{n-2} s_1 \mathrm{d}s_1$$

$$\iff \int_{-1}^{1} H_f(x,0,\dots,0)(1-x^2)^{\frac{n-3}{2}}\,\mathrm{d}x = \int_0^{\pi} f(s_1)\sin^{n-2} s_1 \mathrm{d}s_1 \tag{10}$$

(the latter equation without the abbreviation in H_f). If f is an arbitrary continuous boundary function only depending on s_1, then an approximation argument on the basis of Weierstraß' approximation theorem and the maximum principle show that (10) still holds.

Next, we drop the assumption that f is rotationally invariant with respect to the x_1-axis. We denote by $\tilde{f}$ the "symmetrization" of f, that is,

$$\tilde{f}(x) := \int_{SO(n-1)} f(Ax)\,\mathrm{d}A \qquad \text{for } x \in \partial E_R\,,$$

where $SO(n-1)$ stands for the group of rotations about the x_1-axis and $\mathrm{d}A$ for its normalized Haar integral. Since rotations preserve harmonicity and because of the relation $H_f \circ A = H_{f\circ A}$ for $A \in SO(n-1)$, the function

$$z \longmapsto \int_{SO(n-1)} H_f(Az)\,\mathrm{d}A$$

is harmonic and has boundary values equal to $\tilde{f}$. Therefore,

$$\int_{SO(n-1)} H_f(Az)\,\mathrm{d}A = H_{\tilde{f}}(z)\,,$$

which implies that $H_f(x,0,\dots,0) \equiv H_{\tilde f}(x,0,\dots,0)$. Now (10) gives:

$$\int_{-1}^{1} H_f(x,0,\dots,0)(1-x^2)^{\frac{n-3}{2}}\mathrm{d}x = \int_0^\pi \tilde f(s_1)\sin^{n-2} s_1\,\mathrm{d}s_1$$
$$= \int_0^\pi \int_{SO(n-1)} f\left(A(\cosh R\cos s_1, \sinh R\sin s_1, 0,\dots,0)^T\right)\sin^{n-2} s_1\,\mathrm{d}A\,\mathrm{d}s_1$$
$$= \frac{1}{\Omega_{n-1}}\int_0^\pi \dots \int_0^\pi \int_0^{2\pi} f(\cosh R\cos s_1, \sinh R\sin s_1\cos s_2, \dots,$$
$$\sinh R\sin s_1\dots\sin s_{n-1})\cdot \sin^{n-2}s_1\sin^{n-3}s_2\dots\sin s_{n-2}\mathrm{d}s_{n-1}\mathrm{d}s_{n-2}\dots\mathrm{d}s_2\mathrm{d}s_1,$$

since the Haar integral induces the rotation invariant measure (= surface area measure) on the sphere (Ω_{n-1} stands for the area of the unit sphere in $\mathbb{R}^{n-1}$).

For an arbitrary prolate ball $\frac{x_1^2}{a^2}+\frac{x_2^2+\dots+x_n^2}{b^2} < 1$ ($a > b$) with foci at $(-c,0,\dots,0)$ and $(c,0,\dots,0)$ ($c=\sqrt{a^2-b^2}$) the similarity $x \mapsto cx$ has to be employed. So, if h is a harmonic function on an open neighbourhood of the closed prolate ball, it holds:

$$\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi}c^{n-2}}\int_{-c}^{c} h(x,0,\dots,0)(c^2-x^2)^{\frac{n-3}{2}}\mathrm{d}x$$
$$(11)\quad = \frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}}\int_0^\pi \dots \int_0^\pi \int_0^{2\pi} h(a\cos s_1, b\sin s_1\cos s_2, \dots, b\sin s_1\dots\sin s_{n-1})$$
$$\sin^{n-2} s_1\sin^{n-3} s_2\dots\sin s_{n-2}\,\mathrm{d}s_{n-1}\mathrm{d}s_{n-2}\dots\mathrm{d}s_1$$

(the constants have been introduced according to $\Omega_k = \frac{2\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}$ for each k and so that each side equals one for $h = 1$). This equation generalizes (1) and presents the mean value property in all dimensions.

It is important to find out the geometric meaning of the multiple integral on the right side of (11). Let E denote the above prolate ball. The vector

$$\left(\frac{x_1}{a^2}, \frac{x_2}{b^2}, \dots, \frac{x_n}{b^2}\right)$$

is orthogonal to the boundary ellipsoid at its point $(x_1,\dots,x_n)$. Thus, the distance from the centre to the tangent plane at $(x_1,\dots,x_n)$ is equal to

$$\frac{\frac{x_1^2}{a^2}+\frac{x_2^2+\dots+x_n^2}{b^2}}{\sqrt{\frac{x_1^2}{a^4}+\frac{x_2^2+\dots+x_n^2}{b^4}}} = \frac{1}{\sqrt{\frac{x_1^2}{a^4}+\frac{1}{b^2}\left(1-\frac{x_1^2}{a^2}\right)}} = \frac{1}{\sqrt{\frac{1}{b^2}-\frac{c^2x_1^2}{a^4b^2}}}$$
$$= \frac{b}{\sqrt{1-\frac{c^2x_1^2}{a^4}}} = \frac{b}{\sqrt{1-\frac{c^2\cos^2 s_1}{a^2}}} = \frac{ab}{\sqrt{b^2+c^2\sin^2 s_1}}.$$

The surface element of ∂E is given by

$$\sqrt{b^2 + c^2 \sin^2 s_1}\, b^{n-2} \sin^{n-2} s_1 \sin^{n-3} s_2 \dots \sin s_{n-2} \,\mathrm{d}s_1 \dots \mathrm{d}s_{n-1}$$

(in the normalized situation at the beginning of this section it is equal to $\det (g_{ij})^{1/2}_{2\le i,j\le n} \,\mathrm{d}s_1 \dots \mathrm{d}s_{n-1}$). Therefore, the integral on ∂E that measures the volume to the centre is given by

$$\frac{1}{n} a b^{n-1} \sin^{n-2} s_1 \sin^{n-3} s_2 \dots \sin s_{n-2} \,\mathrm{d}s_1 \dots \mathrm{d}s_{n-1}$$

for $s_1, \dots, s_{n-2} \in [0, \pi]$ and $s_{n-1} \in [-\pi, \pi]$. Up to a constant factor, this is exactly the integral in (11). The same observation can be made in the two-dimensional case (1), where the integral $\mathrm{d}s$ is proportional to the area that the segment from the origin to the point $(a\cos s, b\sin s)$ traces. Is this phenomenon the key to discover mean value properties for other centrally symmetric domains?

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