UNIFORM APPROXIMATION BY POLYNOMIALS ON REAL NON-DEGENERATE WEIL POLYHEDRON

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Abstract. It is proved, that on real non-degenerate polynomial Weil polyhedron G any function, holomorphic in G and continuous on its closure, can be uniformly approximated by polynomials.

1. Introduction

A bounded domain $G \subset \mathbb{C}^n$ is called analytic polyhedron if there are some functions $\chi_1, \dots, \chi_N$ holomorphic in neighborhoods V of $\overline{G}$, such that

$$G = \{z \in V\colon\ |\chi_i(z)| < 1, \quad i = 1, 2, \dots, N\}. \tag{1}$$

The boundary ∂G of G consists of the "edges"

$$\sigma_i = \{z \in \partial G\colon\ |\chi_i(\zeta)| = 1\}$$

intersecting along the k-dimensional "ribs"

$$\sigma_{i_1,\dots,i_k} = \sigma_{i_1} \cap \dots \cap \sigma_{i_k}.$$

An analytic polyhedron is called Weil polyhedron if $N \geq n$, all edges σ_i are $(2n-1)$-dimensional manifolds and the dimensions of all ribs $\sigma_{i_1,\dots,i_k}$ $(2 \leq k \leq n)$ are at most $2n - k$. The union of all these n-dimensional ribs is

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the distinguished boundary of G. The domain G is called polynomial polyhedron if all determining functions χ_i are polynomials in (1).

The main result of this paper (Theorem 3.1) states that if G is a Weil polyhedron of "general position" in the sense of real analysis (see. Definition 2.1), then any function holomorphic in G and continuous in $\overline{G}$ can be uniformly approximated by functions holomorphic in some neighborhoods of $\overline{G}$. In the particular case of polynomial polyhedrons, it is proved (Theorem 3.2) that such functions can be approximated by polynomials.

We use some improvement of a method, which is applied in [1] for strictly pseudoconvex domains, and is based on some uniform estimates of solutions of the $\overline{\partial}$-equation

$$\overline{\partial} u = g, \tag{2}$$

where $g = \sum_{k=1}^{n} g_k \, d\overline{z}_k$ is a $\overline{\partial}$-closed in G differential form of $(0,1)$ type.

We use the following uniform estimate which for $n = 2$ is obtained in [2] and for arbitrary n in [3]: *in a real non-degenerate Weil polyhedron the equation (2) has a solution $u_0(z)$ such that*

$$\|u_0\|_G \leq \gamma \|g\|_G,$$

where $\gamma = \gamma(G)$ is a constant independent of g and $\|\cdot\|_G$ is the sup*-norm:*

$$\|u\|_G = \sup_{z \in G} u(z), \quad \|g\|_G = \sum_{k=1}^{n} \|g_k\|_G.$$

Note that there is no theorem on approximation for arbitrary Weil polyhedrons. By a different method, the author [4] has proved an approximation theorem under the complex non-degeneracy condition (meaning that in the general position of complex analysis sense the appropriate edges intersect in the points of distinguished boundary).The class of real non-degenerate polyhedrons is wide enough to provide approximation of any domain of holomorphy by real non-degenerate polyhedrons, which is not true in the case, when the polyhedrons are complexly non-degenerate, i. e. if their edges intersect in a general position, (in the complex analysis sense).

2. Local approximation

Definition 2.1. We call a polyhedron (1) real non-degenerate if for any collection $i_1, \dots, i_k$ the matrix

$$(\mathrm{grad}_{\mathbf{R}} |\chi_{i_1}(z)|, \dots, \mathrm{grad}_{\mathbf{R}} |\chi_{i_k}(z)|)$$

attains its maximal rank in all points $z \in \sigma_{i_1,\dots,i_k}$.

Here

$$\mathrm{grad}_{\mathbf{R}}\, f(z) = {}^t\big(D_1 f(z), \dots, D_n f(z), \overline{D}_1 f(z), \dots, \overline{D}_n f(z)\big)\,,$$

where t before the bracket means transposition and

$$D_k f(z) = \frac{\partial f(z)}{\partial z_k}, \quad \overline{D}_k f(z) = \frac{\partial f(z)}{\partial \overline{z}_k}, \quad k = 1, \dots, n.$$

Geometrically, Definition 2.1 means that the edges $\sigma_{i_1}, \dots, \sigma_{i_k}$ intersect in a general position (in the real analysis sense).

We start by proving the following geometrical property of non-degenerate polyhedrons.

Proposition 2.2. *Let G be a real non-degenerate polyhedron (1) and let $N \le 2n$. Then for any point $\zeta \in \partial G$ there exist a neighborhood B_ζ and a vector ν_ζ, such that $z + \delta\nu_\zeta \in G$ if $z \in \overline{B}_\zeta \cap \overline{G}$ for $\delta > 0$ small enough.*

Proof. Denote $\varphi_i = |\chi_i| - 1$ and assume that $\zeta \in \partial G$ belongs to the edge $\sigma_{i_1,\dots,i_k}$, i.e. $\varphi_{i_1}(\zeta) = 0, \dots, \varphi_{i_k}(\zeta) = 0$ and

(3) $$\varphi_s(\zeta) < 0, \quad s \neq i_1, \dots, i_k.$$

By $k \leq 2n$ and our assumptions, the vectors $\operatorname{grad}_{\mathbf{R}} \varphi_{i_1}(\zeta), \ldots, \operatorname{grad}_{\mathbf{R}} \varphi_{i_k}(\zeta)$ are linearly independent. Hence there is a point w such that

$$\sum_{m=1}^{n} D_m\varphi_j(\zeta)(w_m - \zeta_m) + \sum_{m=1}^{n} \overline{D}_m\varphi_j(\zeta)(\overline{w}_m - \overline{\zeta}_m) < 0, \quad j = i_1, \ldots, i_k.$$

Due to the continuity of $D_m\varphi_j(\zeta)$ and $\overline{D}_m\varphi_j(\zeta)$, there is a neighborhood B_ζ, such that for all points $z \in \overline{B}_\zeta$ the inequalities

$$\sum_{m=1}^{n} D_m\varphi_j(z)(w_m - \zeta_m) + \sum_{m=1}^{n} \overline{D}_m\varphi_j(z)(\overline{w}_m - \overline{\zeta}_m) < 0, \quad j = i_1, \ldots, i_k. \tag{4}$$

are true. Let $z \in \overline{B}_\zeta$, $\delta > 0$. Then

$$\varphi_j(z + \delta(w - \zeta)) = \varphi_j(z) + 2\delta \operatorname{Re} \sum_{m=1}^{n} D_m\varphi_j(z)(w_m - \zeta_m) + o(\delta). \tag{5}$$

Denoting $\nu_\zeta = w - \zeta$ and taking in account that $\varphi_j(z) \leq 0$ for $z \in \overline{G}$, from (4) and (5) we conclude that there exists some $\delta_0 > 0$ such that for $\delta < \delta_0$

$$\varphi_j(z + \delta\nu_\zeta) < 0, \quad j = i_1, \ldots, i_k, \quad z \in \overline{B}_\zeta \cap \overline{G}, \tag{6}$$

By continuity of φ_j, it follows from (3) that one can choose a neighborhood B_ζ and a number δ_0 such that for $\delta < \delta_0$

$$\varphi_s(z + \delta\nu_\zeta) < 0, \quad s \neq i_1, \ldots, i_k, \quad z \in \overline{B}_\zeta \cap \overline{G}.$$

Hence, by (6) we conclude that $z + \delta\nu_\zeta \in G$. □

The following lemma relates to local approximation.

Lemma 2.3. *There exists a finite covering* $\{U_k\colon k = 0, 1, \dots, p\}$ *of* $\overline{G}$ *by open sets, such that for any* $\varepsilon > 0$ *and any* $f \in A(G)$ *there are holomorphic in* $\overline{U_k} \cap \overline{G}$ *functions* f_k *for which*

$$\sup_{z \in \overline{U}_k \cap \overline{G}} |f(z) - f_k(z)| < \varepsilon. \tag{7}$$

Proof. Let $f \in A(G)$, $\zeta \in \partial G$ and let B_ζ be a neighborhood satisfying the conditions of Proposition 2.2. Then the family of open sets $\{B_\zeta\colon \zeta \in \partial G\}$ covers the compact ∂G, and a a finite subcovering $\{B_{\zeta_k},\ k = 1, \dots, p\}$ can be chosen. By Proposition 2.2, the functions $f(z + \delta\nu_{\zeta_k})$ are holomorphic in $\overline{B}_{\zeta_k} \cap \overline{G}$ for any $\delta > 0$ small enough. By uniform continuity of f in $\overline{G}$,

$$\sup_{z \in \overline{B}_{\zeta_k} \cap \overline{G}} |f(z + \delta\nu_{\zeta_k}) - f(z)| \to 0 \quad \text{as} \quad \delta \to 0.$$

Now, choosing a small enough $\delta > 0$ and denoting $U_k = B_{\zeta_k}$, $f_k(z) = f(z + \delta\nu_{\zeta_k})$, we get (7) for $k = 1, \dots, p$. Further, we take a compact subdomain $U_0 \subset G$ such that the system $\{U_k\colon k = 0, 1, \dots, p\}$ is an open covering of $\overline{G}$ and put $f_0(z) = f(z)$. Then obviously (7) is true also for $k = 0$. □

3. Global approximation

Recalling that a function is said to be holomorphic in a compact set K if it is holomorphic in some neighborhood of K, we prove

Theorem 3.1. *Let* G *be a real non-degenerate Weil polyhedron* (1) *and let* $N \leq 2n$. *Then any function* $f \in A(G)$ *can be uniformly approximated in* $\overline{G}$ *by functions holomorphic in* $\overline{G}$.

Proof. Let $\varepsilon > 0$, let $f \in A(G)$ and let $\{U_k\colon k = 0, 1, \dots, p\}$ that of Lemma 2.3. Then by Lemma 2.3, there are functions f_k holomorphic in $\overline{U}_k \cap \overline{G}$, such that

$$\|f_k - f\|_{U_k \cap G} < \varepsilon, \quad k = 0, 1, \dots, p. \tag{8}$$

Let $\{e_k(z),\ k=0,1,\dots,p\}$ be a partition of unity, i.e. a system of infinitely differentiable, nonnegative, finite functions such that

(a) $\operatorname{Supp} e_k \subset U_k, \quad k=0,1,\dots,p,$
(b) $\sum_{k=0}^{p} g_k(z) \equiv 1$ in some neighborhood of $\overline{G}$.

Choose some number $\eta(\varepsilon)>0$ small enough to provide the holomorphy of f_k in the sets

$$V_k = U_k \cap G^{\varepsilon}, \quad k=0,1,\dots,p,$$

where

$$G^{\varepsilon} = \{z \in V\colon\ |\chi_i(z)| < 1+\eta(\varepsilon), \quad i=1,2,\dots,N\}.$$

Obviously

$$\|f_k - f_i\|_{U_k\cap U_i\cap G} \le \|f_k - f\|_{U_k\cap G} + \|f_i - f\|_{U_i\cap G} < 2\varepsilon,\ i,k=0,1,\dots,p, \tag{9}$$

and, if necessary, taking smaller $\eta(\varepsilon)>0$, by continuity we can get

$$\|f_k - f_i\|_{V_k\cap V_i} < 3\varepsilon, \quad k,i=0,1,\dots,p. \tag{10}$$

Now consider the functions

$$\begin{aligned} h_{ik}(z) &= \begin{cases} [f_i(z)-f_k(z)]e_k(z) & \text{if } \ z\in V_i\cap V_k; \\ 0 & \text{if } \ z\in V_i\setminus V_k, \end{cases} \\ h_i(z) &= \sum_{k=0}^{p} h_{ik}(z). \end{aligned} \tag{11}$$

The support of $g_k(z)$ belongs to the set B_k (by the assumption (a)), and the set $V_i^\varepsilon \bigcap \partial V_k^\varepsilon$ does not intersect with that support. Therefore, the functions h_{ik}^ε and h_i^ε are infinitely differentiable in V_i^ε, and by (10)

$$|h_i(z)| \leq \sum_{k=0}^{p} |f_i(z) - f_k(z)| e_k(z) < 3\varepsilon \sum_{k=0}^{p} g_k(z) = 3\varepsilon. \tag{12}$$

for all $z \in V_i^\varepsilon \bigcap \overline{G^\varepsilon}$. Further, for $z \in V_i \cap V_j$

$$\begin{aligned} h_i(z) - h_j(z) &= \sum_{k=0}^{p} [f_i(z) - f_k(z)]\, e_k(z) - \sum_{k=0}^{p} [f_j(z) - f_k(z)]\, e_k(z) \\ &= \sum_{k=0}^{p} [f_i(z) - f_j(z)]\, e_k(z) = f_i(z) - f_j(z), \end{aligned}$$

i. e.

$$f_i(z) - h_i(z) = f_j(z) - h_j(z), \quad i, j = 0, 1, \ldots, p.$$

This means that the function

$$\psi(z) = f_i(z) - h_i(z) \quad z \in V_i, \tag{13}$$

is globally given in G^ε and moreover, $h \in C^\infty(G^\varepsilon)$. Using the inequalities (12) and (8), from (13) we obtain

$$|\psi(z) - f(z)| \leq |h_i(z)| + |f_i(z) - f(z)| < 4\varepsilon, \quad z \in U_i \cap \overline{G}.$$

Consequently,

$$\|\psi - f\|_G < 4\varepsilon. \tag{14}$$

Considering the differential form $g = \overline{\partial}\psi$ in the domain G^ε, we see that obviously $\overline{\partial} g = 0$. Besides, using (11) and taking in account that f_i is holomorphic in V_i, we get

$$g = \overline{\partial}\psi(z) = \overline{\partial}h_i(z) = \sum_{k=0}^{p} \overline{\partial}h_{ik}(z) = \sum_{k=0}^{p}(f_i(z) - f_k(z))\overline{\partial}e_k(z) \tag{15}$$

for $z \in V_i \cap \overline{G^\varepsilon}$. In addition, denoting $\gamma_0 = \gamma_0(G) = \max_{0\le k\le p} \|\overline{\partial}e_k\|_{U_k}$, by (15) and (10) we obtain

$$\|g\|_{G^\varepsilon} \le \sum_{k=0}^{p} \|f_i - f_k\|_{G^\varepsilon} \|\overline{\partial}e_k\|_{U_k} \le 3\gamma_0\varepsilon. \tag{16}$$

Now, let u_0 be a solution of the equation

$$\overline{\partial}u = g$$

in the domain G^ε, satisfying the uniform estimate

$$\|u_0\|_{G^\varepsilon} \le \gamma(G^\varepsilon)\|g\|_{G^\varepsilon}. \tag{17}$$

Then it follows from the proof of the estimate (17) in [**2**, **3**] that the constants $\gamma(G^\varepsilon)$ are bounded, i.e.

$$\gamma(G^\varepsilon) \le \gamma = \gamma(G). \tag{18}$$

Besides, (17), (16) and (18) imply

$$\|u_0\|_{G^\varepsilon} \le 3\gamma_0\gamma\varepsilon. \tag{19}$$

Further, the function $F(z) = \psi(z) - u_0(z)$ is holomorphic in the domain G^ε since $\overline{\partial}\psi - \overline{\partial}u_0 = g - \overline{\partial}u_0 = 0$. Besides, by (14) and (19)

$$\|f - F\|_G \le \|\psi - f\|_G + \|u_0\|_G < 4\varepsilon + 3\gamma_0\gamma\varepsilon = \gamma_1\varepsilon, \tag{20}$$

where the constant γ_1 depends only on G. □

A stronger assertion than Theorem 3.1 is true for polynomial polyhedrons. Before proving that assertion, recall that a compact set K is said to be polynomially convex if for any point $\zeta \notin K$ there is a polynomial P_ζ such that $|P_\zeta(\zeta)| > \max_{z \in K} |P_\zeta(z)|$. Besides, Oka-Weil's theorem (see., e.g. [5]), states that *any function holomorphic in a neighborhood of a polynomially convex compact set* K *can be uniformly approximated on* K *by polynomials.*

Theorem 3.2. *Let* G *be a real non-degenerate polynomial polyhedron* (1) *and let* $N \leq 2n$. *Then any function* $f \in A(G)$ *can be uniformly approximated on* $\overline{G}$ *by polynomials.*

Proof. Let $\zeta \notin \overline{G}$. By the definition of the polyhedron G, $|\chi_i(\zeta)| > 1$ for some i, which means that $\overline{G}$ is polynomially convex compact set. It suffices to see that the desired assertion follows from Theorem 3.1 and Oka-Weil's theorem. □

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