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DIFFERENTIAL SANDWICH THEOREMS FOR SOME SUBCLASSES OF ANALYTIC FUNCTIONS INVOLVING A LINEAR OPERATOR

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ABSTRACT. By making use of the familiar Carlson–Shaffer operator,the authors derive derive some subordination and superordination results for certain normalized analytic functions in the open unit disk. Relevant connections of the results, which are presented in this paper, with various other known results are also pointed out.

1. INTRODUCTION

Let $\mathcal{H}$ be the class of functions *analytic* in the *open* unit disk

$$\Delta := \{z : |z| < 1\}.$$

Let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots .$$

Let

$$\mathcal{A}_m := \left\{f \in \mathcal{H}, f(z) = z + a_{m+1} z^{m+1} + a_{m+2} z^{m+2} + \cdots\right\}$$

and let $\mathcal{A} := \mathcal{A}_1$. With a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in Δ. Then we say that the function f is *subordinate* to g if there exists a Schwarz function ω, analytic in Δ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \Delta),$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \Delta).$$

We denote this subordination by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z).$$

In particular, if the function g is univalent in Δ, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

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Let $p, h \in \mathcal{H}$ and let $\phi(r, s, t; z) : \mathbb{C}^3 \times \Delta \to \mathbb{C}$. If p and $\phi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z), \tag{1.1}$$

then p is a solution of the differential superordination (1.1). (If f is subordinate to F, then F is called to be superordinate to f.) An analytic function q is called a *subordinant* if $q \prec p$ for all p satisfying (1.1). An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.1) is said to be the best subordinant. Recently Miller and Mocanu [**6**] obtained conditions on h, q and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z).$$

with the results of Miller and Mocanu [**6**], Bulboacă [**3**] investigated certain classes of first order differential superordinations as well as superordination-preserving integral operators [**2**]. Ali et al. [**1**] used the results obtained by Bulboacă [**3**] and gave sufficient conditions for certain normalized analytic functions f to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$$

where q_1 and q_2 are given univalent functions in Δ with $q_1(0) = 1$ and $q_2(0) = 1$. Shanmugam et al. [**7**] obtained sufficient conditions for a normalized analytic functions f to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2f'(z)}{(f(z))^2} \prec q_2(z).$$

where q_1 and q_2 are given univalent functions in Δ with $q_1(0) = 1$ and $q_2(0) = 1$. Recently, the first author combined with the third and fourth authors of this paper obtained sufficient conditions for certain normalized analytic functions f to satisfy

$$q_1(z) \prec \frac{z}{L(a,c)f(z)} \prec q_2(z)$$

where q_1 and q_2 are given univalent functions with $q_1(0) = 1$ and $q_2(0) = 1$ (see [**8**] for details; also see [**9**]). A detailed investigation of starlike functions of complex order and convex functions of complex order using Briot-Bouquet differential subordination technique has been studied very recently by Srivastava and Lashin [**10**].

Let the function $\varphi(a, c; z)$ be given by

$$\varphi(a, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \qquad (c \neq 0, -1, -2, \dots; z \in \Delta),$$

where $(x)_n$ is the *Pochhammer symbol* defined by

$$(x)_n := \begin{cases} 1, & n = 0; \\ x(x+1)(x+2)\dots(x+n-1), & n \in \mathbb{N} := \{1, 2, 3, \dots\}. \end{cases}$$

Corresponding to the function $\varphi(a, c; z)$, Carlson and Shaffer [**4**] introduced a linear operator $L(a, c)$, which is defined by the following Hadamard product (or convolution):

$$L(a, c)f(z) := \varphi(a, c; z) * f(z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} a_n z^{n+1}.$$

We note that

$$L(a, a)f(z) = f(z), \quad L(2, 1)f(z) = zf'(z), \quad L(\delta + 1, 1)f(z) = D^{\delta} f(z),$$

where $D^{\delta} f$ is the *Ruscheweyh derivative* of f.

The main object of the present sequel to the aforementioned works is to apply a method based on the differential subordination in order to derive several subordination results involving the Carlson Shaffer Operator. Furthermore, we obtain the previous results of Srivastava and Lashin [**10**] as special cases of some of the results presented here.

2. Preliminaries

In order to prove our subordination and superordination results, we make use of the following known results.

Definition 1. [**6**, Definition 2, p. 817] Denote by Q the set of all functions f that are analytic and injective on $\overline{\Delta} - E(f)$, where

$$E(f) = \{\zeta \in \partial\Delta : \lim_{z \to \zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta - E(f)$.

Theorem 1. [**5**, Theorem 3.4h, p. 132] *Let the function q be univalent in the open unit disk Δ and θ and ϕ be analytic in a domain D containing $q(\Delta)$ with $\phi(w) \neq 0$ when $w \in q(\Delta)$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$. Suppose that*

1. *Q is starlike univalent in Δ, and*
2. $\Re \left(\dfrac{zh'(z)}{Q(z)} \right) > 0$ *for $z \in \Delta$.*

If

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),$$

then $p(z) \prec q(z)$ and q is the best dominant.

Theorem 2. [**3**] *Let the function q be univalent in the open unit disk Δ and ϑ and φ be analytic in a domain D containing $q(\Delta)$. Suppose that*

1. $\Re \dfrac{\vartheta'(q(z))}{\varphi(q(z))} > 0$ *for $z \in \Delta$,*
2. *$zq'(z)\varphi(q(z))$ is starlike univalent in Δ.*

If $p \in \mathcal{H}[q(0), 1] \cap Q$, with $p(\Delta) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in Δ, and

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)),$$

then $q(z) \prec p(z)$ *and* q *is the best subordinant.*

3. Subordination and Superordination for Analytic Functions

We begin by proving involving differential subordination between analytic functions.

Theorem 3. *Let* $\left(\dfrac{L(a+1,c)f(z)}{z}\right)^{\mu} \in \mathcal{H}$ *and let the function* $q(z)$ *be analytic and univalent in* Δ *such that* $q(z) \neq 0, (z \in \Delta)$. *Suppose that* $\dfrac{zq'(z)}{q(z)}$ *is starlike univalent in* Δ. *Let*

$$\Re\left\{1 + \frac{\xi}{\beta}q(z) + \frac{2\delta}{\beta}(q(z))^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0 \tag{3.1}$$

$$(\alpha,\ \delta,\ \xi,\ \beta \in \mathbb{C};\ \beta \neq 0)$$

and

$$\begin{aligned}\Psi(a,c,\mu,\xi,\beta,\delta,f)(z) := \alpha + \xi\left[\frac{L(a+1,c)f(z)}{z}\right]^{\mu} + \delta\left[\frac{L(a+1,c)f(z)}{z}\right]^{2\mu} \\ + \beta\mu(a+2)\left[\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} - 1\right].\end{aligned} \tag{3.2}$$

If q *satisfies the following subordination:*

$$\Psi(a,c,\mu,\xi,\beta,\delta,f)(z) \prec \alpha + \xi q(z) + \delta(q(z))^2 + \beta\frac{zq'(z)}{q(z)}$$

$$(\alpha,\ \delta,\ \xi,\ \beta, \mu \in \mathbb{C};\ \mu \neq 0;\ \beta \neq 0),$$

then

$$\left(\frac{L(a+1,c)f(z)}{z}\right)^{\mu} \prec q(z) \quad (\mu \in \mathbb{C};\ \mu \neq 0) \tag{3.3}$$

and q *is the best dominant.*

Proof. Let the function p be defined by

$$p(z) := \left(\frac{L(a+1,c)f(z)}{z}\right)^{\mu} \qquad (z \in \Delta;\ z \neq 0;\ f \in \mathcal{A}),$$

so that, by a straightforward computation, we have

$$\frac{zp'(z)}{p(z)} = \mu\left[\frac{z(L(a+1,c)f(z))'}{L(a+1,c)f(z)} - 1\right].$$

By using the identity:

$$z(L(a,c)f(z))' = (1+a)L(a+1,c)f(z) - aL(a,c)f(z),$$

we obtain

$$\frac{zp'(z)}{p(z)} = \mu\left[(a+2)\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} - (a+2)\right].$$

By setting

$$\theta(\omega) := \alpha + \xi\omega + \delta\omega^2 \qquad \text{and} \qquad \phi(\omega) := \frac{\beta}{\omega},$$

it can be easily observed that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C} \setminus \{0\}$ and that

$$\phi(\omega) \neq 0 \quad (\omega \in \mathbb{C} \setminus \{0\}).$$

Also, by letting

$$Q(z) = zq'(z)\phi(q(z)) = \beta\frac{zq'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = \alpha + \xi q(z) + \delta(q(z))^2 + \beta\frac{zq'(z)}{q(z)},$$

we find that $Q(z)$ is starlike univalent in Δ and that

$$\Re\left(\frac{zh'(z)}{Q(z)}\right) = \Re\left\{1 + \frac{\xi}{\beta}q(z) + \frac{2\delta}{\beta}(q(z))^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0,$$

$$(\alpha,\ \delta,\ \xi,\ \beta \in \mathbb{C};\ \beta \neq 0).$$

The assertion (3.3) of Theorem 3 now follows by an application of Theorem 1. $\square$

For the choices $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$ and $q(z) = \left(\frac{1+z}{1-z}\right)^\gamma$, $0 < \gamma \leq 1$, in Theorem 3, we get the following results (Corollaries 1 and 2 below).

Corollary 1. *Assume that* (3.1) *holds. If* $f \in \mathcal{A}$, *and*

$$\Psi(a,c,\mu,\xi,\beta,\delta,f)(z) \prec \alpha + \xi\frac{1+Az}{1+Bz} + \delta\left(\frac{1+Az}{1+Bz}\right)^2 + \frac{\beta(A-B)z}{(1+Az)(1+Bz)}$$

$$(z \in \Delta;\ \alpha,\ \delta,\ \xi,\ \beta, \mu \in \mathbb{C};\ \mu \neq 0; \beta \neq 0),$$

where $\Psi(a,c,\mu,\xi,\beta,\delta,f)$ *is as defined in* (3.2), *then*

$$\left(\frac{L(a+1,c)f(z)}{z}\right)^\mu \prec \frac{1+Az}{1+Bz} \qquad (\mu \in \mathbb{C};\ \mu \neq 0)$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Corollary 2. *Assume that* (3.1) *holds. If* $f \in \mathcal{A}$, *and*

$$\Psi(a,c,\mu,\xi,\beta,\delta,f)(z) \prec \alpha + \xi\left(\frac{1+z}{1-z}\right)^\gamma + \delta\left(\frac{1+z}{1-z}\right)^{2\gamma} + \frac{2\beta\gamma z}{(1-z^2)}$$

$$(\alpha,\ \delta,\ \xi,\ \beta, \mu \in \mathbb{C};\ \mu \neq 0; \beta \neq 0)$$

where $\Psi(a,c,\mu,\xi,\beta,\delta,f)(z)$ *is as defined in* (3.2), *then*

$$\left(\frac{L(a+1,c)f(z)}{z}\right)^\mu \prec \left(\frac{1+z}{1-z}\right)^\gamma \quad (\mu \in \mathbb{C};\ \mu \neq 0)$$

and $\left(\frac{1+z}{1-z}\right)^\gamma$ *is the best dominant.*

For a special case $q(z) = \mathrm{e}^{\mu Az}$, with $|\mu A| < \pi$, Theorem 3 readily yields the following.

Corollary 3. *Assume that* (3.1) *holds. If* $f \in \mathcal{A}$, *and*

$$\Psi(a,c,\mu,\xi,\beta,\delta,f)(z) \prec \alpha + \xi\,\mathrm{e}^{\mu Az} + \delta\,\mathrm{e}^{2\mu Az} + \beta A\mu z$$
$$(\alpha,\ \delta,\ \xi,\ \beta, \mu \in \mathbb{C};\ \mu \neq 0; \beta \neq 0)$$

where $\Psi(a,c,\mu,\xi,\beta,\delta,f)(z)$ *is as defined in* (3.2)*, then*

$$\left(\frac{L(a+1,c)f(z)}{z}\right)^{\mu} \prec \mathrm{e}^{\mu Az} \qquad (\mu \in \mathbb{C},\ \mu \neq 0)$$

and $\mathrm{e}^{\mu Az}$ *is the best dominant.*

For a special case when $q(z) = \dfrac{1}{(1-z)^{2b}}$ $\quad (b \in \mathbb{C} \setminus \{0\})$, $a = c = 1$, $\delta = \xi = 0$, $\mu = \alpha = 1$ and $\beta = \dfrac{1}{b}$, Theorem 3 reduces at once to the following known result obtained by Srivastava and Lashin [**10**].

Corollary 4. *Let* b *be a non zero complex number. If* $f \in \mathcal{A}$, *and*

$$1 + \frac{1}{b}\frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z},$$

then

$$f'(z) \prec \frac{1}{(1-z)^{2b}}$$

and $\frac{1}{(1-z)^{2b}}$ *is the best dominant.*

Next, by appealing to Theorem 2 of the preceding section, we prove Theorem 4 below.

Theorem 4. *Let* q *be analytic and univalent in* Δ *such that* $q(z) \neq 0$ *and* $\dfrac{zq'(z)}{q(z)}$ *be starlike univalent in* Δ*. Further, let us assume that*

$$\Re\left[\frac{2\delta}{\beta}(q(z))^2 + \frac{\xi}{\beta}q(z)\right] > 0, \quad (\delta,\ \xi,\ \beta \in \mathbb{C};\ \beta \neq 0). \tag{3.4}$$

If $f \in \mathcal{A}$,

$$0 \neq \left(\frac{L(a+1,c)f(z)}{z}\right)^{\mu} \in \mathcal{H}[q(0),1] \cap Q,$$

and $\Psi(a,c,\mu,\xi,\beta,\delta,f)$ *is univalent in* Δ, *then*

$$\alpha + \xi q(z) + \delta(q(z))^2 + \beta\frac{zq'(z)}{q(z)} \prec \Psi(a,c,\mu,\xi,\beta,\delta,f)$$
$$(z \in \Delta;\ \alpha,\ \delta,\ \xi,\ \beta, \mu \in \mathbb{C};\ \mu \neq 0; \beta \neq 0),$$

implies

$$q(z) \prec \left(\frac{L(a+1,c)f(z)}{z}\right)^{\mu} \qquad (\mu \in \mathbb{C};\ \mu \neq 0) \tag{3.5}$$

and q *is the best subordinant where* $\Psi(a,c,\mu,\xi,\beta,\delta,f)(z)$ *is as defined in (3.2).*

Proof. By setting

$$\vartheta(w) := \alpha + \xi w + \delta w^2 \qquad \text{and} \qquad \varphi(w) := \beta\frac{1}{w},$$

it is easily observed that ϑ is analytic in $\mathbb{C}$. Also, φ is analytic in $\mathbb{C}\setminus\{0\}$ and that

$$\varphi(w) \neq 0, \qquad (w \in \mathbb{C}\setminus\{0\}).$$

Since q is convex (univalent) function it follows that,

$$\Re\frac{\vartheta'(q(z))}{\varphi(q(z))} = \Re\left[\frac{2\delta}{\beta}(q(z))^2 + \frac{\xi}{\beta}q(z)\right] > 0,$$

$$(\delta,\, \xi,\, \beta \in \mathbb{C};\, \beta \neq 0).$$

The assertion (3.5) of Theorem 4 follows by an application of Theorem 2. □

We remark here that Theorem 4 can easily be restated, for different choices of the function q. Combining Theorem 3 and Theorem 4, we get the following sandwich theorem.

Theorem 5. *Let q_1 and q_2 be univalent in Δ such that $q_1(z) \neq 0$ and $q_2(z) \neq 0$, $(z \in \Delta)$ with $\frac{zq_1'(z)}{q_1(z)}$ and $\frac{zq_1'(z)}{q_1(z)}$ being starlike univalent. Suppose that q_1 satisfies* (3.4) *and q_2 satisfies* (3.1). *If $f \in \mathcal{A}$,*

$$\left(\frac{L(a+1,c)f(z)}{z}\right)^{\mu} \in \mathcal{H}[q(0),1]\cap Q \quad \textit{and} \quad \Psi(a,c,\mu,\xi,\beta,\delta,f)(z)$$

is univalent in Δ, then

$$\alpha + \xi q_1(z) + \delta(q_1(z))^2 + \beta\frac{zq_1'(z)}{q_1(z)} \prec \Psi(a,c,\mu,\xi,\beta,\delta,f)(z)$$
$$\prec \alpha + \xi q_2(z) + \delta(q_2(z))^2 + \beta\frac{zq_2'(z)}{q_2(z)}$$
$$(\alpha,\, \delta,\, \xi,\, \beta, \mu \in \mathbb{C};\, \mu \neq 0; \beta \neq 0),$$

implies

$$q_1(z) \prec \left(\frac{L(a+1,c)f(z)}{z}\right)^{\mu} \prec q_2(z) \quad (\mu \in \mathbb{C};\, \mu \neq 0)$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

Corollary 5. *Let q_1 and q_2 be univalent in Δ such that $q_1(z) \neq 0$ and $q_2(z) \neq 0$ $(z \in \Delta)$ with $\frac{zq_1'(z)}{q_1(z)}$ and $\frac{zq_1'(z)}{q_1(z)}$ being starlike univalent. Suppose q_1 satisfies* (3.4) *and q_2 satisfies* (3.1). *If $f \in \mathcal{A}$, $(f')^{\mu} \in \mathcal{H}[q(0),1]\cap Q$ and let*

$$\Psi_1(\mu,\xi,\beta,\delta,f) := \alpha + \xi\left[f'(z)\right]^{\mu} + \delta\left[f'(z)\right]^{2\mu} + \frac{3}{2}\beta\mu\frac{zf''(z)}{f'(z)}$$

is univalent in Δ, *then*

$$\alpha + \xi q_1(z) + \delta(q_1(z))^2 + \beta\frac{zq_1'(z)}{q_1(z)} \prec \Psi_1(\mu, \xi, \beta, \delta, f)$$

$$\prec \alpha + \xi q_2(z) + \delta(q_2(z))^2 + \beta\frac{zq_2'(z)}{q_2(z)}$$

$$(\alpha,\, \delta,\, \xi,\, \beta, \mu \in \mathbb{C};\ \mu \neq 0; \beta \neq 0),$$

implies

$$q_1(z) \prec (f')^{\mu} \prec q_2(z) \quad (\mu \in \mathbb{C};\ \mu \neq 0)$$

and q_1 *and* q_2 *are respectively the best subordinant and the best dominant.*

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